Unstable Particles in S-Matrix Theory*†

DANIEL ZWANZIGER
Department of Physics, University of California, Berkeley, California and Istituto di Fisica, Università di Roma, Istituto Nazionale di Fisica Nucleare, Sezione di Roma, Roma, Italy

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The consequences of the existence of complex poles in scattering amplitudes, corresponding to resonances or unstable particles, are investigated. Specific examples show that corresponding to such poles are normal threshold cuts lying near the physical region and which cause "wooly cusps" in scattering cross sections. The discontinuity across such a cut is expressed by a unitarity-like relation in terms of unphysical amplitudes with unstable external particles defined by the residues of the complex poles. More generally, it is shown that the Landau equations for the singularities must be extended to include all unstable as well as stable particles. The Cutkosky formulas specify the corresponding discontinuities in terms of the physical and unphysical amplitudes.

I. INTRODUCTION

In the description of scattering phenomena provided by the theory of the analytically continued S matrix, a stable particle appears as a fixed pole in the scattering amplitude on the positive real axis of the complex energy plane. Similarly, a resonance corresponds to a fixed pole at a complex value of the energy, on a second, "unphysical," sheet at a point near to the physical value of the energy at which the resonance is observed. The angular momentum in both cases is determined by the residue which contributes generally to only one partial wave in an angular momentum projection of the amplitude. An investigation is made of the consequences of the existence of poles on unphysical sheets. It is found quite generally that properties of scattering amplitudes which are known to hold for stable particles also hold for unstable particles. The following two sections are devoted to special cases and the last section to general results.

In Sec. II the known properties of complex poles on the second sheet of elastic two-body scattering amplitudes are first reviewed. In a pair theory it is shown that there are poles in the amplitude for two particles $\rightarrow$ four particles which is coupled to the elastic amplitude by unitarity. Unphysical amplitudes in which one or more external particles have complex masses are defined in terms of the residue at the complex pole in the physical $2\rightarrow 4$ amplitude. Such unphysical amplitudes are shown to be simply proportional to the projection onto the angular momentum state of the unstable particle of the physical amplitude, evaluated at an energy equal to the complex mass of the unstable particle. Analyticity and unitarity properties of the unphysical amplitudes follow.

In Sec. III we investigate the amplitude $3\rightarrow 3$ which is related to the $2\rightarrow 4$ amplitude by crossing. We study the analytic continuation across the three-particle cut from the physical sheet onto the adjacent unphysical sheet. A two-body cut is found there corresponding to a normal threshold of a stable and an unstable particle. Such a branch point on an unphysical sheet was first pointed out by Blankenbecler *et al.*[1] The discontinuity across this cut is expressed by a unitarity-like relation in terms of the unphysical amplitudes defined earlier. As a simple application we derive the known formula[2,3] for the "wooly cusp" which appears in an elastic scattering cross section at the onset of unstable particle production. The suggestion that a prominent known resonance in a scattering cross section strongly influences the cross section at an energy corresponding to the appearance of the resonant state in a competing inelastic channel was first put forward by Peierls[4] and developed by that author and co-workers[5] into the isobar model of inelastic processes. That model is the approximation to the physical three-body unitarity condition which is obtained by retaining the resonant pole terms. This approximate physical unitarity condition closely resembles the exact unitarity-like relation obtained in this section for the unphysical discontinuity on the second sheet. Ball, Frazer, and Nauenberg,[6] to whom the reader is referred for an extensive bibliography on production amplitudes and unstable particles, have also made a detailed study of the approximation which results from the retention of resonant terms in the three-body unitarity condition, and obtained a similar approximate unitarity formula. The analysis of three-body unitarity leading to Eq. (21) of this section is modeled on that of these authors and of Stapp.[7]

In Sec. IV it is shown that the Landau equations

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for the singularities of scattering amplitudes must be extended to include particles of complex mass. The Cutkosky formula[8] for the corresponding discontinuities is shown to hold, provided the set of amplitudes is extended to include amplitudes in which external particles are unstable.

