PHYSICAL REVIEW VOLUME 130, NUMBER 6 15 JUNE 1963

Singularities in the Unphysical Sheets and the Isobar Model*

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(Received 11 February 1963)

Scattering amplitudes involving two- and three-particle states are continued across the inelastic section of the unitarity cut, and the singularities in the unphysical sheets thus reached are determined. Associated with an unstable particle is a complex unitarity cut, through which we further continue into another unphysical sheet. Dynamical singularities associated with a Born-type diagram for the three-particle scattering amplitude are found to be present in the elastic scattering and the production amplitudes also because of the coupling by unitarity. These singularities are isolated from the physical region by the complex unitarity cut and thus cannot be directly responsible for any resonance phenomenon. However, if the interaction force is favorable, a resonance pole can occur in a different Riemann sheet in the vicinity of these singularities. This resonance pole is directly accessible from the physical region and can therefore give rise to a resonance.

I. INTRODUCTION

STUDY of the analytic properties of scattering amplitudes has recently been extended to processes involving multiparticle final states. Attempts have been made[1,2] to incorporate unstable particles into the S-matrix theory. Specifically, several authors[3–6] have considered the problems concerning pion-nucleon scattering and production processes in the approximation that different pairs of particles in a three-particle channel are replaced by their respective resonant states; thus, one treats essentially a multichannel system, each channel consisting of only two particles.

In this paper, we also consider scattering amplitudes involving two- and three-particle states, but instead of studying the analyticity in the physical sheet only with the energy of a resonant state being *replaced* by the complex mass of the associated unstable particle, we extend our consideration of the analyticity of the amplitudes to the unphysical sheets, which are connected with the physical sheet by the three-particle unitarity cut. The analytic continuation is effected by use of the unitarity condition appropriate for energies above the production threshold. Without making specifically the "isobar approximation," we find the locations of the singularities of the amplitudes in the Riemann surface; in this way, we can determine the relevance of some of the singularities to resonances observed at higher energies.

Peierls[7] has proposed an isobar model to account for the second pion-nucleon resonance. In this model, the principal mechanism that generates the resonance is the singularity associated with a diagram where π and N^* are scattered via the exchange of a nucleon, N^* being the 3-3 resonance state. Later, Goebel[8] raised the question concerning the location of that singularity in the Riemann surface. It was claimed that for amplitudes involving stable incident particles, the singularity is not near the physical region and, therefore, cannot be responsible for a resonance in a physical process.

One of our aims in this work is to examine in detail Peierls' isobar mechanism and Goebel's criticism. We find that the singularity under discussion is located at such a position in an unphysical sheet that it cannot directly enhance the scattering amplitude in the physical region, but for a state involving suitable dynamical forces a resonance pole can arise in another unphysical sheet, which can be reached directly from the physical region and is, indeed, responsible for an observable resonance.

In Sec. II we describe the choice of scattering amplitudes for processes involving two- and three-particle states, and derive the unitarity condition that is valid for energies below the four-particle threshold. The integral equations stating the unitarity condition are then used in Sec. III in the analytic continuation of the amplitudes across the three-particle cut into the unphysical sheets, in which are found complex "unitarity" cuts corresponding to channels that have two particles each, one stable and the other unstable. In Sec. IV we investigate the dynamical singularities associated with simple diagrams; they give rise to other singularities in the unphysical sheets, their locations being determined in Sec. V. Further continuation across the complex unitarity cut into other unphysical sheets is considered in Sec. VI, and the existence of resonance poles there is studied in Sec. VII. The last section contains some concluding remarks about this work.

II. UNITARITY CONDITION

To be specific, let us consider the pion-nucleon scattering problem. Since the energy range of interest to us does not extend much beyond the one-pion production region, we are concerned mainly with the two channels: $\pi+N$ and $\pi+\pi+N$. For simplicity, we ignore the spin

* Research supported in part by the U. S. Air Force Office of Scientific Research.
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of the nucleon and the isotopic spins of all the particles; these assumptions do not in any way affect the essence of our study of the dynamics of the problem.

For a three-particle state $\pi_1(k_1)+\pi_2(k_2)+N(p)$, where the symbols inside the parentheses denote the four-momenta, we define the invariant variables as follows:

$$\begin{aligned} s&=-(p+k_1+k_2)^2,\\ \sigma&=-(p+k_1)^2,\\ \omega&=-(k_1+k_2)^2. \end{aligned} \tag{2.1}$$

Thus, s is the square of the total c.m. energy of the three-particle system, σ the c.m. energy squared of the nucleon and the first pion (we assume that the pions are distinguishable), and ω the same for the two-pion system. We use s, σ, and the appropriately chosen angular variables[9] to specify the configuration of the three-particle system.

The unitarity condition may be expressed in various forms depending upon the choice of the transition amplitudes.[1,2,5,6,10,11] The amplitude T_{33}, describing the transition between one three-particle state and another, may be decomposed into connected and disconnected parts:

$$T_{33}=T_{33}{}^{C}+\sum_{i=1}^{3} T_{33}{}^{D_i},$$

where $T_{33}{}^{D_i}$ corresponds to the diagram in which the ith particle does not interact with the others. For definiteness let π_1, π_2, and N be labeled by $i=1$, 2, and 3, respectively. With s and σ being chosen as the energy variables, it is convenient to isolate only the $T_{33}{}^{D_2}$ amplitude and define

$$T_{33}{}^{C'}=T_{33}-T_{33}{}^{D_2}. \tag{2.2}$$

In terms of $T_{33}{}^{C'}$ the unitarity condition takes the form[5,6] (subscripts of T indicating the number of particles in the initial and final states)

$$\begin{aligned} T_{22}(s+)-T_{22}(s-)&=2i\sum T_{22}(s+)T_{22}(s-)\\ &\quad+2i\sum T_{23}(s+,\sigma''+)T_{32}(s-,\sigma''-), \end{aligned} \tag{2.3}$$

$$\begin{aligned} T_{32}(s+,\sigma)-T_{32}(s-,\sigma)&=2i\sum T_{32}(s+,\sigma)T_{22}(s-)\\ &\quad+2i\sum T_{33}{}^{C'}(s+,\sigma,\sigma''+)T_{32}(s-,\sigma''-), \end{aligned} \tag{2.4}$$

$$\begin{aligned} &T_{33}{}^{C'}(s+,\sigma,\sigma')-T_{33}{}^{C'}(s-,\sigma,\sigma')\\ &\quad=2i\sum T_{32}(s+,\sigma)T_{23}(s-,\sigma')\\ &\quad+2i\sum T_{33}{}^{C'}(s+,\sigma,\sigma''+)T_{33}{}^{C'}(s-,\sigma''-,\sigma'), \end{aligned} \tag{2.5}$$

where the angular variables have been suppressed and the $+$ $(-)$ signs following s and σ imply that the values are to be taken just above (below) the real axes. The summation signs denote integrations over the phase space of the intermediate states. The derivation of (2.4) and (2.5) is based on the assumption that T_{32} and T_{33} satisfy the following subsidiary conditions, when s, σ and σ' have physical values,

$$\begin{aligned} &T_{32}(s,\sigma+)-T_{32}(s,\sigma-)\\ &\qquad=2iT_{33}{}^{D_2}(s,\sigma+,\sigma''+)T_{32}(s,\sigma''-), \end{aligned} \tag{2.6}$$

$$\begin{aligned} &T_{33}(s,\sigma+,\sigma')-T_{33}(s,\sigma-,\sigma')\\ &\qquad=2iT_{33}{}^{D_2}(s,\sigma+,\sigma''+)T_{33}(s,\sigma''-,\sigma'), \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} &T_{33}(s,\sigma,\sigma'+)-T_{33}(s,\sigma,\sigma'-)\\ &\qquad=2iT_{33}(s,\sigma,\sigma''+)T_{33}{}^{D_2}(s,\sigma''-,\sigma'-), \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} T_{33}{}^{D_2}(s,\sigma,\sigma')&=T_{33}{}^{D_2}(s,\sigma,\sigma)\\ &=(2\pi)^3\delta^3(\mathbf{k}_2-\mathbf{k}_2')T_{22}(\sigma). \end{aligned} \tag{2.9}$$

Some simplification of the unitarity condition may be achieved by expressing the transition amplitudes in the angular momentum representation. Just as in the case of a two-particle state which can be expanded in the Legendre series in the usual way, the same can be done for the π_1+N system in its own rest frame. In this way, the three-particle state may be regarded as having only two particles, one being π_2, the other having rest mass $\sigma^{1/2}$ and spin l, where l is the orbital angular momentum of the π_1+N system. The helicity amplitudes for transitions involving three-particle states can then be obtained in the way as shown in reference 5. For given values of total c.m. energy $s^{1/2}$ and total angular momentum J, a two-particle elastic scattering amplitude is characterized by no other variables or quantum numbers, whereas a production amplitude is specified in addition by σ, l, and m, the last being the z component of l. (Conservation of such quantum numbers as strangeness and baryon number is taken to be understood.) Thus, the unitarity condition, when expressed in terms of the partial-wave amplitudes, obtains the very simple form where only the three-particle intermediate states involve integrations over σ and summations over l and m.

Among all the possible three-particle states, let us consider only that in which the π_1+N system can have a resonance in a particular l state, and assume that the amplitudes for all the other l values are negligible in strength. For the sake of clarity, let us further assume that the resonance occurs in the $l=0$ state so that among all the production and three-particle scattering amplitudes, we need only retain the ones corresponding to $l=0$ and $m=0$. These assumptions are made only for the purpose of simplifying the notations in the problem, and do not affect any of our conclusions about the locations of the dynamical singularities in the Riemann surface.

