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Low-Energy Pion-Pion Scattering*

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This calculation starts with the partial-wave dispersion relations for pion-pion scattering as deduced from the Mandelstam representation for the scattering amplitude. The contribution of the "left-hand cut" was calculated from $\lambda\phi^4$ perturbation theory (to second order in λ). The s-wave and p-wave amplitudes were obtained numerically and found to agree closely with those obtained by Chew, Mandelstam, and Noyes who proceeded from a different viewpoint.

I. INTRODUCTION

THE formulation of a quantitative theory of pion-pion scattering should be the easiest problem of strong-coupling physics. That this is likely seems to follow from the uniquely small mass of the π meson and the relatively large mass splitting between the low-energy two-pion system and the next heavier system with which a pion pair can interact (four π mesons). One can then argue that it should be a good first approximation to consider the low-energy two-pion system to be decoupled from the rest of the universe. That is, the forces that govern the interactions of a pair of pions with each other at low energy should be expressible in terms of pion-pion scattering itself.

Several attempts have been made to exploit this situation within the framework of an "extremist" s-matrix viewpoint.[1,2] The rules of this game are that one is permitted to call only upon crossing symmetry, unitarity, and analytic continuation of the s matrix in order to obtain relations among the partial-wave scattering amplitudes. In actual application, however, one finds it necessary to construct an expression for the scattering amplitude at unphysical values of the energy variable. Crossing symmetry is of limited use in enabling one to do this and approximations must be made at this point. It appears that the particular manner in which such approximations are introduced then determines the nature of the solution. In particular, it appears that in order to obtain a resonant p-wave solution, which has been the goal of most previous investigators, one must insert a p-wave resonance into the formulation of the problem. *Note added in proof.* The last sentence of the second paragraph is unfair to Dr. Moffat. It seems probable that his p-wave resonant solutions are a consequence of his choice of asymptotic behavior for his amplitudes. We are indebted to Dr. Gatland and Dr. Moffat for a lively discussion of this point.

We prefer to take the position that the scattering amplitude should be completely specified by the "dynamics" of the pion-pion interactions. The dynamics will be specified a priori in terms of a Lagrangian for the pion field. In addition to specifying a Lagrangian, we shall also insist that the partial-wave amplitudes satisfy the same analyticity and unitarity conditions that are imposed by the s-matrix theorists. We shall not, however, attempt to insert crossing symmetry (which, at any rate, cannot be put in exactly) in order to express the scattering amplitude in terms of partial-wave amplitudes when the arguments of the scattering amplitude are unphysical. Instead, the "unphysical" amplitude will be calculated from perturbation theory and will enter as a generalized "potential" that determines the scattering. In the present paper we shall only include terms up to second order in the pion-pion coupling constant.

At this point it is desirable to enter a disclaimer. We neither hope nor expect that the calculation described here will predict the observed p-wave (ρ particle) resonance.[3] It is our feeling that the order of perturbation theory to which we calculate here should be appropriate to the description of *low*-energy pion-pion scattering. If a comparison with experiment then suggests that the coupling constant is sufficiently small to justify our use of a coupling-constant expansion for the "potential," we will then permit ourselves to hope that a higher order calculation might predict the observed pion-pion scattering for moderately high energies. Of course, one should also keep in mind the possibility that the ρ is not a "dynamical" resonance but must be inserted as a "fundamental" unstable particle. We shall not belabor this question here.

In Sec. II we derive integral equations for the s- and p-wave scattering amplitudes. Section III is a digression in which the model of nonrelativistic scattering by a Yukawa potential is used to illustrate the computational technique that we use in the relativistic problem. Section IV contains a description of the numerical results and a remark concerning the "pole approximation" for the "potential." In Sec. V we investigate

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[1] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960); G. Chew, S. Mandelstam, and H. P. Noyes, *ibid.* **119**, 478 (1960), referred to as CMN. Our coupling constant λ is the negative of the one used by these authors.

[2] M. Cini and S. Fubini, Ann. Phys. (N. Y.) **10**, 352 (1960); B. H. Bransden and J. W. Moffat, Nuovo Cimento **21**, 505 (1961); A. Efremov, V. Serebryakov, and D. Shirkov, in *Proceedings of the 1962 Annual International Conference on High-Energy Physics at CERN*, edited by J. Prentki (CERN, Geneva, 1962), p. 163; L. A. P. Balazs, Phys. Rev. **128**, 1939 (1962).

