CONCLUSION

The three determinations of the branching ratio are shown as follows:

(a) Branching ratio from $\mathbf{R}_{\max} = (0.82 \pm 0.13) \times 10^{-3}$;
(b) branching ratio from δ rays $= (0.70 \pm 0.31) \times 10^{-3}$; and
(c) branching ratio from kinematics $= (0.62 \pm 0.22) \times 10^{-3}$.

It should be noted that the events used to determine (a) are completely separate from those used to determine (b) and (c). Different parts of the electron-energy spectrum are sampled by the three categories. It can be seen that all the values agree within the errors.

The best value obtained from this experiment for the branching ratio is that obtained from $\mathbf{R}_{\max}$ events. The 15% error quoted contains both the statistical uncertainty and the errors on the scanning efficiencies. We feel that any systematic error would be considerably less than this.

The confirmation of the branching ratio from $\mathbf{R}_{\max}$ by the δ-ray and kinematic methods, in spite of much larger uncertainties, is valuable because the only common link in determining the ratios is the Monte Carlo calculation of detection efficiencies. Formerly, the best estimate of the ratio was that of Aubert *et al.*,[4] who found $(3.0^{+1.5}_{-1.2}) \times 10^{-3}$ on the basis of 8 events. The value found in our work is clearly in disagreement with the prediction of 16×10^{-3} made by Feynman and Gell-Mann.[1]

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Unitary Impulse Approximation*

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With the aid of multiple scattering expansions, Blankenbecler's generalized unitarity relation is derived for a multichannel three-body potential scattering problem. A matrix representation of the scattering amplitudes is obtained in the form of Heitler's integral equation, with the **K** matrix replaced by a matrix **N**, so that generalized unitarity is automatically satisfied if **N** has no physical cut in the total energy variable. A simple, physically reasonable, choice for **N** leads to representations of the inelastic amplitudes which have the form of initial- (or final-) state interaction corrections to the impulse approximation. With the elastic amplitude given, no sums over three-body phase space appear. The elastic amplitude itself is obtained as the solution of an integral equation which sums all diagrams which are iterations of the basic impulse approximation diagram. It is explicitly demonstrated that the partial-wave amplitudes thus obtained must satisfy unitarity even when the impulse (or strip) approximation is nonunitary. A convergent iterative solution is presented which treats the effects of longer ranged forces first and should be appropriate for high-energy diffraction scattering. Rearrangement collisions are treated in a similar way.

I. INTRODUCTION

IT has become increasingly clear that relativistic field theory and potential scattering theory display a number of interesting similarities, particularly when both are formulated in the language of dispersion relations. This has the consequence that new techniques developed in the context of one theory can find expression in the other, thereby establishing a useful interplay. We are concerned here with the development of approximation techniques which take into account the effects of inelastic scattering processes. We have chosen as our model a three-body potential scattering problem in which an energetic particle is incident on a target consisting of two other particles in a bound state. The problem then is to determine the amplitudes for elastic, breakup, and rearrangement processes. The individual

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particles involved may be nucleons or, more generally, tightly bound clusters (cores) of nucleons. The approach adopted here borrows freely from recent developments in the S-matrix theory of strong interactions, and it will be obvious how to transcribe the results obtained back to field-theoretic language. One such feasible application, to the calculation of the higher $\pi-N$ resonances with the aid of a unitary strip approximation, is briefly discussed in Sec. 4.

The full complexity of the three-body problem shows up even in the case of elastic scattering where virtual inelastic transitions must be taken into account. In a previous paper[1] we have shown how the basic idea of the strip approximation,[2] in which the complexity of many-particle intermediate states is reduced by expressing their effects in terms of physical two-body scattering amplitudes, can be formulated in a useful way in a potential model. Specifically, the discontinuity of the elastic amplitude across the cut in the t (momentum transfer squared) variable is given in a certain interval along the cut, starting at the anomalous threshold, by an expression involving the physical two-body amplitude and the asymptotic form of the target wave function. This provides a dispersion-theoretic formulation of the impulse approximation which, in correctly reproducing the nearby singularities in t, is expected to provide a good approximation to the scattering amplitude for low momentum transfers. Our concern in the present paper is to obtain the most important multiple scattering corrections to the simple impulse approximation.

Now it is suggested by the N/D techniques developed in field theory that an effective way of approximating the sum of a set of rescattering diagrams is obtained by imposing the constraints of unitarity. We present here a simple variation of the multichannel N/D procedure described by Blankenbecler.[3] With the aid of a multiple scattering expansion of the amplitudes, Blankenbecler's generalized unitarity relation, in which imaginary parts are replaced by absorptive parts, is derived (see Appendix) for the problem at hand. A matrix representation of the scattering amplitudes is then obtained in the form of Heitler's integral equation, with the **K** matrix replaced by a matrix **N**, so that generalized unitarity is automatically satisfied if **N** has no physical cut in s, the total energy variable. This equation [see Eq. (3.14)], therefore, forms a convenient basis for approximations. One simple, physically reasonable choice for **N** leads to representations of the inelastic amplitudes which have the form of initial- (or final-) state interaction corrections to the impulse approximation. The elastic amplitude in this approximation satisfies a linear integral equation which can be solved in a number of ways assuming that the amplitude in the impulse approximation is known. Rearrangement collisions can be treated in a similar way, and we shall derive unitarity corrections to the Born approximation for pickup and stripping processes[4] in our three-body model.

All results obtained here have a simple physical interpretation. They may be viewed as a natural generalization of the basic idea behind the impulse approximation, namely, that the potential V_T which binds the target system in initial and/or final states may be ignored in all intermediate stages of the scattering process. However, V_T may cause binding in intermediate states and this effect, for consistency, should not be neglected. We have summed an infinite subclass of diagrams in which V_T enters only through the fact that it allows for the existence of states, intermediate as well as initial and final, in which the target is bound. These diagrams correspond to the "most peripheral" collisions. In fact, our construction of the elastic amplitude as a disperion integral in the t variable shows close analogy with formulations of the multiple peripheral collision idea in field theory.[5]

2. MULTIPLE SCATTERING EXPANSIONS

In order to concentrate on the essential dynamics of the problem, we continue to discuss the simplified model three-body problem described in I. We take spinless, neutral particles interacting in pairs through a central potential V_{ij} of the form

$$rV_{ij}(r)=\int_{\mu}^{\infty}\sigma_{ij}(\nu)e^{-\nu r}d\nu, \tag{2.1}$$

with $V_{13}=0$. V_{23} supports one bound s state of energy $-(\hbar^2/m)\epsilon$, where m is the common mass of the three particles. It is first assumed that V_{12} cannot bind; this restriction is dropped in Sec. 4 in a brief treatment of rearrangement collisions.