In terms of the partial-wave amplitudes,

$$T_{fi}{}^{J}=2\pi\int_{-1}^{+1} T_{fi}(\theta,\varphi=0)P_J(\cos\theta)d\cos\theta, \tag{2.10}$$

[9] The angular variables are those that specify the directions of the vector $\mathbf{k}_1+\mathbf{p}$ in the c.m. system of the s channel and of the vector $\mathbf{p}$ in the c.m. system of the σ channel.

[10] R. E. Cutkosky, J. Math. Phys. **1**, 429 (1960).

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where θ is the scattering angle between nucleons [or $(\pi_1 N)$] in the over-all center-of-mass system, let us define

$$\begin{aligned} A(s)&=T_{22}{}^J(s),\quad a(\sigma)=T_{22}{}^{l=0}(\sigma),\\ B(s,\sigma)&=T_{23}{}^J(s,\sigma)/a(\sigma),\\ C(s,\sigma,\sigma')&=T_{33}{}^{C',J}(s,\sigma,\sigma')/a(\sigma)a(\sigma'). \end{aligned} \tag{2.11}$$

Time-reversal and space-reflection invariances imply that $B(s,\sigma)$ is also equal to $T_{32}{}^J(s,\sigma)/a(\sigma)$ and that $C(s,\sigma,\sigma')=C(s,\sigma',\sigma)$. By construction, $B(s,\sigma)$ and $C(s,\sigma,\sigma')$ have no discontinuities in σ and σ' when these variables are in the physical region. With the masses of the nucleon and the pion denoted by N and μ, respectively, the unitarity condition now becomes

$$A(s+)-A(s-)=2i\rho_1(s+)\theta[s-(N+\mu)^2]A(s+)A(s-)+2i\int_{\sigma_0}^{\sigma_1(s+)}d\sigma''\rho_2(s+,\sigma'')\times\theta[s-(N+2\mu)^2]a(\sigma''+)a(\sigma''-)B(s+,\sigma'')B(s-,\sigma''), \tag{2.12}$$

$$B(s+,\sigma)-B(s-,\sigma)=2i\rho_1(s+)\theta[s-(N+\mu)^2]B(s+,\sigma)A(s-)+2i\int_{\sigma_0}^{\sigma_1(s+)}d\sigma''\rho_2(s+,\sigma'')\times\theta[s-(N+2\mu)^2]a(\sigma''+)a(\sigma''-)C(s+,\sigma,\sigma'')B(s-,\sigma''), \tag{2.13}$$

$$C(s+,\sigma,\sigma')-C(s-,\sigma,\sigma')=2i\rho_1(s+)\theta[s-(N+\mu)^2]B(s+,\sigma)B(s-,\sigma')+2i\int_{\sigma_0}^{\sigma_1(s+)}d\sigma''\rho_2(s+,\sigma'')\times\theta[s-(N+2\mu)^2]a(\sigma''+)a(\sigma''-)C(s+,\sigma,\sigma'')C(s-,\sigma'',\sigma'), \tag{2.14}$$

where the limits of the integrations are

$$\sigma_0=(N+\mu)^2,\quad \sigma_1(s)=(s^{1/2}-\mu)^2. \tag{2.15}$$

The phase-space factors are

$$\rho_1(s)=k(s,N^2)/32\pi^2 s^{1/2}, \tag{2.16}$$

$$\rho_2(s,\sigma)=k(s,\sigma)q(\sigma)/32(2\pi)^5(s\sigma)^{1/2}, \tag{2.17}$$

where

$$k(s,m^2)=[s-(m+\mu)^2]^{1/2}[s-(m-\mu)^2]^{1/2}/2s^{1/2},\quad m^2=N^2 \text{ or } \sigma, \tag{2.18}$$

$$q(\sigma)=[\sigma-(N+\mu)^2]^{1/2}[\sigma-(N-\mu)^2]^{1/2}/2\sigma^{1/2}. \tag{2.19}$$

III. UNPHYSICAL SHEETS

As a function of the complex variable s, $\rho_1(s)$ has square-root branch points at $s=(N+\mu)^2$ and $(N-\mu)^2$. We cut the s plane along the real axis from $(N+\mu)^2$ to $+\infty$ and from $(N-\mu)^2$ to $-\infty$; thus $\rho_1(s)$ satisfies the reflection property

$$\rho_1(s)=-\rho_1{}^*(s^*). \tag{3.1}$$

For a fixed value of σ, $\rho_2(s,\sigma)$ likewise has branch points at $s=(\sigma^{1/2}\pm\mu)^2$, which we join with $\pm\infty$ by branch lines in a similar manner. The corresponding reflection property is

$$\rho_2(s,\sigma)=-\rho_2{}^*(s^*,\sigma^*). \tag{3.2}$$

On the other hand, for a fixed value of s, $\rho_2(s,\sigma)$ has four branch points in the σ plane; they are $\sigma=(s^{1/2}\pm\mu)^2$ and $(N\pm\mu)^2$. We cut the σ plane by three branch lines which join $-\infty$ to $(N-\mu)^2$, $(N+\mu)^2$ to $(s^{1/2}-\mu)^2$, and $(s^{1/2}+\mu)^2$ to $+\infty$. It is evident that (3.2) is thereby also satisfied in the σ plane.

The physical branches are defined to be those in which the values of $\rho_1(s=N^2)$, $\rho_2(s=\sigma,\sigma)$, and $\rho_2(s,\sigma=s)$ are all positive imaginary. In these branches physical values of $\rho_1(s)$ and $\rho_2(s,\sigma)$ are obtained when s approaches the real axis from above and is greater than $(N+\mu)^2$ and $(\sigma^{1/2}+\mu)^2$, respectively, provided that in the second case σ is real and greater than $(N+\mu)^2$. In the σ plane, the physical region is just *below* the cut between $(N+\mu)^2$ and $(s^{1/2}-\mu)^2$. Let us use σ_r (σ_l) to indicate the value of σ on the right (left) side of this cut as one views in the direction from $(N+\mu)^2$ to $(s^{1/2}-\mu)^2$; this definition is useful even when s is complex. It is clear that in (2.12)–(2.14) the integrations in σ'' should be instructed to be performed on the right side of the cut in $\rho_2(s+,\sigma'')$; thus σ_r'' is implied in the argument.

We note that the left-hand sides of the equations, (2.12)–(2.14), are antisymmetric under the interchange of $s+$ and $s-$. In view of (3.1), (3.2), and the fact that the phase-space factors are real in the physical region, we see that the right-hand sides of (2.12) and (2.14) are also antisymmetric provided that

$$\rho_2(s+,\sigma_r'')\leftrightarrow\rho_2(s-,\sigma_l'') \tag{3.3}$$

when $s+$ and $s-$ are interchanged. Indeed, (3.3) is required if the contour of integration is to remain unchanged. Considerations of this nature lead to an alternative form of (2.13)

$$B(s+,\sigma)-B(s-,\sigma)=-2i\rho_1(s-)\theta[s-(N+\mu)^2]B(s-,\sigma)A(s+)-2i\int_{\sigma_0}^{\sigma_1(s-)}d\sigma''\rho_2(s-,\sigma_l'')\times\theta[s-(N+2\mu)^2]a(\sigma''+)a(\sigma''-)C(s-,\sigma,\sigma'')B(s+,\sigma''). \tag{3.4}$$

We now wish to continue the partial-wave scattering amplitudes across the physical cut in the s plane into the unphysical sheets by means of the unitarity relations (2.12)–(2.14). Let us use the subscript u to label the amplitudes in the first unphysical sheet that is reached by a clockwise continuation, while v refers to a counterclockwise continuation. Since the procedures involved in effecting these two types of continuations are essentially the same, we confine our discussions mainly to the amplitudes in sheet u.

Analytic continuations of two-particle amplitudes across two-particle unitarity branch cuts have been considered quite extensively in the literature.[12–17] The problem is generally not made more complicated with the inclusion of multiparticle amplitudes in the consideration, if one assumes the usual unphysical unitarity condition, and if no dynamical singularities enter the two-particle region. The simplicity is evident from (2.12)–(2.14) where, for s in the region $(N+\mu)^2<s<(N+2\mu)^2$, the integral terms are irrelevant. The results of the continuation are that the two-particle branch cut connects two Riemann sheets and that the amplitudes in the unphysical sheet are

$$A_u(s)=A(s)[1+2i\rho_1(s)A(s)]^{-1}, \tag{3.5a}$$

$$B_u(s,\sigma)=B(s,\sigma)[1+2i\rho_1(s)A(s)]^{-1}, \tag{3.5b}$$

$$C_u(s,\sigma,\sigma')=C(s,\sigma,\sigma')-2i\rho_1(s)B(s,\sigma)B(s,\sigma')\times[1+2i\rho_1(s)A(s)]^{-1}. \tag{3.5c}$$

Clearly, these amplitudes have poles at those values of s where $1+2i\rho_1(s)A(s)$ vanishes. If the poles are near the section $(N+\mu)^2<s<(N+2\mu)^2$ of the real axis, they represent unstable particles.

Before we proceed to consider the continuation across the inelastic section of the unitarity cut, let us first make the obvious remark that the two-particle elastic amplitude $a(\sigma)$ can, in the same manner as $A(s)$, be continued across the elastic section of the unitarity cut in the σ plane. The amplitude in the unphysical sheet is

$$a_u(\sigma)=a(\sigma)[1+2i\rho_1(\sigma)a(\sigma)]^{-1}. \tag{3.6}$$

We assume that the πN channel has an unstable particle in the $l=0$ state so that $a_u(\sigma)$ has a conjugate pair of poles at $\sigma=M^2$ and M^{2*}, where[18]

$$1+2i\rho_1(M^2)a(M^2)=1+2i\rho_1(M^{2*})a(M^{2*})=0. \tag{3.7}$$

In a realistic situation, this unstable particle would correspond to the resonance in the $I=\frac{3}{2}$, $J=\frac{3}{2}$ state of the πN system.