[3] A. R. Erwin, R. H. March, W. D. Walker, and E. West, Phys. Rev. Letters **6**, 628 (1961).

the extent to which crossing symmetry is violated and compare our results with those of Chew, Mandelstam, and Noyes.[1] In the process of doing this we are able to make a parenthetical remark concerning "bootstrap" calculations. Finally, Sec. VI includes a discussion of some of the implications of experiment. Computational details are relegated to Appendices.

II. DERIVATION

We specify our dynamics by invoking an interaction Lagrangian that is proportional to $(\boldsymbol{\phi}\cdot\boldsymbol{\phi})^2$. The value of the scattering amplitude at the symmetry point

$$s=t=u=\tfrac{4}{3} \tag{1}$$

is defined to be the renormalized coupling constant. Here s, t, and u have their usual meanings ($\hbar=c=m_\pi=1$):

$$s=4(q^2+1), \tag{2a}$$

$$t=-2q^2(1-\cos\theta), \tag{2b}$$

$$u=-2q^2(1+\cos\theta), \tag{2c}$$

where $\mathbf{q}$ is the barycentric momentum of one of the pions and θ is the barycentric scattering angle. Then to lowest order in the coupling constant, the amplitudes A^I of isotopic spin I may be written[4]

$$A^0=5\lambda,\quad A^1=0,\quad A^2=2\lambda. \tag{3}$$

The second-order expressions may now be obtained from elastic unitarity, crossing, and the renormalization condition. This is done in Appendix A, and the results are found to be

$$A^0=5\lambda\{1+5\lambda F(s)+3\lambda[F(t)+F(u)]\}, \tag{4a}$$

$$A^1=5\lambda^2[F(t)-F(u)], \tag{4b}$$

$$A^2=2\lambda\left\{1+2\lambda F(s)+\frac{9\lambda}{2}[F(t)+F(u)]\right\}. \tag{4c}$$

For this calculation we shall not go beyond the second order.

The next step is to project out the partial-wave amplitudes for specified angular momenta. In the case of the isospin-zero s wave, for example, we obtain

$$A_0{}^0(s)=5\lambda+25\lambda^2F(s)+\frac{15\lambda^2}{2}\int_{-1}^{1}d\cos\theta\{F(t)+F(u)\}. \tag{5}$$

We now observe that the representation of $F(s)$ given in the Appendix exhibits regularity in the complex s plane if the latter is cut along the segment

$$4\leqslant s<\infty \tag{6}$$

on the real axis. It follows that $A_0{}^0(s)$, given by Eq. (5), is regular in the complex s plane except for the cut (6) and an additional cut along the negative real axis contained in the integral in Eq. (5). This is, of course, the expected analytic behavior of a π-π partial-wave amplitude.[1]

We now adopt the viewpoint that the dynamics is given by the terms having the left-hand branch cut (and the subtraction constant 5λ) which plays the role of a potential. The exact discontinuity across the right-hand branch cut is known from unitarity in terms of the amplitude $A_0{}^0(s)$ itself. Incidentally, the unitarity condition used in the Appendix implies the $A_l{}^I$ must have the representation

$$A_l{}^I(s)=\left(\frac{q^2+1}{q^2}\right)^{1/2}\exp(i\delta_l{}^I)\sin\delta_l{}^I, \tag{7}$$

and our approximation of keeping only terms of order λ^2 (two-meson intermediate states) implies that the $\delta_l{}^I$ are real. We, therefore, rewrite Eq. (5) with the second term on the right replaced by a statement of the unitarity conditions so that it reads

$$A_0{}^0(s)=5\lambda+\frac{(s-s_0)}{\pi}\int_4^\infty\frac{ds'}{(s'-s_0)(s'-s)}\left(\frac{s'-4}{s'}\right)^{1/2}\times|A_0{}^0(s')|^2+\frac{15}{2}\lambda^2\int_{-1}^{1}d\cos\theta\{F(t)+F(u)\}. \tag{8}$$

The subtraction at the symmetry point where $s_0=\frac{4}{3}$ is required by the definition of the renormalized coupling constant.

This equation, and the corresponding equations for the isospin-two s wave and the isospin-one p wave are then solved by the familiar N/D techniques.[1] This leads to an integral equation for the denominator function D. The equations are written down in Appendix B. We remind the reader that the equations in that Appendix could be modified by the addition to the D functions of poles (with positive residues) on the real, positive, q^2 axis.[5] Our calculation is made unique by excluding such poles.