II. DEFINITION OF UNPHYSICAL AMPLITUDE

Let us consider the interactions of particles of mass m and spin zero that are produced in pairs. The analyticity properties of the amplitude $A(s_1,\cos\theta)$ for the elastic scattering, through an angle θ of a pair of these particles of center-of-mass square energy s_1, has been discussed by several authors.[9] For $\cos\theta$ in its physical range, $-1\leq\cos\theta\leq 1$, $A(s_1,\cos\theta)$ is assumed to be analytic in the cut s_1 plane with cuts extending along the real axis from $s_1=4m^2$ to $s_1=+\infty$ and from $s_1=-\infty$ to $s_1=0$. A is real in the real interval $0\leq s_1\leq 4m^2$. The discontinuity across the right hand cut is specified by the elastic unitarity condition[10] in the elastic interval $4m^2\leq s_1\leq 16m^2$.

$$[A(s_{1+},\hat{n}_f\cdot\hat{n}_i)-A(s_{1-},\hat{n}_f\cdot\hat{n}_i)]/2i$$
$$=\frac{1}{8}\left[\frac{s_1-4m^2}{s_1}\right]^{1/2}\int A(s_{1+},\hat{n}_f\cdot\hat{n})A(s_{1-},\hat{n}\cdot\hat{n}_i)d\hat{n},\quad(1)$$

where $\hat{n}_i$, $\hat{n}_f$, and $\hat{n}$ are unit vectors and $A(s_{1+})$ and $A(s_{1-})$ are, respectively, the values of A above and below the cut. As has been shown in detail,[9] A possesses a square-root-type branch point at $s_1=4m^2$ and a continuation along the elastic interval onto the adjacent sheet. We designate the values of A on the original "physical" sheet and on the adjacent "unphysical" sheet as $A(s_1^{\mathrm{I}},\cos\theta)$ and $A(s_1^{\mathrm{II}},\cos\theta)$, respectively. In Ref. 9 it is shown that $A(s_1^{\mathrm{II}},\cos\theta)$ is analytic in the cut s_1 plane with cuts on the real axis from $s_1=-\infty$ to $s_1=0$ and from $s_1=4m^2$ to $s_1=+\infty$ (we are keeping $\cos\theta$ in its physical range) with the possible exception of discrete poles at fixed values of s_1. Stapp[7] has argued that, apart from cases of degeneracy which are not expected to occur, each such pole is simple and appears in only one partial wave amplitude. Its residue is consequently proportional to a Legendre polynomial $P_l(\cos\theta)$. Our considerations shall only apply to poles of this type. If such a pole at $s_1=s_r$ lies sufficiently near the physical points, it causes a resonance at a center-of-mass energy $E=\mathrm{Re}(s_r^{1/2})$ of width $\Gamma=2\,\mathrm{Im}(s_r^{1/2})$ in the lth partial wave. We interpret these poles as unstable particles of mass $m_r=(s_r^{1/2})$ and spin l. The ρ meson may be thought of as of this type. The uniqueness of this definition and the charge conjugation invariance of unstable particles is discussed in Ref. 7. The reality property of the amplitude requires that if a pole of residue $\gamma P_l(\cos\theta)$ is present at s_r, there will be a second pole of residue $\gamma^* P_l(\cos\theta)$ at s_r^*.

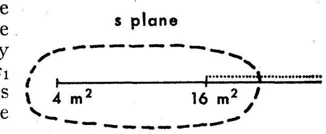


FIG. 1. The dashed line indicates the boundary of the assumed domain of analyticity of $A_{2,4}(s_1)$ in the complex s_1 plane. The solid line indicates a cut, the dots indicate the physical points.

In the physical region $16m^2\leq s_1\leq 36m^2$ a second channel is energetically allowed and the unitarity condition couples the elastic-scattering amplitude $A_{2,2}$ to the production amplitude $A_{2,4}$. The subscripts indicate the number of initial and final particles. The analyticity properties of production amplitudes are not well understood at present. We suppose that as a function of square c.m. energy s_1, $A_{2,4}(s_1)$ has a cut along the elastic interval $4m^2\leq s_1\leq 16m^2$, that it is analytic in some neighborhood which surrounds the cut, as shown in Fig. 1, and that the discontinuity across this cut is specified by the physical unitarity condition in which only the elastic intermediate states are included. This is equivalent to the assumption that the unitarity condition is valid down to the vanishing of the phase-space factors. Because of these assumptions, the discussion of this and the following section must be regarded as heuristic. The main burden of these two sections is to show that the existence of normal thresholds on the physical sheet and resonance poles on an unphysical sheet implies the existence of certain further singularities on unphysical sheets. The existence of other singularities on the physical sheet does not alter this conclusion but only makes the arguments more lengthy. In the last section the problem is considered anew from a general point of view which does not require these unjustified assumptions.

