Treatment of Exchange in Scattering Processes*

M. BOLSTERLI
Institute for Theoretical Physics, University of Copenhagen, Copenhagen, Denmark and School of Physics, University of Minnesota, Minneapolis, Minnesota

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The scattering amplitude for processes in which either the initial (or final) state involves a single particle plus a bound group is split into direct and exchange parts. It is shown that these can be determined from the direct wave function for the initial (or final) state, which satisfies a single-particle differential equation without exchange, and the asymptotic plane-wave part of the wave function of the final (or initial) state.

1. INTRODUCTION

IN a preceding paper[1] it was shown that the N to $N+1$ particle amplitudes (units $\hbar=2m=1$)

$$\langle N\alpha\mathbf{K}|\psi(\mathbf{x})|N+1\,\beta\mathbf{K}'\rangle = (N+1)^{1/2}\int \Psi_{\alpha\mathbf{K}}^*(\mathbf{x}_1\cdots\mathbf{x}_N)\Psi_{\beta\mathbf{K}'}(\mathbf{x},\mathbf{x}_1\cdots\mathbf{x}_N)\times d\mathbf{x}_1\cdots d\mathbf{x}_N \quad (1)$$

have the form

$$\langle N\alpha\mathbf{K}|\psi(\mathbf{x})|N+1\,\beta\mathbf{K}'\rangle = (2\pi)^{-3/2}\exp[i(\mathbf{K}'-\mathbf{K})\cdot\mathbf{x}]\tilde{\psi}_{\alpha\beta}\left(\frac{N}{N+1}\mathbf{K}'-\mathbf{K}\right), \quad (2)$$

where $\tilde{\psi}_{\alpha\beta}$ is invariant under translations in velocity and configuration space. The Fourier transforms

$$\psi_{\alpha\beta}(\mathbf{x}) = (2\pi)^{-3/2}\int \exp(i\mathbf{k}\cdot\mathbf{x})\tilde{\psi}_{\alpha\beta}(\mathbf{k})d\mathbf{k} \quad (3)$$

satisfy the set of differential equations

$$\left(-\frac{N+1}{N}\nabla^2+\mathcal{E}_\alpha-\mathcal{E}_\beta\right)\psi_{\alpha\beta}(\mathbf{x})+S_\gamma v_{\alpha\gamma}(\mathbf{x})\psi_{\alpha\beta}(\mathbf{x})=0. \quad (4)$$

Here $v_{\alpha\gamma}(\mathbf{x})$ is the invariant potential defined in Ref. 1, and $\mathcal{E}_\alpha$ and $\mathcal{E}_\beta$ are the internal energies of states α and β, respectively. The boundary condition on $\psi_{\alpha\beta}(\mathbf{x})$ for continuum states β is

$$\lim_{|\mathbf{x}|\to\infty} \psi_{\alpha\beta^{\text{in}}}(\mathbf{x}) = \varphi_{\alpha\beta^{\text{in}}}(\mathbf{x}) + \text{outgoing spherical waves}, \quad (5)$$

where $\varphi_{\alpha\beta^{\text{in}}}(\mathbf{x})$ is the plane-wave part of $\psi_{\alpha\beta^{\text{in}}}(\mathbf{x})$. The functions $\varphi_{\alpha\beta}(\mathbf{x})$ are given in I for some states α and β. As was noted in I, the (anti)symmetry of the system can only be imposed on the continuum functions $\psi_{\alpha\beta}(\mathbf{x})$ by using the exact form of $\varphi_{\alpha\beta}(\mathbf{x})$, since the usual coordinate (anti)symmetrization procedure cannot be applied owing to the fact that $\psi_{\alpha\beta}(\mathbf{x})$ may contain as few as one of the $N+1$ particle coordinates. For bound states the (anti)symmetry is ensured by selecting only those bound-state solutions of (4) which satisfy a supplementary condition given in I. Bound states will not be considered further in this paper.