The elastic scattering amplitude T_{aa} is given by[6]

$$T_{aa}=\langle\Phi_{23}^{f}|\,V_{12}|\,\Psi^{i}\rangle, \tag{2.2}$$

where Ψ^i is that solution of the Schrödinger equation

$$(K+V_{12}+V_{23}-E)\Psi=0 \tag{2.3}$$

(K is the kinetic energy operator and E is the total energy of the system) corresponding to an initial state in which particles 2 and 3 are bound and particle 1 is in a plane-wave state with momentum $\hbar k_1^i$ (entrance channel a). Φ_{23}^f is the "free" solution [obtained by setting $V_{12}=0$ in Eq. (2.3)] in which particle 1 has momentum $\hbar k_1^f$ (exit channel a). It will be convenient

[1] L. Rosenberg, Phys. Rev. **129**, 968 (1963), referred to in the following as I.

[2] G. F. Chew and S. C. Frautschi, Phys. Rev. **123**, 1478 (1961); R. Cutkosky, Phys. Rev. Letters **4**, 624 (1960).

[3] R. Blankenbecler, Phys. Rev. **122**, 983 (1961).

[4] For a different approach to this problem, see R. D. Amado, Phys. Rev. **127**, 261 (1962).

[5] D. Amati, S. Fubini, A. Stanghellini, and M. Tonin, Nuovo Cimento **22**, 569 (1961); M. Baker and R. Blankenbecler, Phys. Rev. **128**, 415 (1962).

[6] B. A. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950).

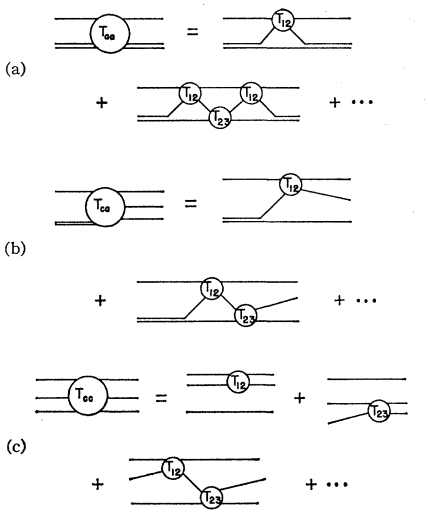

FIG. 1. Leading terms in the diagrammatic expansions of the amplitudes. The double line represents the two-body bound system. The potential V_{13} is taken to be zero.

to refer to the multiple scattering expansion for T_{aa} which is easily derived from the three-body scattering formalism developed by Faddeev.[7,8] As shown by Faddeev, the wave function Ψ can be represented as

$$\Psi=\Psi^{(1)}+\Psi^{(2)}+\Psi^{(3)}, \tag{2.4}$$

and satisfies the matrix-integral equation

$$\begin{pmatrix}\Psi^{(1)}\\ \Psi^{(2)}\\ \Psi^{(3)}\end{pmatrix}=\begin{pmatrix}\Phi_{23}\\ 0\\ 0\end{pmatrix}-G_0(z)\begin{pmatrix}0 & T_{23}(z) & T_{23}(z)\\ T_{13}(z) & 0 & T_{13}(z)\\ T_{12}(z) & T_{12}(z) & 0\end{pmatrix}\begin{pmatrix}\Psi^{(1)}\\ \Psi^{(2)}\\ \Psi^{(3)}\end{pmatrix}, \tag{2.5}$$

with $z=E+i\eta$, $\eta\to 0+$, and

$$\begin{aligned}T_{ij}(z)&=V_{ij}-V_{ij}G_0(z)T_{ij}(z),\\ G_0(z)&=(K-z)^{-1}.\end{aligned} \tag{2.6}$$

If, according to our simplifying assumption, we set $V_{13}=0$, and solve Eq. (2.5) by iteration, we obtain a series development for Ψ, and, hence, for T_{aa}. These two series can then be formally resummed, leading to the relations

$$\Psi=(1-G_0T_{12})[1-G_0T_{23}G_0T_{12}]^{-1}\Phi_{23}, \tag{2.7}$$

$$T_{aa}=\langle\Phi_{23}{}^f|\,T_{12}[1-G_0T_{23}G_0T_{12}]^{-1}|\,\Phi_{23}{}^i\rangle. \tag{2.8}$$

Equation (2.8) may be interpreted simply as a shorthand way of writing the expansion[9]

$$T_{aa}=\langle\Phi_{23}{}^f|\,T_{12}+T_{12}G_0T_{23}G_0T_{12}+\cdots|\,\Phi_{23}{}^i\rangle. \tag{2.9}$$

A diagrammatic representation for the expansion of T_{aa} appears in Fig. 1(a). A similar (though in detail more complicated) procedure can be carried through when $V_{13}\neq 0$. Again, all possible diagrams of the form shown in Fig. 1(a) are to be drawn, with the restriction that two successive scatterings never take place between the same pair of particles.

In a similar way, a multiple scattering expansion for the inelastic amplitude T_{ca}, which describes the breakup process, and for T_{cc} in which all three particles are free in initial and final states [see Figs. 1(b) and 1(c)], are obtained:

$$T_{ca}=\langle\Phi_0{}^f|\,(T_{12}-T_{23}G_0T_{12})[1-G_0T_{23}G_0T_{12}]^{-1}|\,\Phi_{23}{}^i\rangle, \tag{2.10}$$

$$\begin{aligned}T_{cc}=\langle\Phi_0{}^f|\,&T_{12}+T_{23}-T_{12}G_0T_{23}[1-G_0T_{12}G_0T_{23}]^{-1}\\ &\times(1-G_0T_{12})-T_{23}G_0T_{12}[1-G_0T_{23}G_0T_{12}]^{-1}\\ &\times(1-G_0T_{23})|\,\Phi_0{}^i\rangle,\end{aligned} \tag{2.11}$$

where

$$(K-E)\Phi_0=0. \tag{2.12}$$

Equation (2.11) displays the connected part of T_{cc}, i.e., $T_{cc}{}^C=T_{cc}-T_{cc}{}^{(1)}$, with

$$T_{cc}{}^{(1)}=\langle\Phi_0{}^f|\,T_{12}+T_{23}|\,\Phi_0{}^i\rangle. \tag{2.13}$$

The elastic amplitude in the center-of-mass system M_{aa} is defined by

$$T_{aa}=(2\pi)^3\delta^3(\mathbf{K}^i-\mathbf{K}^f)M_{aa}, \tag{2.14}$$

where $\mathbf{K}^i$ and $\mathbf{K}^f$ are the initial and final values of the total momentum. Center-of-mass amplitudes M_{ca} and M_{cc} are defined in a similar way. We have a multiple scattering expansion for M_{aa} corresponding to Eq. (2.9); $M_{aa}=\sum_i M_{aa}{}^{(i)}$. The leading term $M_{aa}{}^{(1)}$ is just the impulse approximation to which the discussion in I was confined. (A more suitable notation has been adopted in the present paper.) We are interested in finding approximations to the sum of this series, and the corresponding series for M_{ca} and M_{cc}. As it will turn out that the impulse approximation amplitude $M_{aa}{}^{(1)}$ plays a key role in this analysis we will now briefly review the results obtained in I.