III. THE YUKAWA POTENTIAL

In the preceding section we ascribed the dynamics to the last term on the right of Eq. (8), the term which we consider to be the "generalized potential" for the scattering problem. It is interesting to see how this interpretation is justified by examining a simple nonrelativistic scattering problem.

The Born approximation amplitude for scattering by a Yukawa potential of range $1/\mu$ and strength $\sigma\kappa_0{}^2$ is

$$f_B(t)=-\sigma\kappa_0{}^2\mu^{-1}(t-\mu^2)^{-1}. \tag{9}$$

We choose $\kappa_0{}^2/\mu^2$ to have the value 1.68 so that when

[4] See, for example, S. Gasiorowicz, Fortschr. Physik 8, 665 (1960).

[5] L. Castillejo, R. H. Dalitz, and F. J. Dyson, Phys. Rev. 101, 453 (1956), referred to as CDD.

the dimensionless parameter σ is equal to unity the potential gives rise to a zero-energy s-wave bound state.[6] The scattering amplitude $f(t)$ is normalized to have the partial-wave expansion

$$f(t)=\sum_{l=0}^{\infty}(2l+1)\exp(i\delta_l)\left(\frac{\sin\delta_l}{q}\right)P_l(\cos\theta)$$

$$\equiv\sum_{l=0}^{\infty}(2l+1)a_l(q)P_l(\cos\theta), \quad (10)$$

where q is the wave number in the Schrödinger equation, and t is defined by Eq. (2b).

The integral equation for the s-wave amplitude, obtained by requiring unitarity for positive q^2 and by using the s-wave projection of $f_B(t)$ for the contribution from the left-hand cut, is

$$\mu a_0(x)=+\left\{\left(\frac{\sigma\kappa_0{}^2}{\mu^2}\right)\left[\frac{\ln(1+4x)}{4x}\right]+\frac{1}{\pi}\int_0^{\infty}\frac{dx'}{x'-x}(x')^{1/2}|a_0(x')|^2\right\}, \quad (11)$$

where we have defined

$$x=q^2/\mu^2.$$

It is easy to obtain an approximate solution of this equation if the term $[\ln(1+4x)]/4x$ is replaced by a single pole on the negative x axis. We choose the position of the pole and the residue to give the correct value and slope at threshold where $x=0$. The approximate equation then becomes

$$\mu a_0(x)=\sigma\left(\frac{\kappa_0}{\mu}\right)^2(1+2x)^{-1}+\frac{1}{\pi}\int_0^{\infty}\frac{dx'(x')^{1/2}}{(x'-x)}|a_0(x')|^2. \quad (12)$$

The solution is found to have the threshold value

$$\mu a_0(0)=\sigma\left(\frac{\kappa_0}{\mu}\right)^2\left[1-\frac{\sigma}{2\sqrt{2}}\left(\frac{\kappa_0}{\mu}\right)^2\right]^{-1}$$

$$=1.68\sigma[1-0.594\sigma]^{-1}. \quad (13)$$

TABLE I. Scattering lengths for a Yukawa potential.

σ	Scattering length Exact	Eq. (13)	Born approx.
-4	-1.88	-1.98	-6.72
-2	-1.39	-1.53	-5.03
-1	-0.955	-1.05	-1.68
-0.1	-0.155	-0.159	-0.17
0.1	0.184	0.179	0.17
0.5	1.53	1.20	0.84
0.95	12.3	3.67	1.60

[6] R. D. Levee and R. L. Paxton, Lawrence Radiation Laboratory (Livermore) Report UCRL-7155-T (unpublished). We are indebted to Professor Dalitz for bringing this work to our attention.

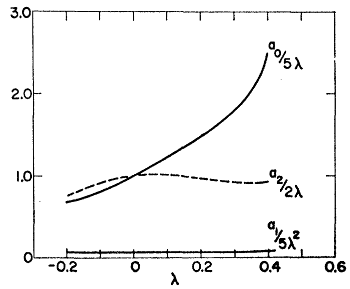

FIG. 1. The s-wave and p-wave scattering lengths as functions of the coupling constant λ.