In the unphysical sheet u reached by a clockwise continuation across the section $(N+2\mu)^2<s<(N+3\mu)^2$ of the unitarity cut,[19] the amplitudes are required to satisfy along that section of the real axis the following conditions:

$$A_u(s-)=A(s+),\quad B_u(s-,\sigma)=B(s+,\sigma),\quad C_u(s-,\sigma,\sigma')=C(s+,\sigma,\sigma'). \tag{3.8a}$$

The limits of integrations over σ'' in (2.12)–(2.14) restricts the value of σ'' to be less than $(N+2\mu)^2$ when s is less than $(N+3\mu)^2$; consequently, we may write

$$a(\sigma''+)=a_u(\sigma''-), \tag{3.8b}$$

where $a_u(\sigma'')$ is given by (3.6). With ρ_1 and ρ_2 expressed in terms of $s-$, (2.12) becomes, for $(N+2\mu)^2<s<(N+3\mu)^2$,

$$A_u(s-)-A(s-)=-2i\rho_1(s-)A(s-)A_u(s-)-2i\int_{\sigma_0}^{\sigma_1(s-)}d\sigma''\rho_2(s-,\sigma_l'')\times a(\sigma''-)a_u(\sigma''-)B(s-,\sigma'')B_u(s-,\sigma''). \tag{3.9}$$

Similar expressions follow for (2.13), (2.14), and (3.4). We can analytically continue these equations into the complex s plane, and obtain

$$A_u(s)=[1+2i\rho_1(s)A(s)]^{-1}\left[A(s)-2i\int_{\sigma_0}^{\sigma_1(s)}d\sigma''\rho_2(s,\sigma_l'')a(\sigma'')a_u(\sigma'')B(s,\sigma'')B_u(s,\sigma'')\right], \tag{3.10}$$

$$B_u(s,\sigma)=[1+2i\rho_1(s)A(s)]^{-1}\left[B(s,\sigma)-2i\int_{\sigma_0}^{\sigma_1(s)}d\sigma''\rho_2(s,\sigma_l'')a(\sigma'')a_u(\sigma'')C_u(s,\sigma,\sigma'')B(s,\sigma'')\right], \tag{3.11}$$

$$B_u(s,\sigma)=B(s,\sigma)[1-2i\rho_1(s)A_u(s)]-2i\int_{\sigma_0}^{\sigma_1(s)}d\sigma''\rho_2(s,\sigma_l'')a(\sigma'')a_u(\sigma'')C(s,\sigma,\sigma'')B_u(s,\sigma''), \tag{3.12}$$

$$C_u(s,\sigma,\sigma')=C(s,\sigma,\sigma')-2i\rho_1(s)B(s,\sigma)B_u(s,\sigma')-2i\int_{\sigma_0}^{\sigma_1(s)}d\sigma''\rho_2(s,\sigma_l'')a(\sigma'')a_u(\sigma'')C(s,\sigma,\sigma'')C_u(s,\sigma'',\sigma'). \tag{3.13}$$

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[15] W. Zimmermann, Nuovo Cimento **21**, 249 (1961).
[16] P. G. O. Freund and R. Karplus, Nuovo Cimento **21**, 519, 531 (1961).
[17] R. C. Hwa and D. Feldman, Ann. Phys. (N. Y.) **21**, 453 (1963).
[18] For definiteness we let M have a negative imaginary part.
[19] This sheet is, of course, different from the one defined by (3.5). However, we continue to use the label u for this sheet, since we shall not be concerned again with that sheet reached by continuation across the elastic unitarity cut of the s channel.

It has been shown by Stapp,[1] Dragt and Karplus[20] that the singularities at the thresholds of states involving an even number of particles are square-root branch points, while the ones at the thresholds of states involving an odd number of particles are logarithmic branch points. Thus, the branch cut in the s plane between $(N+\mu)^2$ and $(N+2\mu)^2$ connects two Riemann sheets, while the cut between $(N+2\mu)^2$ and $(N+3\mu)^2$ connects an infinite number of sheets. This is due essentially to the fact that $\rho_1(s)$ has a square-root branch point at $s=(N+\mu)^2$, but $\rho_2(s,\sigma_l'')$ returns to its original value after s is taken in a circuit around the point $(N+2\mu)^2$, as can most easily be seen in the σ'' plane.

It is evident from (3.10)–(3.13) that the amplitudes $A_u(s)$, $B_u(s,\sigma)$, and $C_u(s,\sigma,\sigma')$ have end-point singularities at $s=s_a\equiv(M+\mu)^2$, where the upper limit, $\sigma_1(s)$, of the integration over σ'' reaches the pole of $a_u(\sigma'')$ at $\sigma''=M^2$. This is the threshold of a branch cut which we place in parallel to the real axis. It corresponds to the πM channel in the intermediate state, and therefore is referred to as the complex unitarity cut. Continuation across this cut is considered in a later section.

There are also other branch cuts in the complex plane of sheet u due to the singularities of the amplitudes in the physical sheet. A discussion of them is deferred until after we have considered the pertinent dynamical singularities of $A(s)$, $B(s,\sigma)$, and $C(s,\sigma,\sigma')$.

We now want to find a formal solution to the coupled integral equations (3.10)–(3.13) and determine the condition under which a pole may appear in sheet u. We assume that all the amplitudes on the physical sheet are known. Because of (3.6), $a_u(\sigma)$ is also considered as known. Writing $B_u(s,\sigma)$ as

$$B_u(s,\sigma)=\beta(s,\sigma)[1-2i\rho_1(s)A_u(s)], \qquad (3.14)$$

we see that, because of (3.12), $\beta(s,\sigma)$ satisfies the equation

$$\beta(s,\sigma)=B(s,\sigma)-2i\int_{\sigma_0}^{\sigma_1(s)} d\sigma''\rho_2(s,\sigma_l'') \times a(\sigma'')a_u(\sigma'')C(s,\sigma,\sigma'')\beta(s,\sigma''), \qquad (3.15)$$

to which the solution is

$$\beta(s,\sigma)=B(s,\sigma)+\int_{\sigma_0}^{\sigma_1(s)} d\sigma''\Gamma(s,\sigma,\sigma'')B(s,\sigma''), \qquad (3.16)$$

where $\Gamma(s,\sigma,\sigma'')$ is the resolvent for the kernel

$$K(s,\sigma,\sigma'')=-2i\rho_2(s,\sigma_l'')a(\sigma'')a_u(\sigma'')C(s,\sigma,\sigma''). \qquad (3.17)$$

The resolvent can have poles at isolated values of s where the Fredholm determinant vanishes; clearly, at these values of s, $\beta(s,\sigma)$ also has poles. Similarly, from (3.13), we obtain

$$C_u(s,\sigma,\sigma')=\gamma(s,\sigma,\sigma')-2i\rho_1(s)\beta(s,\sigma)B_u(s,\sigma'), \qquad (3.18)$$

where

$$\gamma(s,\sigma,\sigma')=C(s,\sigma,\sigma')+\int_{\sigma_0}^{\sigma_1(s)} d\sigma''\Gamma(s,\sigma,\sigma'')C(s,\sigma'',\sigma'). \qquad (3.19)$$

Now, using (3.10), (3.14), (3.18), and the definition that

$$F(s)=\int_{\sigma_0}^{\sigma_1(s)} d\sigma''\rho_2(s,\sigma_l'') \times a(\sigma'')a_u(\sigma'')B(s,\sigma'')\beta(s,\sigma''), \qquad (3.20)$$

we have

$$A_u(s)=[A(s)-2iF(s)]/D(s), \qquad (3.21)$$

$$B_u(s,\sigma)=\beta(s,\sigma)/D(s), \qquad (3.22)$$

$$C_u(s,\sigma,\sigma')=\gamma(s,\sigma,\sigma')-2i\rho_1(s)\beta^2(s,\sigma)/D(s), \qquad (3.23)$$

where

$$D(s)=1+2i\rho_1(s)[A(s)-2iF(s)]. \qquad (3.24)$$

It is evident from these equations that the amplitudes in the unphysical sheet u have poles when $D(s)$ vanishes, and that there are no other poles whose positions depend only on s. That the latter statement is true also for $C_u(s,\sigma,\sigma')$ can be more easily seen in (3.11) than in (3.23). Thus, a pole in the resolvent $\Gamma(s,\sigma,\sigma'')$ does not lead to any singularities in the amplitudes.

Some parenthetical remarks may be made to compare this problem with one in which both of the two channels under consideration have only two particles each. If we use $\rho_2(s)$ to denote the phase-space factor in the second two-particle channel and $C(s)$ to represent the partial-wave-scattering amplitude in that channel, then it can be shown that the quantity $\Gamma(s)$, which corresponds to the resolvent $\Gamma(s,\sigma,\sigma'')$ in the present problem, is $-2i\rho_2(s)C(s)/[1+2i\rho_2(s)C(s)]$. Thus, a pole in $\Gamma(s)$ is due to a zero in the S matrix of the second channel; this pole does not appear in the unphysical sheet that is reached by continuation across the two-channel cut (which corresponds to sheet u in the present problem) but is in sheet IV defined by Oehme.[13] Vanishing of the denominator $D(s)$ corresponds to the zeros of the determinant of the two-channel S matrix; if they occur near the real axis above the second threshold, the corresponding poles in the amplitudes are associated with unstable particles which are coupled to both channels.