In Sec. 2, the integral equations for the invariant amplitudes are given. Section 3 gives the derivation of a new expression for the scattering amplitude or T matrix. Section 4 contains a discussion of the results.

2. GREEN'S FUNCTIONS AND INTEGRAL EQUATIONS

It is useful to transform (4) into a set of integral equations, by using the Green's functions $G_{\alpha\gamma}(\mathbf{x},\mathbf{x}';E)$ defined by

$$\left[-\frac{N+1}{N}\nabla^2+\mathcal{E}_\alpha-(E\pm i0)\right]G_{\alpha\gamma}{}^{\text{in,out}}(\mathbf{x},\mathbf{x}';E) = -\delta(\mathbf{x}-\mathbf{x}')\mathbf{1}_{\alpha\gamma}{}^N. \quad (6)$$

Here α and γ are N-particle internal states and $\mathbf{1}_{\alpha\gamma}{}^N$ is the generalization of the unit matrix in the subspace of N-particle internal states:

$$S_\gamma\mathbf{1}_{\alpha\gamma}f_\gamma=f_\alpha. \quad (7)$$

Note that $G_{\alpha\beta}$ is zero if α and γ are different types of states; that is, $G_{\alpha\gamma}$ is zero unless α and γ contain the same sets of bound groups. Only the relative momenta can differ in α and γ. Note also that no (anti)symmetry conditions are imposed on G, since the requirements of (anti)symmetry can only be imposed on $\psi_{\alpha\beta}(\mathbf{x})$ through its asymptotic form. G is the invariant analog of the usual $(E-H_0\pm i0)^{-1}$. With the Green's function of (6), Eqs. (4) become

$$\psi_{\alpha\beta^{\text{in}}}(\mathbf{x}) = \varphi_{\alpha\beta^{\text{in}}}(\mathbf{x}) + S_{\gamma\delta}\int d\mathbf{x}'\,G_{\alpha\gamma}{}^{\text{in}}(\mathbf{x},\mathbf{x}';\mathcal{E}_\beta)\times v_{\gamma\delta}(\mathbf{x}')\psi_{\delta\beta^{\text{in}}}(\mathbf{x}'). \quad (8)$$

An equation similar to (8) holds with "in" replaced by "out" throughout.

Now consider the special case that β^{in} is the internal state $b\mathbf{p}^{\text{in}}$ consisting asymptotically $(t\to-\infty)$ of N-particle bound state b plus a single incident particle with relative momentum $\mathbf{p}$. As shown in I (upper and

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[1] M. Bolsterli, Phys. Rev. **129**, 2830 (1963), hereafter referred to as I.

lower signs for fermions and bosons, respectively)

$$\varphi_{a,b\mathbf{p}^{\text{in}}}(\mathbf{x})=\delta_{a,b}(2\pi)^{-3/2}e^{i\mathbf{p}\cdot\mathbf{x}}, \quad (9a)$$

$$\varphi_{\gamma^{\text{in}}\mathbf{q}^{\text{in}},b\mathbf{p}^{\text{in}}}(\mathbf{x})=(\mp)(2\pi)^{-3/2}N^3e^{iN(\mathbf{q}-\mathbf{p})\cdot\mathbf{x}} \times\tilde{\psi}_{\gamma^{\text{in}}b}\left(N\mathbf{q}-\frac{N^2-1}{N}\mathbf{p}\right), \quad (9b)$$

$$\varphi_{\alpha^{\text{in}}b\mathbf{p}^{\text{in}}}(\mathbf{x})=0,\quad \alpha^{\text{in}}\neq\gamma^{\text{in}}\mathbf{q}^{\text{in}},\quad \alpha^{\text{in}}\neq a, \quad (9c)$$

where a is the N-particle bound state a, and $\gamma^{\text{in}}\mathbf{q}^{\text{in}}$ is the internal state consisting asymptotically of $N-1$ particles in internal state γ^{in} plus a single particle with relative momentum $\mathbf{q}$ [if γ^{in} contains any asymptotic single particles, then a slight modification of (9) is required]. It follows from the linearity of (4) that it is possible to split $\psi_{\alpha,b\mathbf{p}^{\text{in}}}(\mathbf{x})$ into two parts corresponding to the two asymptotic plane waves (9a) and (9b):