The assumed unitarity relation for the elastic interval has the form

$$[A_{2,4}(\Omega_f,s_{1+},\hat{n}_i)-A_{2,4}(\Omega_f,s_{1-},\hat{n}_i)]/2i$$
$$=\frac{1}{8}\left[\frac{s_1-4m^2}{s_1}\right]^{1/2}\int A_{2,4}(\Omega_f,s_{1-},\hat{n})A_{2,2}(s_{1+},\hat{n}\cdot\hat{n}_i)d\hat{n}\quad(2a)$$
$$=\frac{1}{8}\left[\frac{s_1-4m^2}{s_1}\right]^{1/2}\int A_{2,4}(\Omega_f,s_{1+},\hat{n})A_{2,2}(s_{1-},\hat{n}\cdot\hat{n}_i)d\hat{n},\quad(2b)$$

where $\hat{n}$ specifies the two particle configuration and Ω_f represents the set of variables which specify the final four-particle configuration. Now $A_{2,2}(s_{1+})$ and $A_{2,4}(s_{1-})$ possess clockwise continuations $A_{2,2}(s_1^{\mathrm{II}})$ and $A_{2,4}(s_1^{\mathrm{I}})$, respectively. Consequently, Eq. (2a) provides a clockwise continuation of $A_{2,4}(s_{1+})$ into some domain of the second sheet where it has the value

$$A_{2,4}(\Omega_f,s_1^{\mathrm{II}},\hat{n}_i)=A_{2,4}(\Omega_f,s_1^{\mathrm{I}},\hat{n}_i)$$
$$+\frac{2i}{8}\left[\frac{s_1-4m^2}{s_1}\right]^{1/2}\int A_{2,4}(\Omega_f,s_1^{\mathrm{I}},\hat{n})A_{2,2}(s_1^{\mathrm{II}},\hat{n}\cdot\hat{n}_i)d\hat{n}.\ (3)$$

[8] R. E. Cutkosky, J. Math. Phys. 1, 429 (1960).

[9] See, for example, W. Zimmerman, Nuovo Cimento 21, 249 (1961).

[10] The normalization is specified by $S=1+2i\delta^4(P_f-P_i)A$ and the n-particle phase space which is $\varphi=\Pi_{i=1}^{n}\frac{1}{2}(\mathbf{p}_i^2+m_i^2)^{-1/2}d^3\mathbf{p}_i$ so that for elastic scattering of two scalar particles,

$$A(s_1\cos\theta)=(2/\pi)\Sigma_l(2l+1)s\{[s-(m_1+m_2)^2][s-(m_1-m_2)^2]\}^{-1/2}\times\exp(i\delta_l)\sin\delta_l P_l(\cos\theta).$$

It is worth noting that Eq. (3) directly expresses $A_{2,4}(s_1^{\mathrm{II}})$ in terms of $A_{2,4}(s_1^{\mathrm{I}})$ and $A_{2,2}(s_1^{\mathrm{II}})$. This is, in fact, a general feature. Once the continuation of the two-body amplitude across the two-body cut is known, the continuation of production amplitudes across the two-body cut is expressed as a definite integral. If $A_{2,2}(s_1^{\mathrm{II}},\hat{n}\cdot\hat{n}_i)$ possesses a pole in this domain with principal part $g^2P_l(\hat{n}\cdot\hat{n}_i)[\pi(s_r-s_1)]^{-1}$ then the right-hand side of Eq. (3) and, consequently, $A_{2,4}(s_1^{\mathrm{II}})$ also possess a pole at $s_1=s_r$. Crossing symmetry implies that the analytic continuation of $A_{2,4}$ represents the amplitude for a crossed channel process in which s_1 represents the total square mass of a pair of particles in a three- or four-particle state. Then the pole at $s_1=s_r$ represents the effect of the resonance between the pair in the multiparticle initial or final state. The existence of many of the recently discovered unstable particles has, in fact, been revealed by the observation of Breit-Wigner one-pole dependence of production amplitudes on the mass of a pair of particles in a multiparticle final state.