On account of the integrals in $F(s)$, $\beta(s,\sigma)$, and $\gamma(s,\sigma,\sigma')$, there are branch cuts in $A_u(s)$, $B_u(s,\sigma)$, and $C_u(s,\sigma,\sigma')$ that are not present in the amplitudes on the physical sheet. This is a distinct property associated with analytic continuations across multiparticle unitarity cuts.

IV. DYNAMICAL SINGULARITIES

The dynamical, left-hand singularities of two-particle scattering amplitudes generally must lie in the un-

[20] A. J. Dragt and R. Karplus, Nuovo Cimento 26, 168 (1962).

physical region of the physical sheet, i.e., to the left of the physical threshold. This is not true for amplitudes involving three or more particle states, as is well known. In this paper we are interested in studying the singularities that are in the physical region; in particular, we want to investigate those that give rise to singularities in the unphysical sheets which we have considered.

For the amplitude $C(s,\sigma,\sigma')$ consider the diagram of one-nucleon exchange, as shown in Fig. 1. If the disconnected process $T_{33}{}^{D_2}$ were included in the three-particle scattering amplitude, then this diagram would represent a certain term in the unitarity condition corresponding to the connected part of the disconnected diagrams.[1,2] In our case, where the disconnected process is excluded, this diagram must be introduced as a "Born term" of the scattering amplitude. It has the form

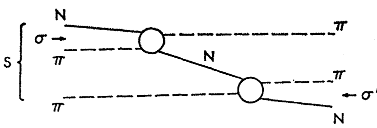

FIG. 1. One-nucleon-exchange diagram for the amplitude $T_{33}{}^{C'}(s,\sigma,\sigma')$.

$$T_{33}{}^{B}(s,\sigma,\sigma')=[a(\sigma)a(\sigma')/k(s,\sigma)k(s,\sigma')]Q_J(z), \quad (4.1)$$

where $Q_J(z)$ is the Legendre function of the second kind, and z is defined by

$$z=-\frac{2s(N^2-\sigma'-\mu^2)+(s+\sigma'-\mu^2)(s-\sigma+\mu^2)}{\{[s-(\sigma^{1/2}+\mu)^2][s-(\sigma^{1/2}-\mu)^2][s-(\sigma'^{1/2}+\mu)^2][s-(\sigma'^{1/2}-\mu)^2]\}^{1/2}}. \quad (4.2)$$

The function $k(s,\sigma)$ is given by (2.18). In view of (2.11), the contribution to the amplitude $C(s,\sigma,\sigma')$ from this Born term is

$$C^B(s,\sigma,\sigma')=[k(s,\sigma)k(s,\sigma')]^{-1}Q_J(z). \quad (4.3)$$

The Legendre function $Q_J(z)$ has branch points at $z=\pm1$. Their locations in the s plane are, according to (4.2),

$$s_\pm(\sigma,\sigma')=g\pm(g^2-h)^{1/2}, \quad (4.4)$$

where

$$g=\tfrac{1}{2}[\sigma+\sigma'-N^2+2\mu^2+(\sigma-\mu^2)(\sigma'-\mu^2)/N^2],$$
$$h=-(\sigma-\mu^2)(\sigma'-\mu^2)+(\sigma+\sigma'-2\mu^2)(\sigma\sigma'-\mu^4)/N^2.$$

In the special case when $\sigma=\sigma'$, we have

$$s_+(\sigma,\sigma)=2(\sigma+\mu^2)-N^2,$$
$$s_-(\sigma,\sigma)=(\sigma-\mu^2)^2/N^2. \quad (4.5)$$

When σ and σ' are both at their respective physical thresholds, $(N+\mu)^2$, s_+ coincides with s_- at the over-all physical threshold, $s=(N+2\mu)^2$. This is to be expected since at this energy the intermediate state of the process in Fig. 1 is on the mass shell. For σ, $\sigma'<(N+\mu)^2$, we have $s_\pm<(N+2\mu)^2$ and $s_+>s_-$. On the other hand, for σ, $\sigma'>(N+\mu)^2$, we have $s_\pm>(N+2\mu)^2$ and $s_+<s_-$. When $\sigma=\sigma'=(N+2\mu)^2$, $s_\pm$ is in the neighborhood of $(N+4\mu)^2$. Thus, we see that for a certain range of physical values of σ and σ' the short-branch cut associated with this Born term is on the real axis above the physical threshold of the three-particle system. Obviously, the unphysical sheets which we have considered in the previous section are reached by analytic continuations that avoid this cut. When the difference between σ and σ' is sufficiently large, $s_\pm(\sigma,\sigma')$ become complex. Our aim is to investigate how singularities such as these affect the two-particle elastic scattering amplitude and the production amplitude.

We note that, when either σ or σ' is at the threshold σ_0, the two branch points $s_\pm(\sigma,\sigma')$ coincide; the common position in the s plane (for $\sigma'=\sigma_0$) is at

$$s_\pm(\sigma,\sigma_0)=[(\sigma+\mu N)\sigma_0{}^{1/2}-\mu^3]/N. \quad (4.6)$$

This equation may be inverted to yield the corresponding position in the σ plane for a given value of s:

$$\sigma_\pm(s,\sigma_0)=(sN+\mu^3)/(N+\mu)-\mu N. \quad (4.7)$$

These expressions are useful later on when we perform analytic continuations.

It is important to ascertain that $C^B(s,\sigma,\sigma')$ has no discontinuities across the real axes of the σ and σ' planes in the physical regions. This property can be obtained most readily in the s plane if we put the kinematical cuts, due to the branch points of $k(s,\sigma)$ and $k(s,\sigma')$, between $(\sigma^{1/2}+\mu)^2$ and $(\sigma'^{1/2}+\mu)^2$, and between $(\sigma^{1/2}-\mu)^2$ and $(\sigma'^{1/2}-\mu)^2$. In the σ plane the branch cut between $\sigma_0=(N+\mu)^2$ and $\sigma_1(s)=(s^{1/2}-\mu)^2$ should not be on the real axis but should be placed either above or below *both* $\sigma+$ and $\sigma-$, depending upon whether s has a positive or negative imaginary part; otherwise, a discontinuity in σ would result. Similar consideration follows for the σ' variables.

In the case of the production amplitude $B(s,\sigma)$ there can also be a one-nucleon-exchange diagram, as shown in Fig. 2. The corresponding singularities in the s plane can be found in just the same way as for $C^B(s,\sigma,\sigma')$; in fact, we need only put σ' equal to N^2 in (4.2)–(4.4). Denoting the branch points for this Born term $B^B(s,\sigma)$ by $s_\pm(\sigma)$, we have

$$s_\pm(\sigma)=\bar{g}\pm(\bar{g}^2-\bar{h})^{1/2}, \quad (4.8)$$

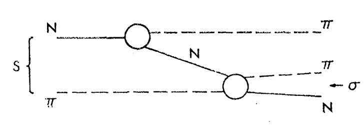

FIG. 2. One-nucleon-exchange diagram for the amplitude $T_{23}(s,\sigma)$.

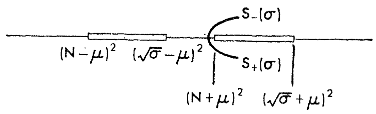

FIG. 3. A branch cut for $B(s,\sigma)$ in the s plane.

where

$$\bar{g}=\tfrac{1}{2}[2\sigma+\mu^2-(\sigma-\mu^2)\mu^2/N^2],$$
$$\bar{h}=(\sigma-\mu^2)^2+\mu^2(\sigma+N^2-2\mu^2)(1-\mu^2/N^2). \tag{4.9}$$

It is interesting to note that if (4.8) is solved for σ as a function of s, we regain the same expressions as (4.8) and (4.9) except that the roles of σ and s are interchanged. That is,

$$\sigma_\pm(s)=\bar{\gamma}\mp(\bar{\gamma}^2-\bar{\eta})^{1/2}, \tag{4.10}$$

where

$$\bar{\gamma}=\tfrac{1}{2}[2s+\mu^2-(s-\mu^2)\mu^2/N^2],$$
$$\bar{\eta}=(s-\mu^2)^2+\mu^2(s+N^2-2\mu^2)(1-\mu^2/N^2). \tag{4.11}$$

For σ in the interval $(N-\mu)^2<\sigma<(N+\mu)^2$, $s_\pm(\sigma)$ are real and are in the same interval $(N-\mu)^2<s<(N+\mu)^2$ in the s plane with $s_+(\sigma)$ being greater than $s_-(\sigma)$. When σ is continued to a value greater than $(N+\mu)^2$, the branch points enter into the complex part of the s plane before reaching the threshold $s=(N+\mu)^2$. If σ has a small positive imaginary part, then $s_+(\sigma)$ is in the lower half plane, while $s_-(\sigma)$ is at a conjugate point in the upper half plane. These points are interchanged if σ has a small negative imaginary part. Now, the Cauchy representation for $B(s,\sigma)$ in the s plane, taking $B^B(s,\sigma)$ as the only left-hand cut, has the form

$$B(s,\sigma)=B^B(s,\sigma)+\frac{1}{\pi}\int_{(N+\mu)^2}^{\infty}ds'\frac{\rho_1(s'+)B(s'+,\sigma)A_u(s'+)}{s'-s}+\frac{1}{\pi}\int_{(N+2\mu)^2}^{\infty}\cdots, \tag{4.12}$$

where the last term represents the dispersion integral beginning at the production threshold, $(N+2\mu)^2$. In view of the fact that the first integral starts at the elastic threshold $(N+\mu)^2$, the dynamical cut that joins $s_+(\sigma)$ with $s_-(\sigma)$ must be distorted to avoid the branch point at $s=(N+\mu)^2$, when σ is continued to a value greater than σ_0. This is illustrated in Fig. 3. It can be verified that the property $B(s,\sigma+)=B(s,\sigma-)$ is preserved.