$$\psi_{\alpha b\mathbf{p}^{\text{in}}}(\mathbf{x})=\psi^{\text{D}}{}_{\alpha b\mathbf{p}^{\text{in}}}(\mathbf{x})\mp\psi^{\text{Ex}}{}_{\alpha b\mathbf{p}^{\text{in}}}(\mathbf{x}), \quad (10)$$

where $\psi^{\text{D}}{}_{b\mathbf{p}^{\text{in}}}$ and $\psi^{\text{Ex}}{}_{b\mathbf{p}^{\text{in}}}$ both satisfy (4) with boundary conditions

$$\lim_{|\mathbf{x}|\to\infty}\psi^{\text{D}}{}_{\alpha^{\text{in}}b\mathbf{p}^{\text{in}}}(\mathbf{x})=\mathbf{1}_{\alpha^{\text{in}}b}\varphi_{bb\mathbf{p}^{\text{in}}}(\mathbf{x}) +\text{outgoing spherical waves}, \quad (11a)$$

$$\lim_{|\mathbf{x}|\to\infty}\psi^{\text{Ex}}{}_{\alpha^{\text{in}},b\mathbf{p}^{\text{in}}}(\mathbf{x})=\mp S_\gamma\int d\mathbf{q}\,\mathbf{1}_{\alpha^{\text{in}},\gamma^{\text{in}}\mathbf{q}^{\text{in}}}\varphi_{\gamma^{\text{in}}\mathbf{q}^{\text{in}},b\mathbf{p}^{\text{in}}}(\mathbf{x}) +\text{outgoing spherical waves}. \quad (11b)$$

Here $\psi^{\text{D}}{}_{b\mathbf{p}^{\text{in}}}(\mathbf{x})$ is the invariant analog of the usual direct wave function. It is the invariant amplitude that would describe the scattering by bound state b of a distinguishable particle with the same mass and interactions as the N particles making up the state b. The function $\psi^{\text{Ex}}{}_{b\mathbf{p}^{\text{in}}}(\mathbf{x})$ is just the difference between the total wave function and this direct wave function.

In general, a continuum-state amplitude has as many distinct plane-wave parts as there are types of bound groups in its asymptotic part, and it can be split into the same number of independent parts each obeying (4) and a boundary condition like (11a) or (11b).

3. A FORM FOR THE SCATTERING AMPLITUDE

Corresponding to the splitting $\psi_{b\mathbf{p}}=\psi^{\text{D}}{}_{b\mathbf{p}}\mp\psi^{\text{Ex}}{}_{b\mathbf{p}}$, the T matrix for elastic scattering, which, according to I, is

$$\langle b\mathbf{q}|T|b\mathbf{p}\rangle=\tilde{\chi}_{b,b\mathbf{p}^{\text{in}}}(\mathbf{q}) =(2\pi)^{-3/2}\int e^{-i\mathbf{q}\cdot\mathbf{x}}S_\gamma v_{b\gamma}(\mathbf{x})\psi_{\gamma b\mathbf{p}^{\text{in}}}(\mathbf{x})d\mathbf{x}, \quad (12)$$

can be split into direct and exchange parts:

$$\langle b\mathbf{q}|T|b\mathbf{p}\rangle=\langle b\mathbf{q}|T|b\mathbf{p}\rangle_{\text{D}}\mp\langle b\mathbf{q}|T|b\mathbf{p}\rangle_{\text{Ex}}, \quad (13)$$

with

$$\langle b\mathbf{q}|T|b\mathbf{p}\rangle_{\text{D,Ex}} =(2\pi)^{-3/2}\int e^{-i\mathbf{q}\cdot\mathbf{x}}S_\gamma v_{b\gamma}(\mathbf{x})\psi^{\text{D,Ex}}{}_{\gamma b\mathbf{p}^{\text{in}}}(\mathbf{x})d\mathbf{x}. \quad (14)$$