From Eq. (3) one also obtains an expression for the principal part of $A_{2,4}(s_1^{\mathrm{II}})$ at $s_1=s_r$:

$$\frac{2ig^2}{8\pi(s_r-s_1)}\left[\frac{s_r-4m^2}{s_r}\right]^{1/2}\int A_{2,4}(\Omega_f,s_r^{\mathrm{I}},\hat{n})P_l(\hat{n}\cdot\hat{n}_i)d\hat{n}. \quad (4)$$

We may write the residue of the pole of $A_{2,2}^{\mathrm{II}}$ in the form

$$\frac{g^2}{\pi}P_l(\hat{n}_f\cdot\hat{n}_i)=\frac{1}{\pi}\sum_\mu\left[g\left(\frac{4\pi}{2l+1}\right)^{1/2}Y_l^\mu(\hat{n}_f)\right]\times\left[g\left(\frac{4\pi}{2l+1}\right)^{1/2}Y_{l,\mu}(\hat{n}_i)\right], \quad (5)$$

where Y_l^μ is the spherical harmonic and $Y_{l,\mu}=Y_l^{\mu*}$. The factors in the brackets are interpreted as the trivial amplitudes A^μ and A_μ for the three-particle process $m_r\leftrightarrow m+m$. Similarly, the residue of the pole term of $A_{2,4}(s_1^{\mathrm{II}})$ given by expression (4) has the form

$$\frac{1}{\pi}\sum\left[\frac{2ig}{8}\left(\frac{s_r-4m^2}{s_r}\right)^{1/2}\left(\frac{4\pi}{2l+1}\right)^{1/2}\int A_{2,4}(\Omega_f,s_r^{\mathrm{I}},\hat{n})\times Y_l^\mu(\hat{n})d\hat{n}\right]\left[g\left(\frac{4\pi}{2l+1}\right)^{1/2}Y_{l,\mu}(\hat{n}_i)\right]. \quad (6)$$

The factor in the first bracket is now interpreted as the amplitude $A_{r,4m}{}^\mu$ for the process $m_r\leftrightarrow 4m$ occurring in a state μ of z component of angular momentum. We have

$$A_{r,4m}{}^\mu(\Omega_f)=\frac{2ig}{8}\left(\frac{s_r-4m^2}{s_r}\right)^{1/2}\left(\frac{4\pi}{2l+1}\right)^{1/2}\times\int A_{2,4}(\Omega_f,s_r^{\mathrm{I}},\hat{n})Y_l^\mu(\hat{n})d\hat{n}, \quad (7)$$

so that the principal part of $A_{2,4}^{\mathrm{II}}$ at $s_1=s_r$ is

$$[\pi(s_r-s_1)]^{-1}\sum_\mu A_{r,4m}{}^\mu(\Omega_f)A_\mu(\hat{n}_i)=[\pi(s_r-s_1)]^{-1}\times\sum_\mu A_{r,4m}{}^\mu(\Omega_f)\left[g\left(\frac{4\pi}{2l+1}\right)^{1/2}Y_{l,\mu}(\hat{n}_i)\right]. \quad (8)$$

Equation (7) defines a nontrivial scattering amplitude with five external particles, one of which is unstable. We shall call amplitudes, where one or more external particles is unstable, "unphysical" scattering amplitudes. Stapp[7] has given a general discussion of the factorization of residues at complex poles and the definition of unphysical amplitudes.

The unphysical amplitude which is defined by Eq. (7) is related to the physical scattering amplitude quite directly. It is, in fact, apart from a constant factor, simply the projection, onto the angular momentum state of the unstable particle, of the physical amplitude evaluated at an energy equal to the complex mass of the unstable particle. If the physical amplitude possesses crossing symmetry so that the unitarity condition expressed in Eqs. (2) and (3) may be continued in the external variables to the physical region for a crossed channel process, then the unphysical amplitude will also possess this crossing. Consequently, the amplitude defined by Eq. (7) also describes the processes $m_r+m\leftrightarrow 3m$, $m_r+2m\leftrightarrow 2m$, and $m_r+3m\leftrightarrow m$. The unitarity property across physical cuts in unphysical amplitudes may also be obtained as a consequence of the direct relation between physical and unphysical amplitudes. For consider a general unitarity condition for physical amplitudes across a physical cut:

$$[A(\Omega_f,s_{1+},\Omega_i)-A(\Omega_f,s_{1-},\Omega_i)]/2i=\int A(\Omega_f,s_{1+},\Omega)A(\Omega,s_{1-},\Omega_i)d\Omega. \quad (9)$$

By setting the mass variable of a pair of external particles which resonate equal to the complex resonant value and projecting onto the appropriate relative angular momentum state, the unitarity condition (9) is established for the corresponding unphysical amplitube.