There are also other Born-type diagrams for the production amplitude. One of them is the one-pion-exchange diagram shown in Fig. 4. A number of studies[5,6,21] has been made on the analyticity of the production amplitude $B(s,\omega)$, taking into account only this interaction as the driving force. It is reasonable in this case to consider the amplitude $B(s,\omega)$, where ω, defined in (2.1), is the square of the c.m. energy of the two-pion system. It has been found that this interaction produces anomalous threshold which extends into the complex s plane.

One can also investigate the effect of this interaction on the amplitude $B(s,\sigma)$, although it is somewhat more complicated because the σ channel does not represent a convenient pair of particles in the final state of that diagram. Since the partial-wave amplitudes correspond to a projection of the full scattering amplitudes, which selects only the part that involves physical scattering angles, they do not contain all the singularities of the full amplitudes. The different pairings in a three-particle state then correspond to considering different regions of the over-all physical sheet spanned by all the independent invariant variables. It is, therefore, expected that the singularities in $B(s,\sigma)$ are different from those in $B(s,\omega)$ even for the same diagram. However, if one is interested in the effect of the production amplitude on the two-particle elastic scattering amplitude, there is no need to consider the same diagram in two different ways. The simpler choice of B suffices, since $A(s)$ is independent of σ and ω. For this reason we do not duplicate here the consideration of the one-pion-exchange diagram.

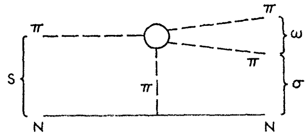

FIG. 4. One-pion-exchange diagram for the production amplitude.

[21] V. L. Teplitz, Physics Department Technical Report No. 270, University of Maryland, 1962 (unpublished).

We briefly discuss two other simple diagrams shown in Fig. 5. The particle m, if unstable, corresponds to a pole in the unphysical sheet of the two-pion scattering amplitude. In the case of Fig. 5(a), $B(s,\sigma)$ is the convenient amplitude to study; the pole is in the usual t variable. The associated branch cuts in the s plane are along the negative real axis and in the unphysical region of the complex plane; these cuts are far to the left of the physical threshold, especially if m is assigned a mass as large as that of the ρ particle. The amplitude for the diagram in Fig. 5(b) has a pole in the ω variable and, therefore, does not contribute to any partial-wave amplitudes other than the ones involving the S state of the ω channel. This pole term, however, gives rise to branch cuts for all waves in the σ channel. In fact, properties of the amplitude for the diagram in Fig. 5(b) can be inferred from those associated with Fig. 5(a), if the variables s and σ are interchanged. For m having the mass of ρ, the trajectories of the branch points in the s plane, as σ is varied from N^2 to a physical value, prescribe approximately a semicircle whose center is in the neighborhood of $s\approx N^2$ and whose radius is as large as $4N^2$; the branch points start at conjugate points in the complex plane and approach the real axis symmetrically toward the right along the semicircle. Since

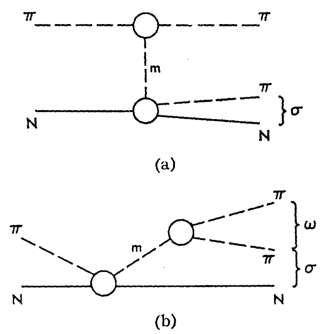

FIG. 5. Two simple diagrams involving particle m.

the singularities associated with both diagrams of Fig. 5 are so far away from the region of interest to us, $(N+2\mu)^2<s<(N+3\mu)^2$, we do not consider them any further.

V. SINGULARITIES IN THE UNPHYSICAL SHEETS

We now examine how the singularities that we have considered in the last section affect the analytic properties of the amplitudes in the unphysical sheets. It is evident from (3.5) that all branch cuts in the physical sheet result in branch cuts also in the unphysical sheet that is reached by continuation across the elastic section of the unitarity cut. However, we are concerned in this paper, exclusively, with the unphysical sheets which are associated with the three-particle intermediate state of the unitarity condition and with the complex unitarity cut.

Consider first the amplitude $C_u(s,\sigma,\sigma')$, which satisfies the integral equation (3.13). The upper limit of integration over σ'' depends upon the value of s; as s is varied, the contour of integration can be deformed so long as it avoids all singularities in the σ'' plane. We have already considered the end-point singularity s_a, where $\sigma_1(s)$ coincides with the pole of $a_u(\sigma'')$; it is the threshold of the πM complex unitarity cut. There are other singularities in the σ'' plane, among which are those due to the Born term of $C(s,\sigma,\sigma'')$ shown in Fig. 1. The locations of the branch points, denoted by $\sigma_\pm''(s,\sigma)$, can be obtained from the equation for $s_\pm(\sigma,\sigma'')$, i.e., Eq. (4.4), by expressing σ'' in terms of s and σ. It is clear from (4.2) that the denominator of $z(s,\sigma,\sigma'')$ vanishes as σ'' approaches $(s^{1/2}-\mu)^2$; hence, except for some unphysical value of σ, the upper limit of integration, $\sigma_1(s)$, when different from σ_0, can never be reached by the branch points $\sigma_\pm''(s,\sigma)$ which correspond to $z=\pm1$. In fact, for physical values of s and σ, $\sigma_\pm''(s,\sigma)$ cannot be greater than $(s^{1/2}-\mu)^2$ in magnitude as can be inferred from (4.7), while on the other hand it must be greater than $(N+\mu)^2$ for $s>(\sigma^{1/2}+\mu)^2$. It, therefore, follows that $\sigma_\pm''(s,\sigma)$ must be in the region between σ_0 and $\sigma_1(s)$.

As we have mentioned in the last section, in order to guarantee that $C^B(s,\sigma,\sigma''+)=C^B(s,\sigma,\sigma''-)$, the branch cut between σ_0 and $\sigma_1(s)$ must not be placed on the real axis of the σ'' plane even in the limit of s approaching a real value. Since (3.13) is obtained as a result of s being continued from $s-$ [cf. (3.9)], this cut should be a continuous deformation of the one placed below the real axis. As instructed by the subscript l of σ'' in the argument of ρ_2 in (3.13), the contour of integration is on the left side of the cut when viewed from σ_0 to $\sigma_1(s)$. The important question to answer is on which side of the contour the singularities $\sigma_\pm''(s,\sigma)$ are located.

Before answering this question, we first recall that the singularities $s_\pm(\sigma,\sigma'')$ in the s plane coincide at $s=(N+2\mu)^2$ when $\sigma=\sigma''=\sigma_0$, and that they become distinct and both increasing as σ and σ'' are continued to physical values; the cut between them lies entirely in the physical region. In determining the locations of the corresponding singularities $\sigma_\pm''(s,\sigma)$ in the σ'' plane, we must make certain that the result is compatible with the properties in the s plane described above. We fix s at a physical value and follow the movement of $\sigma_\pm''(s,\sigma)$ as σ is varied. The continuation should not commence at a value of σ less than σ_0 because the corresponding branch points $\sigma_\pm''(s,\sigma)$ in the σ'' plane are at some complex positions whose real part is greater than $\sigma_\pm(s,\sigma_0)$, which is defined by (4.7) and is slightly less than $\sigma_1(s)$; the cut connecting these branch points must be deformed to avoid the contour of integration ending at $\sigma_1(s)$. As σ is continued to a value above σ_0, this cut encloses the end point $\sigma_1(s)$ while the branch points $\sigma_\pm''(s,\sigma)$ approach the real axis from opposite sides; the resultant branch cut is not compatible with the analyticity in the s plane.

The proper continuation should start at a value of σ in the range $\sigma_\pm(s,\sigma_0)<\sigma<\sigma_1(s)$; at this value of σ, s is greater than $(\sigma^{1/2}+\mu)^2$ and is consequently still in the physical region. The corresponding branch points $\sigma_\pm''(s,\sigma)$ in the σ'' plane are to the left of the threshold σ_0. As σ is decreased, these singularities move to the right and stay on the upper or lower side of the contour of integration depending upon the sign of the small imaginary part of s. For s in the lower half plane, $\sigma_\pm''(s,\sigma)$ are on the lower side of the contour, although these singularities may be on either side of the kinematical cut between σ_0 and $\sigma_1(s)$. When σ reaches σ_0, the branch points coincide at $\sigma_\pm''(s,\sigma_0)$, which, according to (4.7), has a negative imaginary part; the contour of integration on the real axis, therefore, remains on the upper side of the singularities. These properties are compatible with the analyticity in the s plane.

When s is continued to certain complex values in the lower half plane such that the branch points $\sigma_\pm''(s,\sigma)$ coincide with the pole of $a_u(\sigma'')$ at M^2 in the σ'' plane, pinching the contour of integration in between, we have coincidence singularities in the amplitude $C_u(s,\sigma,\sigma')$ as a function of s. Let us denote their locations in the s plane by $s_\pm(\sigma,M^2)$. Since the singularities $\sigma_\pm''(s,\sigma)$ are on the right side of the contour, pinching is possible only when the pole M^2 is on the left. This implies that

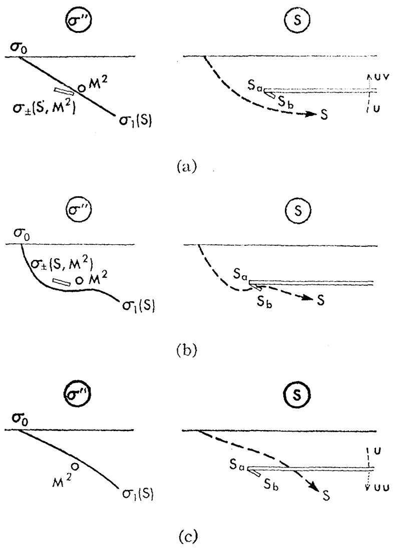

FIG. 6. Paths of continuation in the s plane are indicated by dashed line on the right. On the left are the corresponding contours of integration in the σ'' plane. Open lines indicate branch cuts.