The exchange amplitude can be written in another form by using the integral equation (8), which will be abbreviated

$$\psi^{\text{D,Ex}}{}_{b\mathbf{p}^{\text{in}}}=\varphi^{\text{D,Ex}}{}_{b\mathbf{p}^{\text{in}}}+G^{\text{in}}v\psi^{\text{D,Ex}}{}_{b\mathbf{p}^{\text{in}}}. \quad (15)$$

Iteration gives

$$\psi^{\text{Ex}}{}_{b\mathbf{p}^{\text{in}}}=(1+G^{\text{in}}v+G^{\text{in}}vG^{\text{in}}v+\cdots)\varphi^{\text{Ex}}{}_{b\mathbf{p}^{\text{in}}}, \quad (16)$$

so that

$$\begin{aligned}\langle b\mathbf{q}|T|b\mathbf{p}\rangle_{\text{Ex}} &=\int\varphi^{\text{D}*}{}_{b\mathbf{q}^{\text{out}}}v(1+G^{\text{in}}v+G^{\text{in}}vG^{\text{in}}v+\cdots)\varphi^{\text{Ex}}{}_{b\mathbf{p}^{\text{in}}}\\ &=\int[(1+G^{\text{out}}v+G^{\text{out}}vG^{\text{out}}v+\cdots)\varphi^{\text{D}}{}_{b\mathbf{q}^{\text{out}}}]^*v\varphi^{\text{Ex}}{}_{b\mathbf{p}^{\text{in}}}\\ &=\int\psi^{\text{D}*}{}_{b\mathbf{q}^{\text{out}}}v\varphi^{\text{Ex}}{}_{b\mathbf{p}^{\text{in}}}\\ &=S_{\gamma\alpha}\int d\mathbf{x}\,\psi^{\text{D}*}{}_{\gamma,b\mathbf{q}^{\text{out}}}(\mathbf{x})v_{\gamma\alpha}(\mathbf{x})\varphi^{\text{Ex}}{}_{\alpha,b\mathbf{p}^{\text{in}}}(\mathbf{x}),\end{aligned} \quad (17)$$

where $v^\dagger=v$ and $G^{\text{in}\dagger}=G^{\text{out}}$ have been used. According to this result, the exchange part of the elastic scattering amplitude can be obtained without solving the differential equation for the exchange function ψ^{Ex}. Only the plane-wave part φ^{Ex} of ψ^{Ex} is required, together with the potential v and the direct function ψ^{D}.

An analog of Eq. (17) holds also for inelastic amplitudes. Consider the general T-matrix element for a process in which the final state is $b\mathbf{p}$, the initial state is i; such a process will be called a single-particle process:

$$\begin{aligned}\langle b\mathbf{p}|T|i\rangle&=(2\pi)^{-3/2}S_\gamma\int d\mathbf{x}\,e^{-i\mathbf{p}\cdot\mathbf{x}}v_{b\gamma}(\mathbf{x})\psi_{\gamma i^{\text{in}}}(\mathbf{x})\\ &=S_{\alpha\gamma}\int d\mathbf{x}\,\varphi^{\text{D}*}{}_{\alpha,b\mathbf{p}^{\text{out}}}(\mathbf{x})v_{\alpha\gamma}(\mathbf{x})\psi_{\gamma i^{\text{in}}}(\mathbf{x})\\ &=\int\varphi^{\text{D}*}{}_{b\mathbf{p}^{\text{out}}}v\psi_{i^{\text{in}}},\end{aligned} \quad (18)$$

where the last form is in the abbreviated notation. Again

$$\psi_{i^{\text{in}}}=(1+G^{\text{in}}v+G^{\text{in}}vG^{\text{in}}v+\cdots)\varphi_{i^{\text{in}}}, \quad (19)$$

and hence,

$$\langle b\mathbf{p}|T|i\rangle=S_{\gamma\alpha}\int d\mathbf{x}\,\psi^{\text{D}*}{}_{\gamma,b\mathbf{p}^{\text{out}}}(\mathbf{x})v_{\gamma\alpha}(\mathbf{x})\varphi_{\alpha i^{\text{in}}}(\mathbf{x}), \quad (20)$$

so that, again, only the plane-wave part of the functions $\psi_{\alpha i}{}^{\text{in}}(\mathbf{x})$ is required.