Some exact scattering lengths for the Yukawa potential have been calculated by Levee and Paxton.[6] Table I compares these with the predictions of Eq. (13) and with the Born approximation. It is seen that despite the crudeness of the single-pole approximation for the contribution from the left-hand cut the agreement is quite good until the potential begins to be sufficiently attractive to support a bound state.

We close this section by remarking that the N/D solution of Eq. (12) is ghost-ridden if the potential is repulsive ($s<0$). That is, D will vanish for some negative value of x and thereby give the amplitude an unphysical pole. For potentials that are not too strong (but $|s|$ may be substantially greater than unity) the ghost will be far away from the physical region and the N/D solution will still be a good approximation to the correct one at sufficiently low energy. It is not clear to us whether or not ghosts would continue to make their appearance if the correct Born contribution from the left-hand cut, $x^{-1}\ln(1+4x)$, were to be used. However, if they were to appear we would ascribe their presence to the fact that the Born approximation gives an inadequate representation of the distant left-hand singularities. Nevertheless, we could still retain confidence in the low-energy N/D solutions if the ghost pole is at a sufficient distance from threshold.

IV. PROPERTIES OF THE SOLUTIONS

The equations of Appendix B were solved numerically with the aid of the Argonne IBM 704 computer. The coupling parameter λ was allowed to range from -2 to $+2$. The N/D solutions were then fed back into the original equations [Eq. (8) and its counterparts for the other partial waves] in order to confirm that they were indeed solutions of the nonlinear equations.

It was found that the s-wave solutions for both isotopic spins always had ghost singularities on the negative energy axis. If the coupling constant λ was not too large, then the ghosts were always far away from the physical threshold. As λ was increased the ghosts moved in toward threshold. We confirmed, as might be expected, that the energy range over which N/D solutions satisfied the partial-wave dispersion relations (within the limits of precision of the calculation) extending about as far above the threshold as the ghost was below it. In keeping with the remarks at the end

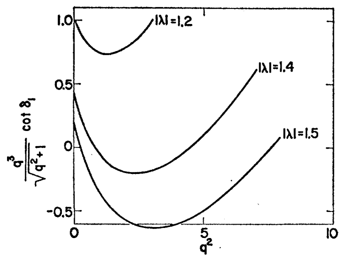

FIG. 2. Development of the p-wave resonance as the coupling strength is increased.

of the preceding section we shall regard the N/D solutions as valid representations of the dynamics over the energy range for which the self-consistency check holds. Figure 1 shows the s- and p-wave scattering lengths as functions of the coupling constant for solutions "valid" out to energies well past the inelastic threshold.

The p-wave solutions were ghost-free (and small) for small values of λ. As the coupling strength was increased well beyond the value where the s-wave solutions were meaningful, the p-wave phase shift began to approach $\pi/2$ at a relatively low energy and eventually did attain this value (Fig. 2). Further increase of the coupling strength moved the resonance to lower and lower energy in a manner completely analogous to what one observes in potential scattering.[7] Just before the resonance turned itself into a bound state, a ghost pole appeared and the self-consistency check failed badly at all energies.

With the numerical solutions at hand, it then seemed reasonable to investigate the possibility of constructing approximate solutions by analytical methods. The standard means for doing this is to approximate the contribution from the left-hand cut by one or two poles. The N/D formulation then reduces to an elementary quadrature. We tested this technique by replacing the contributions from the left-hand cut with a single pole. Just as in the example with the Yukawa potential, the position and residue of the pole were chosen to give the correct slope and value at threshold. The result was not unexpected, namely, that the one-pole approximation predicted the scattering length to within a few percent, the effective range roughly, and was grossly misleading at energies for which the effective-range expansion was no longer applicable.

V. DISCUSSION

The solutions described in the preceding section satisfy, by construction, unitarity and analyticity in the complex energy plane and correspond to a system of dynamics that was specified in advance (ϕ^4 interaction). If the partial-wave amplitudes that we have obtained are physically sensible, they should also satisfy crossing symmetry to the extent that the notion can be made meaningful.