$C_u(s,\sigma,\sigma')$ can have branch points at $s_{\pm}(\sigma,M^2)$ in the lower half of sheet u only below the πM cut, not above. For certain values of σ, $s_{-}(\sigma,M^2)$ may enter into the region above the πM cut from below; this singularity is then in the unphysical sheet uv if we label the sheet reached by a counterclockwise (clockwise) continuation across the πM cut with uv (uu). Symmetry of $C(s,\sigma,\sigma')$ under the interchange of σ and σ' implies that $C_u(s,\sigma,\sigma')$ has branch points also at $s_{\pm}(M^2,\sigma')$ below the πM cut in sheet u and possibly above the πM cut in sheet uv.

The complex unitarity cut is associated with a channel that has only two particles, the pion and the unstable particle M. The corresponding phase-space factor $\rho_2(s,M^2)$ has a square-root branch point at $s=s_a=(M+\mu)^2$. Thus, we expect that the πM cut connects only two sheets. We consider more explicitly the problem of continuation across this cut in the next section and show that the sheets uu and uv are, indeed, identical. Meanwhile, we continue to distinguish between uu and uv, since the distinction should be helpful in clarifying the mechanism that gives rise to the singularities in that unphysical sheet.

The fact that $C_u(s,\sigma'',\sigma')$ is singular at $s_{\pm}(\sigma'',M^2)$ implies that there are branch points in the σ'' plane at $\sigma_{\pm}''(s,M^2)$ for certain values of s. When s is at s_a, $\sigma_{\pm}''(s_a,M^2)$ are on the left side of σ_0 and slightly above the real axis. An end-point singularity results in $C_u(s,\sigma,\sigma')$ when these branch points reach the threshold σ_0. As we have mentioned in connection with (4.6), this occurs at one value of s, which is

$$s_b \equiv s_{\pm}(\sigma_0,M^2)=[(M^2+\mu N)\sigma_0^{1/2}-\mu^3]/N. \quad (5.1)$$

This singularity is very close to s_a, which we join with s_b by a short branch cut.

It is now clear that if s is continued to the region below the πM cut in sheet u along a path which avoids the singularities at s_a and s_b by passing them on their left, then the pole M^2 in σ'' plane is on the left side of the contour of integration viewed in the direction from σ_0 to $\sigma_1(s)$, while the branch points $\sigma_{\pm}''(s,M^2)$ move to the right side of the contour. Thus, pinching of the contour is possible when these singularities coincide, as shown in Fig. 6(a); in the s plane the coincidence singularities are located at $s_{\pm}(M^2,M^2)$ in sheet u. On the other hand, if the path of continuation in the s plane goes across the cut between s_a and s_b, passing s_a on its lower side and s_b over its top, then both M^2 and $\sigma_{\pm}''(s,M^2)$ are on the left side of the contour of integration; pinching is, therefore, impossible. This situation is shown in Fig. 6(b). We are not concerned with the unphysical sheet defined by this continuation.

Consider now the continuation that passes both s_a and s_b on their right. See Fig. 6(c). In this case, we enter into the unphysical sheet uu as we cross the πM cut from above. According to what we have found above, the domain that can be reached by this continuation (i.e., the regions above the πM cut in sheet u and below the πM cut in sheet uu) does not contain any singularities at $s_{\pm}(\sigma,M^2)$ and $s_{\pm}(M^2,\sigma')$. This implies that $\sigma_{\pm}''(s,M^2)$ are not branch points in the σ'' plane for s in this domain, and that there is no pinching of contour as $\sigma_{\pm}''(s,M^2)$ approach M^2. Hence, the points $s_{\pm}(M^2,M^2)$ are regular in sheet uu.

An illustration of the locations of the branch cuts in sheet u is given in Fig. 7, where M is taken to be the complex mass of the 3–3 resonance; σ and σ' are given arbitrary values in the range between $(N+\mu)^2$ and $(N+2\mu)^2$. It is evident from (3.13) that $C_u(s,\sigma,\sigma')$ also has the branch cuts of $B(s,\sigma)$ and $B_u(s,\sigma')$. Poles can only occur at the zeros of $D(s)$, defined in (3.24).

Analyticity in the unphysical sheet v and the others connected with it can be studied in just the same way. Since, in this case, a complex value of s is reached by continuation from $s+$ [in the interval $(N+2\mu)^2<s<(N+3\mu)^2$], the branch points $\sigma_{\pm}''(s,\sigma)$ are on the left side of the contour of integration in the σ'' plane. Thus, the singularities in sheet v are mirror reflections across the real axis of those in sheet u.

Let us now consider the analyticity of $B_u(s,\sigma)$. For the Born term of $B(s,\sigma)$ we take only the one corresponding to the diagram in Fig. 2. The associated branch points are at $s_{\pm}(\sigma)$, as defined in (4.8) and as shown in Fig. 3. It is clear from the inhomogeneous term of (3.11) or (3.12) that $B_u(s,\sigma)$ also has branch points at $s_{\pm}(\sigma)$. A consideration of the contribution from the integral term requires that we turn to the σ'' plane again. Equation (4.10) gives the singularities of $B^B(s,\sigma'')$ in the σ'' plane for a given value of s; let us denote them by $\sigma_{\pm}''(s)$ here. One can show that, when s is real and greater than $(N+2\mu)^2$, $\sigma_{\pm}''(s)$ are at complex-conjugate positions whose real part is greater than $(N+2\mu)^2-(2\mu^3/N)(1+3\mu/4N)$. As s is continued away from the real axis, $\sigma_{\pm}''(s)$ do not go to the region between σ_0 and $\sigma_1(s)$, so they never pinch the contour when they approach M^2. Consequently, $s_{\pm}(M^2)$ are not singularities of $B_u(s,\sigma)$. However, due to the

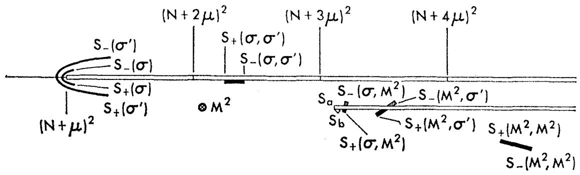

FIG. 7. Branch cuts of $C_u(s,\sigma,\sigma')$ in the s plane.

presence of $C_u(s,\sigma,\sigma'')$ inside the integral in (3.11), $B_u(s,\sigma)$ does have branch points at $s_\pm(\sigma,M^2)$ and $s_\pm(M^2,M^2)$. The amplitude in the unphysical sheet uu, denoted by $B_{uu}(s,\sigma)$, does not have branch points at $s_\pm(M^2,M^2)$ for the same reason that they are not in $C_{uu}(s,\sigma,\sigma')$.

Since $B_u(s,\sigma'')$ is singular at $\sigma''=\sigma_\pm''(s,M^2)$, it is evident by inspection of (3.10) that $A_u(s)$, not $A_{uu}(s)$, has branch points at $s_\pm(M^2,M^2)$. In addition, $A_u(s)$ has, of course, all the branch cuts of $A(s)$, which must be in the unphysical region to the left of the πN threshold.

One of the essential results of this study is that the short branch cut between $s_+(M^2,M^2)$ and $s_-(M^2,M^2)$ is only in sheet u, and not in sheet uu. As a consequence of the coupling between the channels by unitarity, it is not only in the amplitude $C_u(s,\sigma,\sigma')$, but also in $A_u(s)$ and $B_u(s,\sigma)$. Since the region below the πM cut in sheet u can be reached from the real axis only by going around the threshold s_a, it is farther away from the physical section of the unitarity cut than the corresponding region in sheet uu. The singularities $s_\pm(M^2,M^2)$ are, therefore, not expected to have any dominant effects on the elastic and production amplitudes in a direct way; however, as we see in Sec. VII, a significant influence can be exerted indirectly.

VI. CONTINUATION ACROSS THE COMPLEX UNITARITY CUT

We have, in a sense, already considered the continuation across the πM complex unitarity cut. Using (3.10)–(3.13), we have analytically extended the domain of definition of $A_u(s)$, etc., to the region below the πM cut by continuing s from above. In that way we have entered into the unphysical sheet uu and found that the amplitudes are regular at $s=s_\pm(M^2,M^2)$. However, the amplitudes obtained in this procedure are defined by integral equations whose contour of integration, in effect, forces a downward distortion of the πM cut. In this section we find the integral equations which are defined for s entirely in sheet uu.