4. DISCUSSION

The result of all this is that the only differential equation whose solution is required in order to determine the T-matrix elements for single-particle processes is that for the direct elastic invariant function. Only the plane-wave parts of the other invariant functions are required. These latter are known functions involving form factors for fewer than $N+1$ particles, and can be obtained by the methods given in I.

This simplification is analogous to the result, known from consideration of the full $N+1$ particle wave function $\Psi_{b\mathbf{p}^{\text{in}},0}(\mathbf{x}_1\cdots\mathbf{x}_{N+1})$, that only the direct wave function, i.e., that solution of $H\Psi=E\Psi$ with asymptotic form

$$\exp\left[i\mathbf{p}\cdot\left(\mathbf{x}_1-\frac{\mathbf{x}_2+\cdots+\mathbf{x}_{N+1}}{N}\right)\right]\Psi_{b,-\mathbf{p}}(\mathbf{x}_2\cdots\mathbf{x}_{N+1})$$

is needed, since the exchange parts of the wave function can be generated by applying permutation operators to the direct wave function. Equation (20) is essentially the invariant part of the result of the permutations, with a considerable amount of regrouping of terms, of course.

In terms of a more standard formulation, it is clear that (20) is the invariant analog of

$$T_{b\mathbf{p},i}=\int \Psi_{b\mathbf{p}}{}^{(-)*}V\Phi_i, \tag{21}$$

where the equality of (20) and (21) follows from first replacing V by $H-E$, as specified by Ekstein,[2] and then noting that both $\Psi_{b\mathbf{p}}{}^{(-)*}$ and $(H-E)\Phi_i$ are (anti)symmetric, so that $\Psi_{b\mathbf{p}}{}^{(-)*}$ can be replaced by its unsymmetrized part $\Psi^{\text{D}*}{}_{b\mathbf{p}}{}^{(-)}$. An advantage of the present derivation is that the kinematics of the center of mass has been extracted in advance.

Note that (18) and (20) resemble the "post" and "prior" interaction forms of the scattering amplitude in the noninvariant form; they differ from the latter in the treatment of exchange. When the scattering amplitude for a single-particle process is expressed in terms of invariant functions, it is seen that the *same* potential appears in both (18) and (20), although not all matrix elements $v_{\alpha\gamma}$ are the same in both. However, there is absolutely no "post-prior" ambiguity in Born approximation for single-particle processes:

$$\langle b\mathbf{p}|T|i\rangle_{\text{Born}}=(2\pi)^{-3/2}S_\alpha\int d\mathbf{x}\, e^{-i\mathbf{p}\cdot\mathbf{x}}v_{b\alpha}(\mathbf{x})\varphi_{\alpha i}{}^{\text{in}}(\mathbf{x}), \tag{22}$$

which follows from either (18) or (20). The "post-prior" ambiguity can only occur in the noninvariant form of the Born approximation for these processes.

It is perhaps worth noting that (20) is quite useful, since it involves the "best" of both the single-particle state $b\mathbf{p}$ and the initial state, namely, only the direct part of $\psi_{b\mathbf{p}}$ and only the plane-wave part of $\psi_i{}^{\text{in}}$. Hence, exchange can be treated exactly by first calculating the function $\psi^{\text{D}}{}_{b\mathbf{p}}$, the equation for which involves no exchange, and then using only the asymptotic part of ψ_i, but with exchange treated correctly.

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[2] H. Ekstein, Phys. Rev. **101**, 880 (1956).