The significance of the last remark may be better understood by considering the scattering of a pair of neutral scalar bosons. If the scattering amplitude satisfies a Mandelstam double-dispersion relation[8] one is then led to express the discontinuity across the left-hand cut for the lth partial wave as[1]

$$2\,\mathrm{Im}A_l(q^2)=\frac{2}{q^2}\sum_{l'}\int_0^{-q^2-1}dq'^2\,P_l\left(1+2\frac{q'^2+1}{q^2}\right)\times P_{l'}\left(1+2\frac{q^2+1}{q'^2}\right)\mathrm{Im}A_{l'}(q'^2),\quad(14)$$

for $q^2<-1$. The series is known to converge if the magnitude of q^2 is not too large.[1] Now it is easy to see that if q^2 is sufficiently close to -1 so that it can be argued that all the $A_{l'}(q'^2)$ behave as $(q')^{2l'}$ (when q'^2 is small), then the right-hand side of Eq. (14) is well approximated by the lowest one or two partial waves ($l'=0, 2$). When $-q^2$ is large, however, the fact that the argument of $P_{l'}$ ranges from -1 to $-\infty$ will have the consequence that many partial waves with relatively small phase shifts may make important contributions to the sum over l' if they are multiplied by Legendre polynomials of high order. Thus, even if a certain low partial wave (say, the s wave) dominates the physical scattering amplitude, it need no longer be expected to dominate the right-hand side of Eq. (14) except for $-q^2$ close to unity. This is just saying that the convergence of the series gets worse and worse as q^2 approaches the limiting radius of convergence.[9] We remark in passing that this argument suggests that "bootstrap" calculations for resonances at moderately high energies should be viewed with suspicion.

We have already observed that for the range of values of λ for which the present calculation gives sensible s- and p-wave solutions, the s waves are

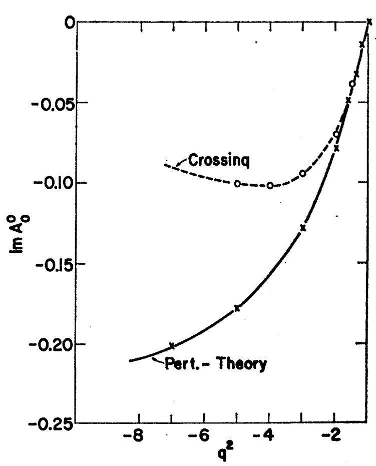

FIG. 3. The discontinuity across the left-hand cut for the isospin-zero s wave from perturbation theory (solid line), and from crossing-symmetry (dashed line) using only the s-wave contributions. The value of λ is -0.1. The agreement becomes poorer as λ is increased.

[7] P. G. Burke, Lawrence Radiation Laboratory Report UCRL-10140, 1962 (unpublished).

[8] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).

[9] A. Efremov, V. Mescheryakov, D. Shirkov, and M. Tzu, Nucl. Phys. **22**, 202 (1961).

observed to be substantially larger than the p wave. The discussion in the preceding paragraph suggests that we should recalculate the discontinuities across the left-hand cuts from our computed s-wave solutions. If these solutions are reasonably consistent with crossing symmetry, then the newly calculated discontinuities should agree with those calculated from perturbation theory in the vicinity of $-q^2=1$. Figure 3 shows that this is indeed so.

At this point it becomes interesting to compare our work with that of Chew, Mandelstam, and Noyes (CMN).[1] These authors, starting from the viewpoint of the s-matrix purists, obtained a system of coupled equations for the partial-wave amplitudes by requiring only unitarity, analyticity, and crossing symmetry. They did not give an explicit specification of the dynamics. The solutions that CMN obtained were, like ours, characterized by large s waves and a small p wave. These solutions were considered to be physically uninteresting because it was thought at that time that the p wave should resonate at a relatively low energy.

Figures 4 and 5 compare our s-wave results with those of CMN.[10] As the preceding discussion might lead one to expect, the low-energy results are practically undistinguishable over a considerable range of values of the coupling constant. We take this as evidence that the CMN formulation of the scattering problem implies a choice of a particular interaction Lagrangian. It has been pointed out elsewhere[11] that the choice is made (in a purely s-matrix formalism) by specifying the nature of the subtractions in dispersion relations for the scattering amplitude.

VI. CONCLUSIONS

We feel justified, on the basis of the foregoing discussion, in concluding that our calculation gives a physically reasonable prediction of low-energy π-π scattering based upon a ϕ^4 interaction Lagrangian.