We deal here, explicitly, with the expression

$$\begin{aligned}M_{aa}{}^{(1)}(s,t)=\int d\mathbf{k}_3'\,&\tilde{\varphi}(\tfrac{1}{2}\mathbf{k}_1{}^i+\mathbf{k}_3')\,\tilde{\varphi}(\tfrac{1}{2}\mathbf{k}_1{}^f+\mathbf{k}_3')\\ &\times t_{12}[\mathbf{k}_1{}^f+\tfrac{1}{2}\mathbf{k}_3',\,\mathbf{k}_1{}^i+\tfrac{1}{2}\mathbf{k}_3';\,E-(3\hbar^2/4m)k_3'{}^2],\end{aligned} \tag{2.15}$$

[7] L. D. Faddeev, Zh. Eksperim. i Teor. Fiz. **39**, 1459 (1960) [translation: Soviet Phys.—JETP **12**, 1014 (1961)].

[8] Multiple scattering expansions have been discussed earlier by a number of authors; see, e.g., G. F. Chew and M. L. Goldberger, Phys. Rev. **87**, 778 (1952). An approximate, variational treatment, applicable to the case of low-energy neutrons scattered by molecularly bound protons, appears in Ref. 6. In that case the leading term, the impulse (or Fermi) approximation, was shown to be extremely accurate.

[9] The convergence properties of the expansion given in Eq. (2.9) have not been well studied. It seems reasonable to assume, however, by analogy with two-body scattering theory, that the expansion may not converge at low energies if three-body bound states exist, but converges, in any case, at high enough energies. It has been pointed out [R. Aaron, R. D. Amado, and B. W. Lee, Phys. Rev. **121**, 319 (1961)] that the Born expansion will diverge even at arbitrarily high energies due to the presence of two-body bound states. That argument, of course, does not apply to Eq. (2.9) since the two-body interaction operators T_{12} and T_{23} are *not* expanded in their Born series. It can be shown, in fact, that Eq. (2.8) remains valid even when the expansion, Eq. (2.9), diverges; see note added in proof following Sec. 3.

where $s=(4m/3\hbar^2)E$ and $t=(\mathbf{k}_1{}^i-\mathbf{k}_1{}^f)^2$. The bound-state wave function for the (23) pair, $\tilde{\varphi}(\mathbf{q})$, has the analytic structure[10]

$$\tilde{\varphi}(\mathbf{q})=\frac{C}{q^2+\epsilon}+\int_{(\epsilon^{\frac{1}{2}}+\mu)}^{\infty}\frac{\sigma(\nu)}{q_i+\nu}d\nu. \tag{2.16}$$

We have also introduced the matrix element

$$t_{ij}(\mathbf{k},\mathbf{k}';z)=\langle\mathbf{k}|t_{ij}(z)|\mathbf{k}'\rangle, \tag{2.17}$$

where $t_{ij}(z)$ is the two-body interaction operator [defined by the two-particle equivalent of Eq. (2.6)]. The approach adopted in I was to study the analytic properties of $M_{aa}{}^{(1)}$ in the variable t, the square of the momentum transfer, assuming that these properties are correctly revealed in perturbation theory (the Born expansion).[11] The singularities on the negative t axis fall into two catagories. Firstly, there are singularities which arise from the analytic structure of the two-particle bound-state wave function. These singularities lie at the positions $t=-16\epsilon$, $t=-16\ (\epsilon^{1/2}+\frac{1}{2}\mu)^2$, and $t=-16(\epsilon^{1/2}+\mu)^2$; with each singularity there is an associated branch cut running along the negative axis to $t=-\infty$. It is interesting to note that the first branch point is just the anomalous threshold singularity obtained by considering the leading diagram of Fig. 1(a) as a relativistic Feynman graph and applying the Landau analysis,[12] with $\epsilon/m\ll 1$. The other two singularities correspond to "pionic" corrections to this graph. In addition there are normal threshold singularities located at $t=-n^2\mu^2$, where n is the order in the Born expansion. In the relativistic case these arise in ladder diagrams where n pions are exchanged between the two colliding systems. It was shown in I that the discontinuity of $M_{aa}{}^{(1)}$ across the cut in t in the anomalous threshold region, $-4\mu^2<t<-16\epsilon$, is given exactly in terms of the wave-function normalization constant C and the absorptive part of the physical two-body scattering amplitude.

The amplitude $M_{ca}{}^{(1)}$, which is just the impulse approximation to M_{ca}, will also be important in the following; it appears as the leading term in the diagrammatic expansion in Fig. 1(b). If we define the variables

$$s_{ij}=k_{ij}{}^2=[\tfrac{1}{2}(\mathbf{k}_i-\mathbf{k}_j)]^2;\quad (ij)=(12)\text{ or }(23), \tag{2.18}$$

we have

$$\begin{aligned}M_{ca}{}^{(1)}&=\tilde{\varphi}(\tfrac{1}{2}\mathbf{k}_1{}^i+\mathbf{k}_3{}^f)t_{12}\left(\mathbf{k}_{12}{}^f,\ \mathbf{k}_1{}^i+\tfrac{1}{2}\mathbf{k}_3{}^f;\frac{\hbar^2}{m}s_{12}{}^f\right)\\&=\tilde{\varphi}(\tfrac{1}{2}\mathbf{k}_1{}^i+\mathbf{k}_3{}^f)(-4\pi\hbar^2/m)\\&\quad\times f[(\tfrac{1}{2}\mathbf{k}_1{}^i+\mathbf{k}_3{}^f)^2+\epsilon+s_{12}{}^f,\ s_{12}{}^f;\ (\mathbf{k}_1{}^i-\mathbf{k}_1{}^f)^2].\end{aligned} \tag{2.19}$$

As pointed out by Fivel[13] the function $f(\bar{s},s;t)$ satisfies $f(s,s;t)=f(s,t)$, where $f(s,t)$ is the physical (on-the-mass-shell) two-body scattering amplitude. Furthermore, $f(\bar{s},s;t)$ can be evaluated in terms of $f(s,t)$ with the aid of unitarity.[14]

We now turn to a discussion of the unitarity conditions and show that they provide a convenient basis for obtaining approximations to the amplitudes which effectively sum a class of multiple scattering diagrams.

3. CONSTRUCTION OF UNITARY AMPLITUDES

The unitarity relation for the elastic amplitude may be obtained by writing

$$\begin{aligned}T_{aa}=&\langle\Phi_{23}{}^f|V_{12}|\Phi_{23}{}^i\rangle\\&-\langle\Phi_{23}{}^f|V_{12}G(E+i\eta)V_{12}|\Phi_{23}{}^i\rangle,\end{aligned} \tag{3.1}$$

where

$$G(z)=(K+V_{12}+V_{23}-z)^{-1}. \tag{3.2}$$

With the aid of the eigenfunction expansion of $G(z)$, and the relation

$$\langle\alpha|G(z)|\beta\rangle^*=\langle\alpha|G(z^*)|\beta\rangle, \tag{3.3}$$

which is essentially the reciprocity property of the Green's function, we find, by taking the imaginary part of Eq. (3.1),

$$-\frac{1}{\pi}\mathrm{Im}M_{aa}=\sum_a M_{aa}{}^*M_{aa}+\sum_c M_{ac}{}^*M_{ca}, \tag{3.4}$$

where

$$\sum_a=\int\frac{d\mathbf{k}_1'}{(2\pi)^3}\delta(E'-E), \tag{3.5}$$

$$\begin{aligned}\sum_c=&\int\int\int\frac{d\mathbf{k}_1'}{(2\pi)^3}\frac{d\mathbf{k}_2'}{(2\pi)^3}\frac{d\mathbf{k}_3'}{(2\pi)^3}\\&\times\delta^3(\mathbf{k}_1'+\mathbf{k}_2'+\mathbf{k}_3')\delta(E'-E).\end{aligned} \tag{3.6}$$