Let $\bar{s}$ be a complex value of s which is situated on the complex unitarity cut. Thus, we have $\mathrm{Re}\bar{s} > \mathrm{Re}(M+\mu)^2$ and $\mathrm{Im}\bar{s}=\mathrm{Im}(M+\mu)^2<0$. Let $\bar{s}+$ ($\bar{s}-$) indicate the value of s just above (below) this cut. Since the existence of this cut is independent of the variables σ and σ', we have for any values of σ and σ'

$$\begin{aligned} A_u(\bar{s}+)&=A_{uu}(\bar{s}-),\\ B_u(\bar{s}+,\sigma)&=B_{uu}(\bar{s}-,\sigma),\\ C_u(\bar{s}+,\sigma,\sigma')&=C_{uu}(\bar{s}-,\sigma,\sigma'),\\ A_u(\bar{s}-)&=A_{uv}(\bar{s}+),\\ B_u(\bar{s}-,\sigma)&=B_{uv}(\bar{s}+,\sigma),\\ C_u(\bar{s}-,\sigma,\sigma')&=C_{uv}(\bar{s}+,\sigma,\sigma'). \end{aligned} \tag{6.1}$$

Suppose we evaluate s in (3.10) at a particular value $\bar{s}+$; the contour of integration in the σ'' plane then passes by the pole M^2 on its upper side. We now push this contour across the pole onto its lower side, thus picking up a term which is[22]

$$\oint_{C_1} d\sigma''\rho_2(\bar{s},\sigma_l'')a(\sigma'')a_u(\sigma'')B(\bar{s},\sigma'')B_u(\bar{s}+,\sigma''), \tag{6.2}$$

where C_1 is a small, closed contour clockwise around $\sigma''=M^2$. Let λ be the residue of the pole of $a_u(\sigma'')$ at M^2, i.e.,

$$a_u(\sigma'')=\lambda/(\sigma''-M^2),\quad \text{as}\quad \sigma''\rightarrow M^2; \tag{6.3}$$

then (6.2) yields

$$-2\pi i\lambda\rho_2(\bar{s}-,M^2)a(M^2)B(\bar{s},M^2)B_{uu}(\bar{s}-,M^2). \tag{6.4}$$

Although s_b is the location of a singularity in sheet u, it is a regular point in sheet uu. Thus, the contour C_2 in the σ'' plane, after moving past the pole at M^2, corresponds to a trajectory in the s plane which leads from the real axis to $\bar{s}-$ and lies entirely in sheet uu. Equation (3.10), originally evaluated at $s=\bar{s}+$, is now expressed in terms of $\bar{s}-$ with the contour of integration lying below M^2; continuing s away from $\bar{s}-$ into sheet uu, we have

$$A_{uu}(s)=[1+2i\rho_1(s)A(s)]^{-1}\Big\{A(s)-4\pi\lambda\rho_2(s,M^2)a(M^2)B(s,M^2)B_{uu}(s,M^2) -2i\int_{\sigma_0}^{\sigma_1(s)} d\sigma''\rho_2(s,\sigma_l'')a(\sigma'')a_u(\sigma'')B(s,\sigma'')B_{uu}(s,\sigma'')\Big\}. \tag{6.5}$$

In exactly the same way, we can obtain the other amplitudes in sheet uu. They are

$$B_{uu}(s,\sigma)=B(s,\sigma)[1-2i\rho_1(s)A_{uu}(s)]-4\pi\lambda\rho_2(s,M^2)a(M^2)C(s,\sigma,M^2)B_{uu}(s,M^2) -2i\int_{\sigma_0}^{\sigma_1(s)} d\sigma''\rho_2(s,\sigma_l'')a(\sigma'')a_u(\sigma'')C(s,\sigma,\sigma'')B_{uu}(s,\sigma''), \tag{6.6}$$

[22] It can be shown from a dispersion relation for $B(s,\sigma)$, such as (4.12) with its last term written out, that $B(s,\sigma)$ has no cut along $\bar{s}$ for real or complex σ.

$$C_{uu}(s,\sigma,\sigma')=C(s,\sigma,\sigma')-2i\rho_1(s)B(s,\sigma)B_{uu}(s,\sigma')-4\pi\lambda\rho_2(s,M^2)a(M^2)C(s,\sigma,M^2)C_{uu}(s,M^2,\sigma')$$
$$-2i\int_{\sigma_0}^{\sigma_1(s)}d\sigma''\rho_2(s,\sigma_l'')a(\sigma'')a_u(\sigma'')C(s,\sigma,\sigma'')C_{uu}(s,\sigma'',\sigma'). \quad (6.7)$$

Evaluating (6.6) and (6.7) at $\sigma=M^2$, we can solve for $B_{uu}(s,M^2)$ and $C_{uu}(s,M^2,\sigma')$, which can then be used to reduce these equations. After some straightforward manipulations we obtain

$$A_{uu}(s)=[1+2i\rho_1(s)A'(s)]^{-1}\left[A'(s)-2i\int_{\sigma_0}^{\sigma_1(s)}d\sigma''\rho_2(s,\sigma_l'')a(\sigma'')a_u(\sigma'')B'(s,\sigma'')B_{uu}(s,\sigma'')\right], \quad (6.8)$$

$$B_{uu}(s,\sigma)=B'(s,\sigma)[1-2i\rho_1(s)A_{uu}(s)]-2i\int_{\sigma_0}^{\sigma_1(s)}d\sigma''\rho_2(s,\sigma_l'')a(\sigma'')a_u(\sigma'')C'(s,\sigma,\sigma'')B_{uu}(s,\sigma''), \quad (6.9)$$

$$C_{uu}(s,\sigma,\sigma')=C'(s,\sigma,\sigma')-2i\rho_1(s)B'(s,\sigma)B_{uu}(s,\sigma')-2i\int_{\sigma_0}^{\sigma_1(s)}d\sigma''\rho_2(s,\sigma_l'')a(\sigma'')a_u(\sigma'')C'(s,\sigma,\sigma'')C_{uu}(s,\sigma'',\sigma'), \quad (6.10)$$

where

$$A'(s)=A(s)+G(s)B^2(s,M^2), \quad (6.11)$$

$$B'(s,\sigma)=B(s,\sigma)+G(s)C(s,\sigma,M^2)B(s,M^2), \quad (6.12)$$

$$C'(s,\sigma,\sigma')=C(s,\sigma,\sigma')+G(s)C(s,\sigma,M^2)C(s,M^2,\sigma'), \quad (6.13)$$

$$G(s)=\frac{-4\pi\lambda\rho_2(s,M^2)a(M^2)}{1+4\pi\lambda\rho_2(s,M^2)a(M^2)C(s,M^2,M^2)}. \quad (6.14)$$

Notice that (6.8)–(6.10) have the same form as that of (3.10)–(3.13), the difference being that the amplitudes on the physical sheet are modified on account of the unstable particle.

It is not obvious from these equations that there are no singularities at $s_\pm(M^2,M^2)$ in sheet uu. Since $G(s)$ involves $C(s,M^2,M^2)$, the functions A', B' and C' are singular at $s_\pm(M^2,M^2)$. However, in view of the procedure by which (6.8)–(6.10) are derived, there is no doubt that the singularities of the inhomogeneous and the integral terms at these points cancel, leaving the amplitudes $A_{uu}(s)$, etc., analytic there.

One can similarly derive the amplitudes in sheet uv which is reached from sheet u by a counterclockwise continuation across the complex unitarity cut. The resultant integral equations are exactly the same as (6.8)–(6.10), with the subscripts uu being replaced by uv. To show that the solution of these coupled equations is unique so that the amplitudes in sheets uu and uv are identical, we merely have to prove that

$$\tilde{B}(s,\sigma)\equiv B_{uu}(s,\sigma)-B_{uv}(s,\sigma)$$

vanishes everywhere in the s plane. The identities of the other amplitudes follow immediately. It is straightforward to establish from (6.8) and (6.9) that

$$\tilde{B}(s,\sigma)=-2i\int_{\sigma_0}^{\sigma_1(s)}d\sigma''\rho_2(s,\sigma_l'')a(\sigma'')a_u(\sigma'')\tilde{B}(s,\sigma'')\left[C'(s,\sigma,\sigma'')-\frac{2i\rho_1(s)B'(s,\sigma)B'(s,\sigma'')}{1+2i\rho_1(s)A'(s)}\right]. \quad (6.15)$$

Nonzero solutions of this equation exist at most for isolated values of s where the resolvent has poles. Since $\tilde{B}(s,\sigma)$ is an analytic function in s, it must therefore vanish identically. Thus, we have shown that the complex unitarity cut associated with the πM channel connects only two sheets. This is not surprising in view of the fact that the channel has only two particles and that the phase–space factor $\rho_2(s,M^2)$ has a square-root branch point at $s=(M+\mu)^2$, as we have already observed in the last section.

Since the integral equations (6.8)–(6.10) are just the same as (3.10), (3.12), and (3.13) except that $A(s)$ is replaced by $A'(s)$, $B(s,\sigma)$ by $B'(s,\sigma)$, and $C(s,\sigma,\sigma')$ by $C'(s,\sigma,\sigma')$, the solutions for the amplitudes in sheet uu are then given by (3.21)–(3.23) with the appropriate change of $A(s)$ to $A'(s)$, etc. Poles in this sheet are located at where $D'(s)$ vanishes, $D'(s)$ being the counterpart of $D(s)$; in general, $D(s)$ and $D'(s)$ do not vanish at the same values of s.

VII. THE RESONANCE POLE

Since the singularities $s_\pm(M^2,M^2)$ are located in the unphysical sheet u, they are not effective in enhancing the scattering amplitudes along the physical region of the real axis in the s plane. In this section we see how these singularities can affect the analyticity in sheet uu; in particular, we investigate the condition under which a resonance pole can occur in the vicinity of these singularities.

The integral equations for the amplitudes in sheet *uu* are given by (6.8)–(6.10). The functions $A'(s)$, $B'(s,\sigma)$, and $C'(s,\sigma,\sigma')$ are expressed in (6.11)–(6.14) in terms of amplitudes on the physical sheet, which are presumed known. A complete calculation cannot be made at present in view of our ignorance about many of the singularities on the physical sheet. However, in the vicinity of the branch cut between $s_+(M^2,M^2)$ and $s_-(M^2,M^2)$, the amplitudes are dominated by the effects of this cut; it is, therefore, possible to make some remarks about the amplitudes there.