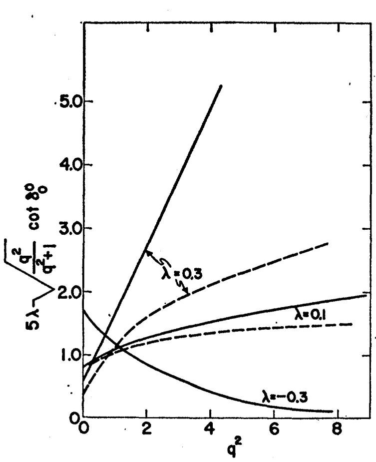

FIG. 4. Comparison of the isospin-zero s-wave amplitudes according to CMN (dashed lines) and the present work (solid lines). When λ is negative (repulsive) the two are indistinguishable.

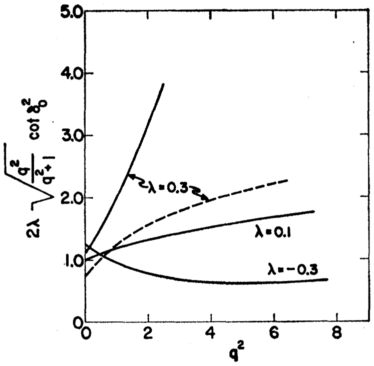

FIG. 5. Comparison of the isospin-two s-wave amplitudes according to CMN (dashed line) and the present work (solid line). For $\lambda \leq 0.1$ the two are indistinguishable.

There are now two further questions that must be investigated. First, does the scattering predicted from a ϕ^4 interaction Lagrangian bear any relationship to pion scattering in the real world? And, second, is the coupling constant λ sufficiently small that a perturbation expansion of the "potential" is reasonable? These questions are clearly related, and we must look to experiment for guidance.

A comparison with experiment is very difficult to make at the present time. The one unambiguous feature of π-π scattering that has been uncovered thus far is the ρ resonance at a total center-of-momentum energy of about 5.5 pion masses, and we have already stated that we do not expect to predict this in a low-energy calculation. Various people have attempted to deduce the low-energy scattering parameters of the π-π system from experiments in which pion pairs are produced. Unfortunately, the theoretical bases upon which these deductions must rest are hardly firm enough to inspire much confidence in the conclusions. Nevertheless, the various analyses of inelastic pion scattering from protons are roughly consistent with each other and will serve to give us a first estimate of the coupling constant λ.

Goebel and Schnitzer[12] have examined pion production from nucleons from the viewpoint of a model based upon static theory, single-pion exchange, and final-state-isobar rescattering corrections. Schnitzer[13] finds scattering lengths (in units of λ_π):

$$a_0 \approx 0.5, \quad a_1 \approx 0.07, \quad a_2 = 0.16.$$

Here we have chosen the Schnitzer solution that gives the same signs for the scattering lengths a_0 and a_2, as would be required in our calculation unless there is a π-π bound state. Batusov *et al.*,[14] using a method of

[10] The p-wave solutions also agree rather well. It should be added that the agreement is not a trivial consequence of the small coupling constant. Even for $|\lambda|=0.1$, the λ^2 contributions to N and D (Appendix B) are substantial.

[11] A. Efremov, H. Tzu, D. Shirkov, Sci. Sinica (Peking) **10**, 812 (1961).

[12] C. J. Goebel and H. Schnitzer, Phys. Rev. **123**, 1021 (1961).

[13] Howard J. Schnitzer, Phys. Rev. **125**, 1059 (1961).

[14] Yu. A. Batusov, S. A. Bunyatov, V. M. Sidorov, and V. A. Yarba, in *Proceedings of the 1960 Annual International Conference on High-Energy Physics at Rochester*, edited by E. C. G. Sudarshan, J. H. Tinlot, and A. C. Melissinos (Interscience Publishers, Inc., New York, 1960), p. 79.

analysis invented by Anselm and Gribov,[15] find

$$a_2 - a_0 = -0.35 \pm 0.30.$$

Kirz *et al.*[16] have performed a Chew-Low extrapolation on $\pi^+ p$ and $\pi^- p$ inelastic scattering to find

$$|a_2| \lesssim 0.15,$$
$$|a_2 + 2a_0| \approx 1.3.$$

The latter value is roughly consistent with the results obtained by Arefev *et al.*[17] in a similar experiment. It is easy to see that the s-wave results are all in approximate agreement with each other. Reference to Fig. 1 shows that the s-wave data can be fitted by making the choice

$$\lambda = +0.09.$$

The corresponding scattering lengths are, then,[18]

$$a_0 \approx 0.5, \quad a_2 \approx 0.17, \quad a_1 \approx 0.003.$$

Let us pretend, for the moment, that our comparison with "experiment" may be taken seriously. We are now able to see how including terms of higher order (in λ) in the potential must affect the scattering amplitude if the ϕ^4 interaction corresponds to real physics and perturbation theory makes sense. The required effect is that the s-wave amplitudes near threshold must remain practically unchanged although the behaviors at medium and high energy might be radially altered. The p-wave "potential" must become considerably more attractive so that at least the next order of perturbation theory must be included before the p-wave behavior begins to stabilize even at threshold. Whether or not these expectations are, in fact, achieved will be the subject of a later investigation.