The imaginary parts of M_{ca}, M_{ac}, and M_{cc} are found in a similar way, leading to the complete unitarity relations which we express in the matrix form[15]

$$-\frac{1}{\pi}\mathrm{Im}\mathbf{M}=\mathbf{M}^*\mathbf{M}. \tag{3.7}$$

In the context of dispersion theory it is more convenient to deal with the discontinuity of an amplitude across a cut rather than its imaginary part. Blankenbecler[3] has written down a generalized unitarity relation for relativistic scattering amplitudes which does deal with absorptive parts rather than imaginary parts.

[10] R. Blankenbecler and L. F. Cook, Phys. Rev. **119**, 1745 (1960).

[11] Analytic properties of three-body scattering amplitudes have not as yet been fully investigated. We regard the assumptions made in I (and carried over in the present paper) as constituting a reasonable ansatz from which new approximation schemes may be generated.

[12] L. D. Landau, Nucl. Phys. **13**, 181 (1959).

[13] D. I. Fivel, Nuovo Cimento **22**, 326 (1961).

[14] See also L. Rosenberg, Nuovo Cimento (to be published) for an alternate and more complete discussion of this point.

[15] Summation over all intermediate states is implied in the matrix multiplication. One could, of course, obtain Eq. (3.7) from the general formalism of Ref. 6.

It is shown in the Appendix that Blankenbecler's prescription can be taken over directly in the nonrelativistic case at hand. The proof makes use of the multiple scattering expansion of the amplitudes; we believe, however, and will assume in the following, that the result obtained is correct whether or not the expansion is valid.

The generalized unitarity relation can be expressed in the matrix form

$$-(2\pi i)^{-1}(\mathbf{M}^{C+}-\mathbf{M}^{C-})=\mathbf{M}^{C*}\mathbf{M}^{C}, \tag{3.8}$$

where $\mathbf{M}^{C+}-\mathbf{M}^{C-}$ represents the discontinuity of $\mathbf{M}^C$ across the physical cut in s, with the other variables held fixed at physical values; the variables s_{12}' and s_{23}' in M_{ca} and M_{cc} are to approach their cuts from below while s_{12}' and s_{23}' in M_{ac} and M_{cc} approach their cuts from above. $\mathbf{M}^C$ is obtained from $\mathbf{M}$ by replacing M_{cc} by its connected part, $M_{cc}{}^C$. We shall suppress the superscript C on $\mathbf{M}$ in the following.

If we follow the prescription of Ref. 3, $\mathbf{M}$ will be chosen as the solution of the coupled integral equations

$$\mathbf{M}\mathbf{D}=\mathbf{N}, \tag{3.9}$$

where the intermediate variables s_{12}' and s_{23}' run below their cuts in D and above their cuts in $\mathbf{M}$. $\mathbf{N}$ is some approximation to $\mathbf{M}$ which, however, contains no unitarity cut in s, while $\mathbf{D}$ satisfies the discontinuity equation

$$\mathbf{D}(s+i\eta)-\mathbf{D}(s-i\eta)=2\pi i\mathbf{N}. \tag{3.10}$$

There is a difficulty in this approach connected with the existence of two poles in the physical region, in the amplitude M_{cc}, which arise from the double scattering term $M_{cc}{}^{(2)}$ (see Appendix). Only within the approximation that these singularities are ignored will $\mathbf{M}$, as determined above, automatically satisfy Eq. (3.8). However, this approximation, the equivalent of which has been adopted in previous applications of the matrix N/D method,[3,16] should not affect too seriously the determination of M_{aa} and M_{ac}. In the present application no additional restrictions are imposed by these considerations since the diagrams which introduce the above-mentioned poles are to be dropped entirely in the approximate solution of the unitarity equations presented below [see Eqs. (3.16)].

One could now attempt to solve Eq. (3.9) for $\mathbf{M}$ by means of the generalized N/D technique described by Blankenbecler,[3] in which analyticity is fully exploited. However, we prefer at this time to adopt a much simpler approach, which, nevertheless, leads to some interesting results. Firstly, we make the choice

$$\mathbf{D}=\varrho^{-1}\mathbf{1}+I(s)\mathbf{N}, \tag{3.11}$$

where $\mathbf{1}$ stands for a Kronecker δ function in the channel indices a and c and a Dirac δ function in the suppressed intermediate variables; ϱ is a diagonal phase-space matrix, such that

$$\mathbf{M}\varrho^{-1}\mathbf{1}=\mathbf{M}. \tag{3.12}$$

$I(s)$ is defined by

$$I(s)=(s+\tfrac{4}{3}\epsilon)^{1/2}\int_0^\infty \frac{dx'}{(x'-s-\tfrac{4}{3}\epsilon)x'^{1/2}}, \tag{3.13}$$

so that $I(s+i\eta)-I(s-i\eta)=2\pi i$, and Eq. (3.10) is satisfied. As we shall only be interested in values of $s+\frac{4}{3}\epsilon$ just above the real axis, with positive real parts, we replace $I(s)$ in the following by its value in this limit, namely $i\pi$. Thus, we obtain an approximate set of scattering amplitudes, which has the virtue that generalized unitarity is satisfied, by determining $\mathbf{M}$ from the relation

$$\mathbf{M}=\mathbf{N}-i\pi\mathbf{M}\mathbf{N}. \tag{3.14}$$

It follows from our choice of $\mathbf{D}$ that $\mathbf{M}$ will be symmetric, i.e., will satisfy time-reversal invariance, provided $\mathbf{N}$ is chosen to be symmetric. Thus, Eq. (3.14) can be written in the alternate form

$$\mathbf{M}=\mathbf{N}-i\pi\mathbf{N}\mathbf{M}. \tag{3.15}$$

We proceed by making a particular choice for $\mathbf{N}$. Firstly, we set $N_{cc}=0$. This results in the considerable simplification that no sums over three-body states appear in the expressions for the inelastic amplitudes.[17] In particular, we have, from Eqs. (3.14) and (3.15), the approximations

$$\begin{aligned} M_{ca}&=N_{ca}-i\pi\textstyle\sum_a N_{ca}M_{aa},\\ M_{ac}&=N_{ac}-i\pi\textstyle\sum_a M_{aa}N_{ac},\\ M_{cc}{}^C&=-i\pi\textstyle\sum_a M_{ca}N_{ac}. \end{aligned} \tag{3.16}$$

We see here how the constraints imposed by unitarity introduce initial- and final-state interaction corrections to the amplitudes. The elastic amplitude M_{aa} can either be determined from experiment or, as we shall now describe, can be calculated, within the approximation of Eq. (3.14), in terms of the impulse approximation amplitude $M_{aa}{}^{(1)}$; i.e., we can obtain unitarity corrections to $M_{aa}{}^{(1)}$.