From (6.14) we see that $G(s)$ has a pole when the denominator $\Delta(s)$ vanishes, where

$$\Delta(s)=1+4\pi\lambda\rho_2(s,M^2)a(M^2)C(s,M^2,M^2). \quad (7.1)$$

We assume that $\Delta(s)$ has at least one zero in the finite s plane and denote its position by s_p. If s_p is near $s_\pm(M^2,M^2)$, its value can be calculated from (7.1) where $C(s,M^2,M^2)$ may be approximated by $C^B(s,M^2,M^2)$ which is known according to (4.3). The quantity $a(M^2)$ is also known on account of (3.7); the residue λ is directly proportional to the width of the resonance in $a(\sigma)$. Although the exact location of s_p is not crucial in the following analysis, we suppose that it is in the near neighborhood of $s_\pm(M^2,M^2)$. In the static limit, the branch points $s_\pm(M^2,M^2)$ coincide and become a pole in $C^B(s,M^2,M^2)$; the smallness of $|\rho_2(s,M^2)/\rho_1(M^2)|$ in this region implies that $\Delta(s)$ must vanish in the close vicinity of this pole. We do not expect this feature to be grossly altered in the relativistic case.

The vanishing of $\Delta(s)$ results in a pole at s_p in each of the amplitudes $A'(s)$, $B'(s,\sigma)$ and $C'(s,\sigma,\sigma')$. For $s\approx s_p$, we can write

$$A'(s)=R_a/(s-s_p),\quad B'(s,\sigma)=R_b(\sigma)/(s-s_p),$$
$$C'(s,\sigma,\sigma')=R_c(\sigma,\sigma')/(s-s_p), \quad (7.2)$$

where R_a, $R_b(\sigma)$ and $R_c(\sigma,\sigma')$ are the appropriate residues. Our next step is to investigate whether these primed amplitudes can generate a pole in the unphysical sheet *uu* in the neighborhood of s_p.

As we have mentioned at the end of the previous section, the amplitudes $A_{uu}(s)$, etc., have the same formal solutions as those of $A_u(s)$, etc., provided that $A(s)$, $B(s,\sigma)$, and $C(s,\sigma,\sigma')$ are replaced by $A'(s)$, $B'(s,\sigma)$, and $C'(s,\sigma,\sigma')$, respectively. A pole in sheet *uu* occurs when $D'(s)$ is zero, where

$$D'(s)=1+2i\rho_1(s)[A'(s)-2iF'(s)], \quad (7.3)$$

$$F'(s)=\int_{\sigma_0}^{\sigma_1(s)} d\sigma''\rho_2(s,\sigma_l'')$$
$$\times a(\sigma'')a_u(\sigma'')B'(s,\sigma'')\beta'(s,\sigma''), \quad (7.4)$$

$$\beta'(s,\sigma)=B'(s,\sigma)-2i\int_{\sigma_0}^{\sigma_1(s)} d\sigma''\rho_2(s,\sigma_l'')$$
$$\times a(\sigma'')a_u(\sigma'')C'(s,\sigma,\sigma'')\beta'(s,\sigma''). \quad (7.5)$$

We assume that the resolvent for the kernel of the integral equation (7.5) does not have a pole[23] in a region around s_p so that $\beta'(s,\sigma)$ is approximately constant in that region, having the value $\beta'(s_p,\sigma)$ which satisfies the equation

$$R_b(\sigma)-2i\int_{\sigma_0}^{\sigma_1(s_p)} d\sigma''\rho_2(s_p,\sigma_l'')$$
$$\times a(\sigma'')a_u(\sigma'')R_c(\sigma,\sigma'')\beta'(s_p,\sigma'')=0. \quad (7.6)$$

Thus, in the neighborhood of s_p, $F'(s)$ has the form

$$F'(s)=\varphi/(s-s_p), \quad (7.7)$$

where

$$\varphi=\int_{\sigma_0}^{\sigma_1(s_p)} d\sigma''\rho_2(s_p,\sigma_l'')$$
$$\times a(\sigma'')a_u(\sigma'')R_b(\sigma'')\beta'(s_p,\sigma''). \quad (7.8)$$

The zero of $D'(s)$ can now be found by use of (7.3); if it is located near s_p, its position is at s_r, where

$$s_r\approx s_p-2i\rho_1(s_p)[R_a-2i\varphi]. \quad (7.9)$$

This is the position of a pole which is present in each of the amplitudes $A_{uu}(s)$, $B_{uu}(s,\sigma)$, and $C_{uu}(s,\sigma,\sigma')$. If it is not too far away from the real axis, it produces a resonance in the physical amplitudes for elastic scattering and production. If the last term of (7.9) is not small, then that equation is not valid a more elaborate calculation for s_r is necessary.

The exact position of the pole obviously depends upon the values of the residues R_a, $R_b(\sigma)$, and $R_c(\sigma,\sigma')$, which, in turn, depend upon the nature of the interactions between the particles. Indeed, a detailed calculation with the spins and the isotopic spins of the particles properly taken into account remains to be done. However, since $|\rho_2(s_p,M^2)|\ll|\rho_1(M^2)|\ll 1$, we can make the following remarks without carrying out a complete dynamical calculation. We see not only that s_p is expected to be near $s_\pm(M^2,M^2)$, but also that the magnitudes of the residues should be small, so the position of the resonance pole, if it occurs, should not be very far from s_p and $s_\pm(M^2,M^2)$. It is important to notice that the locations of $s_\pm(M^2,M^2)$ depend only on the masses of the particles, whereas s_r depends also on the interactions between the particles. We expect that the pole of the resonant amplitude is in the sheet *uu* between $s_\pm(M^2,M^2)$ and the πM cut; being close to the

[23] The implication of a pole in the resolvent has been discussed at the end of Sec. III.

real axis, it is, therefore, the resonance pole which gives rise to a resonance in the physical region. For the nonresonant amplitudes any such pole must be below $s_\pm(M^2,M^2)$ and significantly farther away from the real axis; its residue must be small so that it results in no enhancement in the physical amplitudes.

VIII. CONCLUSION

Working with scattering amplitudes involving two- and three-particle states, and without making specific approximations with regards to the isobar formation in the final state (thus without replacing the σ variables by M^2, the isobar mass squared, at the outset), we have determined the singularities of the elastic and production amplitudes in the unphysical sheets reached by continuation across the inelastic section of the unitarity cut. We have found the existence of the complex unitarity cut associated with the πM channel in the intermediate state. The dynamical singularities related to the process $\pi+\pi+N \rightarrow \pi+\pi+N$ with the exchange of a nucleon have been studied in detail. It is found that they induce other singularities in the unphysical sheet, among which the most notable ones are $s_\pm(M^2,M^2)$; these branch points are associated with the unphysical process[7] $\pi+M \rightarrow \pi+M$ in which N is exchanged.

A principal result of this investigation is that $s_\pm(M^2,M^2)$ are below the πM cut in the unphysical sheet u, but not in sheet uu, which is reached from the former by continuation across the πM cut. Since the πM cut is situated between $s_\pm(M^2,M^2)$ and the real axis, these branch points cannot directly affect the physical scattering processes, as they otherwise could if they were in sheet uu.

We have further derived the general condition under which a pole can occur in sheet uu. While the actual existence of such a pole depends on the dynamics of problem, i.e., the interaction forces for the various states of the scattering process (an aspect of the problem which we have not considered), we have given arguments, mainly on kinematical grounds (in the sense that M is regarded as a kinematical quantity), that if a resonance pole is to occur, it is most likely to be in the neighborhood of $s_\pm(M^2,M^2)$. Since it can be near the real axis, enhancement of the scattering amplitudes in the physical region is, therefore, possible.

Putting M equal to $1238-i\,45$ MeV which corresponds to the position and width of the 3–3 resonance, we find that

$$s_+(M^2,M^2)=(114.7-i\,11.6)\mu^2, \quad w_+=1485-i\,74 \text{ MeV},$$
$$s_-(M^2,M^2)=(133.3-i\,19.9)\mu^2, \quad w_-=1605-i\,118 \text{ MeV},$$

where $w_\pm$ are defined to be $[s_\pm(M^2,M^2)]^{1/2}$. Let us compare these values to the position and width of the second resonance $(I=\frac{1}{2}, J=\frac{3}{2}-)$ in the πN system: $M'=1512-i\,70$ MeV. We see that a resonance pole in sheet uu just above $s_\pm(M^2,M^2)$ is therefore located at the appropriate position in the s plane to be responsible for the second resonance.

Similar considerations can be applied to other problems involving three-particle states, such as the 3π and the $2\pi+Y$ systems. In each of these problems one can continue across the three-particle unitarity cut and find a complex cut associated with the channel consisting of one original particle plus an unstable particle which can decay into the other two particles. This complex unitarity cut connects only two sheets. An investigation of the singularities in these unphysical sheets can very likely shed some light on the dynamical origin of any resonances observed at energies where these three-particle states are physical. Thus, for example, the ω particle must be attributable to a resonance pole in the unphysical sheet u on the left side of the $\pi\rho$ cut. Its position must be influenced mainly by the branch cut related to the diagram of π-ρ scattering with the exchange of an ω; a self-consistent calculation is evidently needed here. In the case of the $2\pi+Y$ system, it is clear that a resonance pole induced by the singularities at $s_\pm(M''^2,M''^2)$, where M'' is the complex mass of Y_1^* (1385) and Λ is the exchanged particle, can give rise to a higher resonance,[24] which may be identified with the one recently observed[25,26] at 1660 MeV.

ACKNOWLEDGMENTS

The author wishes to thank Dr. V. Teplitz for many helpful discussions. He is also grateful for the hospitality extended by Professor K. M. Watson at the Physics Department of the University of California, and by Dr. David Judd at the Lawrence Radiation Laboratory, Berkeley, where part of this work was performed.

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