Equation (7) which defines the projection is to be understood as the continuation of the projection from a physical value of s_1 to its value s_r. This equation may be further continued in its free variables to the region of a crossed channel. During these continuations, singularities may enter the region of integration. The supposition that the continuations exist means that the contour of integration may be suitably deformed to avoid the advancing singularity. In the projection referred to above, it is the deformed contour which must be followed. Stated differently, the continuation of the projection may not equal the projection of the continuation. In such a case, the former must be used.

III. CONTINUATION THROUGH THE THREE-BODY CUT AND THE SINGULARITY ENCOUNTERED THERE

The amplitude $A_{2,4}$ is related by crossing symmetry to the amplitude $A_{3,3}{}^c$, the connected part of the three-particle scattering amplitude. The complete amplitude for the process $3\rightarrow 3$ is the sum of the connected and disconnected parts, $A_{3,3}=A_{3,3}{}^c+A_{3,3}{}^d$ which are represented in Fig. 2. The disconnected parts contain as factors energy-momentum-conserving delta functions and in addition, as we shall see, the connected part has poles in its physical region. These infinitudes in the physical region are allowed because the square modulus of $A_{3,3}$ is not a physical cross section. Only amplitudes with two particles in the initial state are required to be finite in their physical region.

To simplify the discussion of this section we shall consider a slightly different model, the scattering of three different spin-zero particles of masses m_1, m_2, and m_3 each of which carries a different conserved additive quantum number. By simple notational changes, the equations of the preceding section become applicable to this model. Then for physical energies $s\geqslant m_1+m_2+m_3$, but below production threshold the only allowed process is elastic scattering. The amplitude for this process has the form

$$A=A^c+A^d=A^c+A_1{}^d+A_2{}^d+A_3{}^d, \qquad (10)$$

where the terms on the right are represented diagrammatically in Fig. 2. In the last equation the subscript 3, 3 has been suppressed, as it is in the following.

The physical unitarity condition in the elastic region for this process is

$$\text{Im}A=\int AA^*d\varphi,$$

or

$$\text{Im}A^c+\text{Im}A^d=\int[A^cA^{c*}+A^cA^{d*}+A^dA^{c*}+(A^dA^{d*})^c]d\varphi+\int(A^dA^{d*})^d d\varphi, \qquad (11)$$

where $d\varphi$ represents the three-body phase[10] space

$$d\varphi=\int\delta^4(p_1+p_2+p_3-p)\delta(p_1{}^2-m_1{}^2)\delta(p_2{}^2-m_2{}^2)\times\delta(p_3{}^2-m_3{}^2)d^4p_1d^4p_2d^4p_3. \qquad (12)$$

The four-vector integration d^4p extends over the interior of the future light cone. The various types of terms on the right-hand side of Eq. (11) are represented diagrammatically in Fig. 3. The disconnected parts on

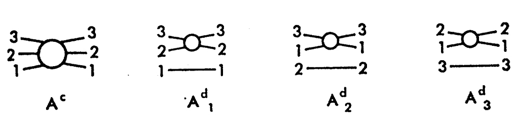


FIG. 2. Connected and disconnected parts of the amplitude $A_{3,3}$.

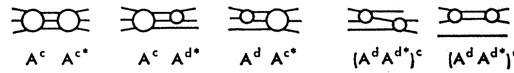


FIG. 3. Terms in the unitarity integral for the scattering of three particles.

the left- and right-hand sides of Eq. (11) are separately equal since they contain energy-momentum conserving delta functions. This condition, $\text{Im}A^d=\int(A^dA^{d*})^d d\varphi$, is satisfied by setting

$$A_1{}^d(p_1,p_2,p_3\rightarrow p_1',p_2',p_3')=2(\mathbf{p}_1{}^2+m_1{}^2)^{1/2}\delta^3(\mathbf{p}_1-\mathbf{p}_1')A_{2,2}(p_2,p_3\rightarrow p_2',p_3'), \qquad (13)$$

and similarly for $A_2{}^d$ and $A_3{}^d$, where $A_{2,2}$ is the elastic two-body scattering amplitude.