Our approximation for M_{aa} becomes, with the aid of Eqs. (3.14) and (3.16),

$$\begin{aligned} M_{aa}=&(N_{aa}-i\pi\textstyle\sum_c N_{ac}{}^*N_{ca})\\ &-i\pi\textstyle\sum_a M_{aa}(N_{aa}-i\pi\sum_c N_{ac}{}^*N_{ca}). \end{aligned} \tag{3.17}$$

If one believes that the impulse approximation provides a good first-order solution to the problem at hand it will be reasonable to make the choices $N_{aa}=\mathrm{Re}M_{aa}{}^{(1)}$

[16] L. F. Cook, Jr., and B. W. Lee, Phys. Rev. **127**, 283 (1962); J. S. Ball, W. R. Frazer, and M. Nauenberg, Phys. Rev. **128**, 478 (1962).

[17] Note that in this way we do not ignore the disconnected part of M_{cc}. As is well known, the ordinary unitarity conditions can be guaranteed by inserting approximations to $\mathbf{K}$, the reaction matrix, in the Heitler integral equation [W. Heitler, Proc. Cambridge Phil. Soc. **37**, 291 (1941)] which is similar in structure to Eq. (3.14). There, however, the choice $K_{cc}=0$ would be quite inconsistent since that *would* imply $M_{cc}{}^{(1)}=0$. This illustrates an advantage of the present formulation.

and $N_{ca}=M_{ca}^{(1)}$. By taking into account the relation

$$-\frac{1}{\pi}\,\mathrm{Im}M_{aa}^{(1)}=\sum_c M_{ac}^{(1)*}M_{ca}^{(1)}, \tag{3.18}$$

Eq. (3.17) then becomes

$$M_{aa}=M_{aa}^{(1)}-i\pi\sum_a M_{aa}M_{aa}^{(1)}. \tag{3.19}$$

Note that with $M_{aa}^{(1)}$ assumed to be given, no integrations over three-body intermediate states appear in Eq. (3.19). The nature of our approximation can be elucidated by considering Zimmerman's irreducible amplitude M_{aa}^{irr}, defined by the relation[18]

$$M_{aa}=M_{aa}^{\mathrm{irr}}-i\pi\sum_a M_{aa}M_{aa}^{\mathrm{irr}},$$

where M_{aa} is now the exact amplitude. Below the inelastic threshold M_{aa}^{irr} coincides with the K-matrix element. It is easily seen[17] that M_{aa}^{irr} has no unitarity cut in the elastic region. That is, it corresponds to an amplitude to whose imaginary part no diagrams with two-body intermediate states contribute. Now it is to be expected (e.g., on the basis of the nearness to the physical region of singularities in the t plane) that just these diagrams constitute the major correction to the impulse approximation. Thus, $M_{aa}^{(1)}$ is a much better approximation to M_{aa}^{irr} than to M_{aa} itself. By looking at the iterative solution of Eq. (3.19) we see that the elastic unitarity cut is introduced through the summation of a chain of diagrams which are iterations of the basic impulse approximation diagram. Similar interpretations hold for our expressions for the inelastic amplitudes. The approximations are pictured diagrammatically in Fig. 2.

Note added in proof. We have recently been able to cast the Lippmann–Schwinger integral equations for the inelastic and inelastic scattering amplitudes in a form which involves the two-body scattering operators T_{ij}, rather than the potentials V_{ij}, for the general case where all three potentials are nonvanishing. Iteration of these equations leads directly to the multiple scattering expansions discussed above, though we now need not rely on the convergence of these expansions. Our basic results, Eqs. (3.16) and (3.19), generalized to include off-energy-shell effects as well as the effects of nonvanishing V_{13}, can be obtained directly by approximating (in the spirit of the simple impulse approximation) the propagator which appears in the integral equations. The unitary nature of these results can be established in a simple way, without recourse to N/D procedures. A more complete discussion of these points will be given in the near future.

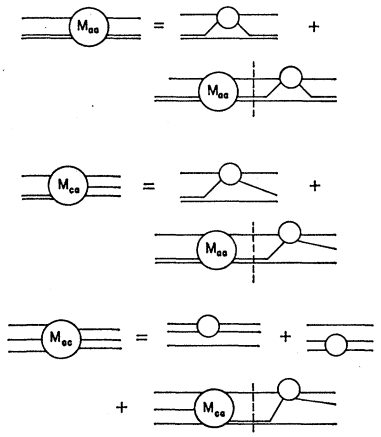

FIG. 2. Diagrammatic representations of the basic approximations, Eqs. (3.16) and (3.19). The dashed vertical lines indicate that only the imaginary (on-the-energy-shell) part of the propagator is retained so that only physical amplitudes are involved.

[18] W. Zimmerman, Nuovo Cimento **21**, 249 (1961).

4. SOLUTION OF THE INTEGRAL EQUATION

There are a number of ways of solving Eq. (3.19) for M_{aa}. For example, by introducing the partial wave decomposition

$$-\frac{1}{4\pi}(4m/3\hbar^2)(s+\tfrac{4}{3}\epsilon)^{1/2}M_{aa}=\sum_{l=0}^{\infty}(2l+1)f_l(s)P_l(\cos\theta), \tag{4.1}$$

and a similar decomposition for $M_{aa}^{(1)}$, Eq. (3.19) leads immediately to

$$f_l(s)=\frac{f_l^{(1)}(s)}{1-if_l^{(1)}(s)}. \tag{4.2}$$

This approximation has the virtue that unitarity is satisfied at all energies, even when inelastic channels are open. To see this we write

$$f_l=\frac{1}{2i}(e^{2i\alpha_l}\eta_l-1), \tag{4.3}$$

with α_l and η_l real, $\eta_l>0$, so that

$$\mathrm{Im}f_l=|f_l|^2+\tfrac{1}{4}(1-\eta_l^2). \tag{4.4}$$

By taking a partial wave projection of the unitarity relation, Eq. (3.4), and noting that, with the aid of the identity

$$P_l(\hat{k}_1^i\cdot\hat{k}_1^f)=(4\pi/2l+1)\sum_{m=-l}^{l}Y_l^m(\hat{k}_1^i)Y_l^{m*}(\hat{k}_1^f), \tag{4.5}$$

the expression

$$\sum_c\int_{-1}^{1}d\cos\theta P_l(\cos\theta)M_{ac}^{*}M_{ca}$$

can be written as an integral over an absolute square, we see that the second term in Eq. (4.4) must be positive, i.e., $\eta_l<1$. It can be demonstrated in a similar

way [see Eq. (3.18)] that $\mathrm{Im}f_l^{(1)}>0$. Now from Eq. (4.2) we have

$$\mathrm{Im}f_l=|f_l|^2+\frac{\mathrm{Im}f_l^{(1)}}{|1-if_l^{(1)}|^2}. \tag{4.6}$$

Since the second term in Eq. (4.6) is bounded from above and below by $\frac{1}{4}$ and zero, respectively, we see that unitarity is automatically preserved, in agreement with the general discussion in the preceding section.