It should be observed that the fact that the "measured" s-wave scattering lengths are roughly in the ratio 5:2 is very suggestive that the ϕ^4 interaction with a small coupling constant is operative in real life. The large p-wave scattering length (and the ρ resonance), on the other hand, could prove to be unobtainable with the inclusion of higher order terms in the potential. One would then have to consider two other possibilities: either the Lagrangian must be made to contain more complicated interactions than the simple, renormalizable ϕ^4, or the ρ resonance dominates the low-energy p-wave behavior but is not itself an outcome of the input dynamics. The ρ would then have to be put into the theory as a CDD[5] zero.

[15] A. A. Anselm and V. N. Gribov, Zh. Eksperim. i Teor. Fiz. **37**, 501 (1959) [translation: Soviet Phys.—JETP **10**, 354 (1960)].

[16] J. Kirz, J. Schartz, and R. D. Tripp, Phys. Rev. **126**, 763 (1962); R. D. Tripp (private communication). We are indebted to Dr. Tripp for permission to quote his results prior to publication.

[17] A. V. Arefev, Yu. D. Bayukov, Yu. M. Zayitsev, M. S. Kozodaev, G. A. Leksin, V. T. Osipenkov, D. A. Suchkov, V. V. Telenkov, and B. V. Fedorov, in *Proceedings of the 1962 Annual International Conference on High-Energy Physics at CERN*, edited by J. Prentki (CERN, Geneva, 1962).

[18] Interestingly enough, the calculations of Moffat (Ref. 2) as well as those of CMN would give about the same values for this choice of λ.

ACKNOWLEDGMENTS

The authors are indebted to Dr. P. T. Matthews and Dr. David Wong for helpful remarks.

APPENDIX A

It is well known to dispersion theorists that the results of conventional Feynman-Dyson perturbation theory may be obtained from repeated application of unitarity and crossing symmetry.[19] However, since some of our non-dispersion-oriented friends seem to be unfamiliar with the technique, we thought it worthwhile to give it additional publicity by writing down the details of our second-order calculation.

The starting point of this exercise rests upon the assumption that each term of perturbation theory satisfies the analyticity properties exhibited by the Mandelstam double-dispersion representation.[8] We then deal with the renormalization question by specifying the values of the three isospin amplitudes at the symmetry point as given by Eq. (3). This is the lowest order approximation.

The amplitudes A^I are chosen to obey the elastic unitarity condition. (We do not discuss inelastic processes which enter into the fourth-order expressions for the elastic amplitudes.) This condition is

$$\mathrm{Im} A^I(q^2, \cos\theta) = \frac{q}{4\pi\omega} \int d\Omega_1 A^I(q^2, \cos\theta_1) A^{I*}(q^2, \cos\theta') \theta(q^2), \quad \text{(A1)}$$

where

$$\cos\theta' = \cos\theta \cos\theta_1 + \sin\theta \sin\theta_1 \cos\phi_1, \quad \text{(A2)}$$

and ω is the energy of a π meson in the barycentric system. The step function $\theta(x)$ is unity for positive argument and vanishes otherwise.

We now insert the expressions from Eq. (3) on the right-hand side of Eq. (A1) to obtain the second-order expressions for the imaginary parts of the amplitudes on the positive energy axis. From our knowledge of the analytic properties we are enabled to write the contribution of the second-order amplitudes as dispersion relations in the variable s [Eq. (2a)] to obtain amplitudes that satisfy two-particle unitarity (to order λ^2) but not crossing symmetry. Changing over to the Mandelstam variables we have

$$A^0(s,t,u) = 5\lambda + 25\lambda^2 F(s), \quad \text{(A3)}$$

$$A^1(s,t,u) = 0, \quad \text{(A4)}$$

$$A^2(s,t,u) = 2\lambda + 4\lambda^2 F(s), \quad \text{(A5)}$$

where $F(s)$ is defined by

$$F(s) = \frac{s - s_0}{\pi} \int_4^\infty \frac{ds'}{(s'-s)(s'-s_0)} \left(\frac{s'-4}{s'}\right)^{1/2}, \quad \text{(A6)}$$