Before proceeding further, it is convenient to introduce variables which describe the three-particle configurations. Let $s=(p_1+p_2+p_3)^2$ and $s_1=(p_2+p_3)^2$, the total mass and the mass of the 2–3 pair, respectively. Let $\hat{\Omega}_1$ be a unit vector in the direction of $\mathbf{p}_1$ in the over-all center-of-mass system and let $\hat{n}_1$ be a unit vector in the direction of $\mathbf{p}_2$ in a center-of-mass frame of the 2–3 pair. s_λ, $\hat{\Omega}_\lambda$, and $\hat{n}_\lambda$ are defined for $\lambda=1$, 2, and 3 by cyclic permutation. Any three-particle configuration in its center-of-mass frame may be specified by s, s_λ, $\hat{\Omega}_\lambda$, and $\hat{n}_\lambda$, where $\lambda=1$, 2, or 3. Making use of the identity

$$\int d^4K\int_0^\infty ds_1\delta^4(p_2+p_3-K)\delta(K^2-s_1)=1, \qquad (14)$$

the integral over the phase space of Eq. (12) may be written as

$$\int d\varphi=\int_{(m_2+m_3)^2}^{(s^{1/2}-m_1)^2}ds_1\times\frac{d\hat{\Omega}_1}{8s}\{[s-(s_1{}^{1/2}+m_1)^2][s-(s_1{}^{1/2}-m_1)^2]\}^{1/2}\times\frac{d\hat{n}_1}{8s_1}\{[s_1-(m_2+m_3)^2][s_1-(m_2-m_3)^2]\}^{1/2}, \qquad (15)$$

or by any cyclic permutation of 1, 2, and 3. It is convenient to define the two-particle phase space integration

$$\int d\varphi_1=\int\frac{d\hat{n}_1}{8s_1}\{[s_1-(m_2+m_3)^2]\times[s_1-(m_2-m_3)^2]\}^{1/2} \qquad (16)$$

and its cyclic permutations.

Let us now equate the connected parts in Eq. (11),

$$\text{Im}A^c=\int[A^cA^{c*}+A^cA^{d*}+A^dA^{c*}+(A^dA^{d*})^c]d\varphi. \qquad (17)$$

The last term on the right $\int(A^dA^{d*})^c d\varphi$ is a sum of

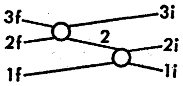


FIG. 4. Pole term of the three-body amplitude $A_{3,3}{}^c$. The circles are the two-body amplitudes $A_{2,2}$.

six terms, each of the form

$$A_{2,2}(s_{f1+})\delta[(p_{f2}+p_{f3}-p_{i3})^2-m_2{}^2]A_{2,2}(s_{i3-}), \quad (18)$$

which is represented in Fig. 4. We have introduced the indices i, f to represent the initial and final configurations, suppressed the angular dependence of the two body amplitudes $A_{2,2}$, and made use of the reality properties of $A_{2,2}$ to replace $A_{2,2}$ and $A_{2,2}{}^*$ by $A_{2,2}(s_+)$ and $A_{2,2}(s_-)$, the amplitude evaluated, respectively, above and below the physical cut. Because s is in the elastic three-body energy range, the three s_λ lie in the elastic two-body energy range. The Dirac delta function which appears in the last expression indicates that A^c has a pole in its physical region in the three-body invariant momentum transfer variable $t\equiv(p_{2f}+p_{3f}-p_{3i})^2=(p_{1i}+p_{2i}-p_{1f})^2$. There are, in fact, six such poles, one for each of the six such invariants.

Our goal is to make use of the unitarity relation, Eq. (17), to effect a continuation across the normal threshold three-body cut in the total three-body energy variable s. However, in contrast to the two-body case, the imaginary part of the amplitude is not the discontinuity in one variable. Each of the three initial and three final two-body energy variables s_λ lies above its two-body normal threshold, giving an imaginary part, as do the six pole terms just mentioned. In addition, we expect that higher order Landau singularities give further imaginary parts.