As an illustration of the practical utility of this result we refer to a calculation of phase shifts for elastic $\pi-N$ scattering performed by Ball and Frazer.[19] The input was η_l, which was calculated by taking a partial wave projection of the amplitude in the strip approximation; the corrected partial wave amplitude was then obtained from a dispersion integral in which the left-hand cut was ignored. We have already noted the formal similarity between the impulse and strip approximations.[1] In the language of the present paper the Ball-Frazer approximation for η_l may be stated as

$$(1-\eta_l^2)/4\cong\mathrm{Im}f_l^{(1)}. \tag{4.7}$$

Ball and Frazer found that this approximation exceeded the unitarity limit by an order of magnitude, discouraging a serious calculational effort along these lines. Equation (4.6) shows that the unitarity violation is easily removed by inclusion of the damping factor $|1-if_l|^{-2}$. It would also be interesting to see if resonances appear as zeros in the denominator in Eq. (4.2) when continued into the complex l plane.

If one is interested in high-energy diffraction scattering, a convenient way to solve Eq. (3.19) is to introduce the Fourier-Bessel transform of the scattering amplitude[20]

$$-\frac{1}{4\pi}(4m/3\hbar^2)M_{aa}(s,t)=\int_0^\infty bdb\,J_0(bt^{1/2})H(b^2,s), \tag{4.8}$$

and a similar relation for $M_{aa}^{(1)}$. With the aid of the high-energy approximation discussed in Ref. 19, Eq. (3.19) is transformed, for large s, to

$$H(b^2,s)=\frac{H^{(1)}(b^2,s)}{1-iH^{(1)}(b^2,s)/2(s+\frac{4}{3}\epsilon)^{1/2}}. \tag{4.9}$$

This representation for $H(b^2,s)$ is precisely of the general form introduced by Blankenbecler and Goldberger in order to guarantee elastic unitarity at high energies; this will be satisfied for any real $H^{(1)}$. When inelastic processes can take place, the optical theorem imposes the restriction

$$\int_0^\infty bdb\,\mathrm{Im}H^{(1)}(b^2,s)>0, \tag{4.10}$$

where correction terms of order $1/s$ have been dropped. Now this condition can be written, with the aid of the relations

$$H^{(1)}(b^2,s)=\int_0^\infty xdx\,J_0(bx)\left(-\frac{1}{4\pi}\frac{4m}{3\hbar^2}\right)M_{aa}^{(1)}(s,x^2), \tag{4.11}$$

and

$$\int_0^\infty bdb\,J_0(bx)=\delta(x)/x, \tag{4.12}$$

as

$$-\frac{1}{4\pi}(4m/3\hbar^2)\,\mathrm{Im}M_{aa}^{(1)}(s,0)>0, \tag{4.13}$$

which is, in fact, true, so that in our approximation the inclusion of inelastic processes corresponds to true absorption, as it should.

We now turn to a discussion of an iterative solution of Eq. (3.19). That is, we seek a solution of the form

$$M_{aa}(s,t)=\lim_{n\to\infty}M_{aa}^{(n)}(s,t), \tag{4.14}$$

where

$$M_{aa}^{(n)}=M_{aa}^{(1)}+\textstyle\sum_a M_{aa}^{(n-1)}M_{aa}^{(1)},\quad n>1. \tag{4.15}$$

The question of the convergence of this successive-approximation procedure can be examined by making use of the analyticity of $M_{aa}(s,t)$ in the variable t, which is implied by Eq. (3.19) and the assumed analyticity of $M_{aa}^{(1)}(s,t)$. That is, we start with the representation

$$M_{aa}^{(1)}(s,t)=\int_{16\epsilon}^\infty\frac{dt'}{\pi}\frac{A^{(1)}(s,t')}{t'+t},\quad s+\tfrac{4}{3}\epsilon>0, \tag{4.16}$$

and seek a solution of the same form,

$$M_{aa}=\int_{16\epsilon}^\infty\frac{dt'}{\pi}\frac{A(s,t')}{t'+t}. \tag{4.17}$$

Equation (3.19) then provides an integral equation for $A(s,t)$,

$$A(s,t)=A^{(1)}(s,t)+i\left[-\frac{1}{4\pi}\frac{4m}{3\hbar^2}\right]\int_{16\epsilon}^\infty\frac{dt_1}{\pi}\int_{16\epsilon}^\infty\frac{dt_2}{\pi}\times A(s,t_1)A^{(1)}(s,t_2)K(s+\tfrac{4}{3}\epsilon,\,t;\,t_1,\,t_2), \tag{4.18}$$

which *can* be solved by a convergent successive-approximation procedure. This follows from the properties of the kernel K, which is given by[21]

$$K(w,\,t;\,t_1,\,t_2)=\frac{\pi}{2}\frac{\theta[t^{1/2}-t_1^{1/2}(1+t_2/4w)^{1/2}-t_2^{1/2}(1+t_1/4w)^{1/2}]}{[w\{t-(t_1^{1/2}+t_2^{1/2})^2\}\{t-(t_1^{1/2}-t_2^{1/2})^2\}-t_1t_2t]^{1/2}}. \tag{4.19}$$

[19] J. S. Ball and W. R. Frazer, Phys. Rev. Letters **7**, 204 (1961).
[20] R. Blankenbecler and M. L. Goldberger, Phys. Rev. **126**, 766 (1962).
[21] R. Blankenbecler, M. L. Goldberger, N. N. Khuri, and S. B. Treiman, Ann. Phys. (N. Y.) **10**, 62 (1960).

It is then seen that at each stage we obtain an approximation $A^{(m)}(s,t)$ which yields the correct function in a region $16\epsilon<t<t^{(m)}$, and that $t^{(m)}\to\infty$ as $m\to\infty$. This in turn guarantees that the limit in Eq. (4.14) exists and yields the correct solution. This type of solution will be particularly appropriate in a study of high-energy diffractive scattering, since the effects of the longer ranged forces are treated by the lower order terms in the sequence of approximations. Furthermore, our assumption, in Eqs. (4.16) and (4.17), that no subtractions are necessary, will be more likely to be true at higher energies. In general, when N subtractions are made, $A(s,t)$ is still determined by Eq. (4.18) but now M_{aa} is given by

$$M_{aa}(s,t)=\sum_{i=0}^{N-1} g_i(s)t^i+t^N\int\frac{dt'}{\pi}\frac{A(s,t')}{t'^N(t'+t)}. \quad (4.20)$$

Once $A(s,t)$ is known, the N functions $g_i(s)$ are determined by the relations

$$\frac{1}{i!}\left[\frac{\partial^i M_{aa}}{\partial t^i}\right]_{t=0}=g_i(s), \quad i=0, 1, \cdots N\text{-}1; \quad (4.21)$$

if M_{aa}, as given by Eq. (4.20), is inserted into the right-hand side of Eq. (3.19), we obtain from Eq. (4.21) N algebraic equations which can be solved for the $g_i(s)$.