[19] Stanley Mandelstam, Phys. Rev. **115**, 1752 (1959). The possibility that this could be done was suggested by J. S. Toll, dissertation, Princeton University, 1952 (unpublished).

and the subtraction at the symmetry point s_0 has been introduced because the value of the amplitude at this point has already been specified (the renormalization condition).

The final step in deriving the set of equations (4) is to make the amplitudes crossing-symmetric (but no longer unitary). Crossing symmetry and the Pauli principle require that A^0 and A^2 be symmetric and A^1 antisymmetric in the variables t and u. Further,

$$A^0(t,s,u)=\tfrac{1}{3}[A^0(s,t,u)+3A^1(s,t,u)+5A^2(s,t,u)]. \quad \text{(A7)}$$

Equations (4) are readily obtained from this.

APPENDIX B

We record here the N and D equations and their relation to the amplitudes.

For the s waves,

$$\mathrm{Re}D_0^I(\nu)=1+\frac{\alpha_I\lambda}{\pi}\left[h(\nu)-2\sqrt{2}\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)\right]+\beta_I\left(\frac{\lambda}{\pi}\right)^2(\nu-\nu_0)\int_1^\infty\frac{d\nu' D(-\nu')}{(\nu'+\nu)(\nu'+\nu_0)}\times[h(-\nu')-h(\nu)]\left(\frac{\nu'-1}{\nu'}\right)^{1/2}\times\left[1-\frac{h(-\nu')}{2\nu'}\right], \quad \text{(B1)}$$

$$\mathrm{Re}N_0^I(\nu)=\alpha_I\lambda-\beta_I\lambda^2\frac{\nu-\nu_0}{\pi}\int_1^\infty\frac{d\nu' D(-\nu')}{(\nu'+\nu)(\nu'+\nu_0)}\left(\frac{\nu'-1}{\nu'}\right)^{1/2}\times\left[1-\frac{h(-\nu')}{2\nu'}\right], \quad \text{(B2)}$$

where

$$\alpha_I=\begin{pmatrix}5\\2\end{pmatrix},\quad \beta_I=2\begin{pmatrix}15\\9\end{pmatrix},\quad \nu\equiv q^2,\quad \nu_0=-\tfrac{2}{3}, \quad \text{(B3)}$$

and

$$h(\nu)=\mathrm{Re}\left(\frac{\nu}{\nu+1}\right)^{1/2}\ln\left(\frac{[\nu/(\nu+1)]^{1/2}+1}{[\nu/(\nu+1)]^{1/2}-1}\right). \quad \text{(B4)}$$

The N and D are related to the scattering amplitude by

$$\left(\frac{\nu}{\nu+1}\right)^{1/2}\cot\delta_0^I=\mathrm{Re}D_0^I(\nu)/N_0^I(\nu),\quad \nu>0. \quad \text{(B5)}$$

For the case of the p-wave amplitude we have

$$\mathrm{Re}D_1(\nu)=1-5\nu\left(\frac{\lambda}{\pi}\right)^2\int_1^\infty\frac{d\nu'}{\nu'^2(\nu'+\nu)}\times D_1(-\nu')\left(\frac{\nu'-1}{\nu'}\right)^{1/2}[h(-\nu')-h(\nu)]\times\left[1-\frac{2\nu'-1}{2\nu'}h(-\nu')\right], \quad \text{(B6)}$$

$$\mathrm{Re}N_1(\nu)=\frac{5\lambda^2}{\pi}\int_1^\infty\frac{d\nu'}{\nu'^2(\nu'+\nu)}D_1(-\nu')\left(\frac{\nu'-1}{\nu'}\right)^{1/2}\times\left[1-\frac{2\nu'-1}{2\nu'}h(-\nu')\right], \quad \text{(B7)}$$

and

$$\left(\frac{\nu^3}{\nu+1}\right)^{1/2}\cot\delta_1=\mathrm{Re}\left[\frac{D_1(\nu)}{N(\nu)}\right],\quad \nu>0. \quad \text{(B8)}$$