Our procedure is easily extended to include rearrangement collisions which are introduced by allowing V_{12} to support a bound state. The leading term in the multiple scattering expansion of the rearrangement amplitude M_{ab} is just the Born approximation $M_{ab}^{(1)}$, given by

$$M_{ab}^{(1)}=-(\hbar^2/m)[(\mathbf{k}_1{}^f+\tfrac{1}{2}\mathbf{k}_3{}^i)^2+\epsilon_{12}]\times\tilde{\varphi}_{12}(\mathbf{k}_1{}^f+\tfrac{1}{2}\mathbf{k}_3{}^i)\tilde{\varphi}_{23}(\mathbf{k}_3{}^i+\tfrac{1}{2}\mathbf{k}_1{}^f). \quad (4.22)$$

It is seen from Eq. (2.16) that, with the energy fixed at a physical value, $M_{ab}^{(1)}$ is analytic in the cut and punctured complex plane of $\omega_{ab}=\hat{k}_3{}^i\cdot\hat{k}_1{}^f$ (the carets denoting unit vectors). The dominant singularity is a pole[22] whose position is determined by the condition $(\mathbf{k}_3{}^i+\frac{1}{2}\mathbf{k}_1{}^f)^2+\epsilon_{23}=0$.

We again adopt Eq. (3.14) as our basic approximation; the matrices $\mathbf{M}$ and $\mathbf{N}$ are now enlarged to accommodate the new channel b. After elimination of M_{ac} (we again set $N_{cc}=0$), we obtain the relations

$$\begin{aligned}M_{aa}&=\bar{M}_{aa}-i\pi\textstyle\sum_a M_{aa}\bar{M}_{aa}-i\pi\sum_b M_{ab}\bar{M}_{ba},\\ M_{ab}&=\bar{M}_{ab}-i\pi\textstyle\sum_a M_{aa}\bar{M}_{ab}-i\pi\sum_b M_{ab}\bar{M}_{bb},\end{aligned} \quad (4.23)$$

where

$$\bar{M}_{\alpha\beta}=N_{\alpha\beta}-i\pi\textstyle\sum_c N_{\alpha c}{}^*N_{c\beta}, \quad \alpha,\beta=a,b. \quad (4.24)$$

These coupled equations for M_{aa} and M_{ab} can be solved by successive approximations which, as we have pointed out with regard to Eq. (3.19), treat the effects of the longer ranged forces first. The convergence of this procedure can be demonstrated if we assume the representations (without subtractions)

$$\bar{M}_{\alpha\beta}=\int d\omega'\frac{\bar{B}_{\alpha\beta}(s,\omega')}{\omega'-\omega_{\alpha\beta}}, \quad (4.25)$$

with $\omega_{aa}=\hat{k}_1{}^i\cdot\hat{k}_1{}^f$, $\omega_{ab}=\hat{k}_3{}^i\cdot\hat{k}_1{}^f$, $\omega_{ba}=\hat{k}_1{}^i\cdot\hat{k}_3{}^f$, and $\omega_{bb}=\hat{k}_3{}^i\cdot\hat{k}_3{}^f$. Equations (4.23) and (4.25) then imply similar analytic properties for the $M_{\alpha\beta}$, with weight functions $B_{\alpha\beta}(s,\omega_{\alpha\beta})$ which are determined by the (assumed known) $\bar{B}_{\alpha\beta}$ on a segment of the $\omega_{\alpha\beta}$ axis. This segment extends to cover the entire cut for arbitrarily high orders of the approximation. These iterative techniques are of course quite similar to the procedure set up in Ref. 20 for two-body potential scattering.

APPENDIX

We here outline a proof of the generalized unitarity relation, Eq. (3.8). The simplifying assumption that $V_{13}=0$ is maintained and use is made of the multiple scattering expansions of the amplitudes given in Sec. 2.

While no distinction exists in the physical region between absorptive and imaginary parts of M_{aa}, the imaginary parts of M_{ca}, M_{ac}, and M_{cc} consist of sums of discontinuities across cuts in the several energy variables upon which these amplitudes depend. Thus, from the Schwarz reflection property we have the relation (the two momentum transfer variables, which do not introduce physical cuts, are suppressed)

$$\begin{aligned}&T_{ca}{}^*(s+i\eta, s_{12}{}^f+i\eta', s_{23}{}^f+i\eta'')\\ &\equiv(+++)^*=T_{ca}(s-i\eta, s_{12}{}^f-i\eta', s_{23}{}^f-i\eta'')\\ &\equiv(---),\end{aligned} \quad (A1)$$

which is in fact valid for each term in the expansion; it follows that

$$\begin{aligned}&2i\,\mathrm{Im}T_{ca}\\ &=(+++)-(---)\equiv[(+++)-(++-)]\\ &\quad+[(++-)-(+--)]+[(+--)-(---)]\\ &=D(s_{23}{}^f)+D(s_{12}{}^f)+D(s).\end{aligned} \quad (A2)$$

Equation (A2) is clearly of the form which arises by applying the identity

$$\mathrm{Im}\prod_{i=1}^{n}a_i=\sum_{i=1}^{n}a_1{}^*a_2{}^*\cdots(\mathrm{Im}a_i)a_{i+1}\cdots a_{n-1}a_n \quad (A3)$$

to each term in the expansion of T_{ca} and then summing all terms. We can, in fact, obtain an explicit expression for the sum of all terms which contribute to $D(s_{12}{}^f)$

[22] The significance of this pole in the analysis of stripping reactions was first discussed by Amado [R. D. Amado, Phys. Rev. Letters 2, 399 (1959)].

$+D(s_{23}^f)$, i.e., which contribute to the discontinuities across the cuts in the variables s_{12}^f and s_{23}^f.[23]

We must collect all terms which contain either the factor

$$2i\,\mathrm{Im}\langle\Phi_0^f|T_{ij}(E+i\eta)|\Phi_0'\rangle$$
$$=\langle\Phi_0^f|T_{ij}(E+i\eta)-T_{ij}(E-i\eta)|\Phi_0'\rangle$$
$$=-(2\pi i)\sum''\langle\Phi_0^f|T_{ij}|\Phi_0''\rangle^*\langle\Phi_0''|T_{ij}|\Phi_0'\rangle\delta(E''-E),$$

or the factor

$$\sum'\langle\Phi_0^f|T_{ij}|\Phi_0'\rangle^*\langle\Phi_0'|G_0(E+i\eta)-G_0(E-i\eta)|\Phi_0'''\rangle$$
$$=(2\pi i)\sum'\sum''\langle\Phi_0^f|T_{ij}|\Phi_0'\rangle^*\langle\Phi_0'|\Phi_0''\rangle$$
$$\times\langle\Phi_0''|\Phi_0'''\rangle\delta(E''-E).$$

The matrix element of T_{ij} introduces a factor $\delta^3(\mathbf{k}_3^f-\mathbf{k}_3'')$ for $(ij)=(12)$; for $(ij)=(23)$ the factor $\delta^3(\mathbf{k}_1^f-\mathbf{k}_1'')$ appears. Thus $\delta(E''-E)$ becomes $\delta[\hbar^2/m\times(s_{ij}^f-s_{ij}'')]$ which justifies the identification of the above class of terms with $D(s_{12}^f)+D(s_{23}^f)$.

Equation (2.10) for T_{ca} may be written in the equivalent form

$$T_{ca}=\langle\Phi_0^f|T_{12}+[T_{12}G_0T_{23}G_0T_{12}-T_{23}G_0T_{12}]\times[1-G_0T_{23}G_0T_{12}]^{-1}|\Phi_{23}^i\rangle. \quad (A4)$$

It follows from the above discussion that

$$-(2\pi i)^{-1}[D(s_{12}^f)+D(s_{23}^f)]$$
$$=\sum'\delta(E'-E)\langle\Phi_0^f|\{T_{12}|\Phi_0'\rangle^*\langle\Phi_0'|-T_{23}|\Phi_0'\rangle^*$$
$$\times\langle\Phi_0'|T_{23}G_0+T_{23}|\Phi_0'\rangle^*\langle\Phi_0'|-T_{12}|\Phi_0'\rangle^*$$
$$\times\langle\Phi_0'|T_{23}G_0\}T_{12}[1-G_0T_{23}G_0T_{12}]^{-1}|\Phi_{23}^i\rangle$$
$$=\sum'\delta(E'-E)\langle\Phi_0^f|T_{12}+T_{23}|\Phi_0'\rangle^*$$
$$\times\langle\Phi_0'|T_{12}-T_{23}G_0T_{12}[1-G_0T_{23}G_0T_{12}]^{-1}|\Phi_{23}^i\rangle. \quad (A5)$$

By combining Eqs. (A2) and (A5) and introducing the center-of-mass amplitudes we obtain the desired result,

$$-\frac{1}{\pi}\mathrm{Im}M_{ca}-\sum_c M_{cc}^{(1)*}M_{ca}$$
$$=-(2\pi i)^{-1}\{M_{ca}(s+i\eta,\ s_{12}^f-i\eta',\ s_{23}^f-i\eta'')$$
$$-M_{ca}(s-i\eta,\ s_{12}^f-i\eta',\ s_{23}^f-i\eta'')\}. \quad (A6)$$

A similar discussion can be given for the amplitude T_{cc}, although an addition complexity arises due to the presence of poles in the physical region at

$$v\equiv(\mathbf{k}_2^i-\mathbf{k}_3^f)\cdot(\mathbf{k}_3^i-\mathbf{k}_3^f)=0,$$

and

$$\tilde{v}\equiv(\mathbf{k}_2^f-\mathbf{k}_3^i)\cdot(\mathbf{k}_3^i-\mathbf{k}_3^f)=0,$$

in the term

$$T_{cc}^{(2)}=\langle\Phi_0^f|-T_{12}G_0T_{23}-T_{23}G_0T_{12}|\Phi_0^i\rangle, \quad (A7)$$

which appears in the expansion of T_{cc}. These poles arise from the vanishing energy denominator in G_0. We therefore define

$$T_{cc}^R=T_{cc}-T_{cc}^{(1)}-T_{cc}^{(2)} \quad (A8)$$

and observe that T_{cc}^R has physical cuts in the five energy variables s_{12}^i, s_{23}^i, s, s_{12}^f, s_{23}^f. With these variables fixed there are no physical singularities in the remaining three momentum-transfer variables, two of which are taken to be v and $\tilde{v}$. Suppressing these latter three variables, we introduce the notation

$$T_{cc}^R(s_{12\pm}^i,s_{23\pm}^i,s_\pm,s_{12\pm}^f,s_{23\pm}^f)=(\pm\pm\pm\pm\pm\pm), \quad (A9)$$

where $\pm$ indicates the sign of the small imaginary part of the corresponding variable. We may then write

$$2i\,\mathrm{Im}T_{cc}^R$$
$$=(++++++)-(------)\equiv[(++++++)$$
$$-(+++++-)]+[(+++++-)-(++++--)]$$
$$+[(++++--)-(++-----)]+[(++----)$$
$$-(+-----)]+[(+-----)-(------)]$$
$$=D(s_{12}^i)+D(s_{23}^i)+D(s)$$
$$+D(s_{12}^f)+D(s_{23}^f). \quad (A10)$$

Observe that $T_{cc}^{(2)}$ has no discontinuity in s (for fixed physical values of the remaining variables) so that with $T_{cc}^C=T_{cc}-T_{cc}^{(1)}$,

$$D(s)=T_{cc}^C(s_{12+}^i,s_{23+}^i,s_+,s_{12-}^f,s_{23-}^f)$$
$$-T_{cc}^C(s_{12+}^i,s_{23+}^i,s_-,s_{12-}^f,s_{23-}^f). \quad (A11)$$

We can now calculate $D(s)$ from Eqs. (A8), (A10), and the unitarity condition for T_{cc}. $D(s_{12}^f)+D(s_{23}^f)$ can be calculated from the expression

$$T_{cc}^R=\langle\Phi_0^f|T_{12}G_0T_{23}G_0T_{12}[1-G_0T_{23}G_0T_{12}]^{-1}$$
$$\times(1-G_0T_{23})+T_{23}G_0T_{12}G_0T_{23}$$
$$\times[1-G_0T_{12}G_0T_{23}]^{-1}(1-G_0T_{12})|\Phi_0^i\rangle, \quad (A12)$$

while $D(s_{12}^i)+D(s_{23}^i)$ is most conveniently obtained from the alternate form

$$T_{cc}^R=\langle\Phi_0^f|(1-T_{12}G_0)[1-T_{23}G_0T_{12}G_0]^{-1}$$
$$\times T_{23}G_0T_{12}G_0T_{23}+(1-T_{23}G_0)$$
$$\times[1-T_{12}G_0T_{23}G_0]^{-1}T_{12}G_0T_{23}G_0T_{12}|\Phi_0^i\rangle. \quad (A13)$$

It is then a straightforward matter to check (we omit the arithmetical details) that

$$-(2\pi i)^{-1}D(s)$$
$$=-\frac{1}{\pi}[\mathrm{Im}T_{cc}-\mathrm{Im}T_{cc}^{(1)}-\mathrm{Im}T_{cc}^{(2)}]$$
$$+(2\pi i)^{-1}[D(s_{12}^i)+D(s_{23}^i)+D(s_{12}^f)+D(s_{23}^f)]$$
$$=(2\pi)^3\delta^3(\mathbf{K}^i-\mathbf{K}^f)[\textstyle\sum_a M_{ca}^*M_{ac}+\sum_c M_{cc}^{C*}M_{cc}^C], \quad (A14)$$

which is the desired discontinuity equation.

[23] Observe that due to the structure of T_{ca} [see Eq. (2.10)] the portion of the amplitude which contains a cut in s_{12}^f, namely,

$$\langle\Phi_0^f|T_{12}[1-G_0T_{23}G_0T_{12}]^{-1}|\Phi_{23}^i\rangle,$$

has no cut in s_{23}^f so that

$$D(s_{12}^f)=(++-)-(+--)=(+++)-(+-+).$$