We adopt the method of assuming a small number of singularities in the amplitude—namely, the resonance poles and the physical threshold cuts—and see what further singularities are required by unitarity. If it be objected that we should also assume all the higher Landau singularities which occur in the absence of resonance poles, let it be pointed out that those singularities were found without assuming resonance poles and the singularities generated by them, which of course change the topological structure of the Riemann surface. The only method of proceeding, in fact, seems to be to begin with those singularities that have a direct physical interpretation and to find the others by consistency. This is how the higher order Landau singularities were themselves obtained[11,12] if one disallows perturbative arguments which are not relevant in the presence of resonance poles. Accordingly, we shall suppose that A^c is formally a real analytic function of its external invariants. Its normal threshold singularities lie on the real axes of the corresponding invariants and in the neighborhood of the elastic physical region these are the poles in the crossed channel variables, the two-particle normal threshold cuts and the three-particle normal threshold cut in s. The arguments of the following section indicate that the presence of other singularities will only complicate the discussion without altering its conclusions.

Consequently, in Eq. (17) A^c and A^{c*} will be written, respectively, as $A^c(s_{f\lambda+},s_+,h_+,t_{\mu+},s_{i\nu+})$ and $A^c(s_{f\lambda-},s_-,h_-,t_{\mu-},s_{i\nu-})$. The arguments of these functions do not refer to independent variables, but to the singularities which contribute imaginary parts in the physical region; $s_{f\lambda}$ and $s_{i\nu}$ (λ, $\nu=1, 2, 3$) represent the two-body cuts in the final and initial two-body energy variables; s the three-body cut, t_μ ($\mu=1,\cdots,6$) the cross-channel poles, and h possible higher order Landau singularities; the subscripts $+$ and $-$ indicate that the function is evaluated, respectively, above and below these discontinuities. We conduct our discussion as if the singularities represented by h were not present. However, we carry h along an argument of A^c to show that its presence does not cause formal difficulties.

We write Eq. (17), explicitly, as

$$[A^c(s_{f+},s_+,h_{fi+},t_{fi+},s_{i+})-A^c(s_{f-},s_-,h_{fi-},t_{fi-},s_{i-})]/2i$$
$$=\int A^c(s_{f+},s_+,h_{fo+},t_{fo+},s_{o+})A^c(s_{o-},s_-,h_{oi-},t_{oi-},s_{i-})d\varphi$$
$$+\sum_{\lambda=1}^{3}\int A^c(s_{f+},s_+,h_{fo+},t_{fo+},s_{i+})A_{2,2}(s_{i\lambda-})d\varphi_\lambda$$
$$+\sum_{\lambda=1}^{3}\int A_{2,2}(s_{f\lambda+})A^c(s_{f-},s_-,h_{oi-},t_{oi-},s_{i-})d\varphi_\lambda$$
$$+\sum A_{2,2}(s_{f+})\delta(t_{fi}-m^2)A_{2,2}(s_{i-}), \quad (19)$$

where the Greek subscripts have been suppressed in the arguments of A^c, i and f label the initial and final external variables, and o labels the internal variables that are integrated over. We note that only the first term on the right is integrated over the full three-body phase space, that the middle two terms are integrated over the two-body phase space, and that the last term contains no integration. The integrals over the two-body phase space may be evaluated by using the unitarity condition across the two-body cut, Eqs. (2a) and (2b), analytically continued from the $2\rightarrow 4$ channel to the $3\rightarrow 3$ channel. The middle two terms on the right-hand side are then given by

$$\begin{aligned}
[3A^c&(s_{f+},s_+,h_{fi+},t_{fi+},s_{i1+},s_{i2+},s_{i3+})\\
&-A^c(s_{f+},s_+,h_{fi+},t_{fi+},s_{i1+},s_{i2+},s_{i3-})\\
&-A^c(s_{f+},s_+,h_{fi+},t_{fi+},s_{i1+},s_{i2-},s_{i3+})\\
&-A^c(s_{f+},s_+,h_{fi+},t_{fi+},s_{i1-},s_{i2+},s_{i3+})]/2i\\
&-[3A^c(s_{f1-},s_{f2-},s_{f3-},s_-,h_{fi-},t_{fi-},s_{i-})\\
&-A^c(s_{f1+},s_{f2-},s_{f3-},s_-,h_{fi-},t_{fi-},s_{i-})\\
&-A^c(s_{f1-},s_{f2+},s_{f3-},s_-,h_{fi-},t_{fi-},s_{i-})\\
&\qquad-A^c(s_{f1-},s_{f2-},s_{f3+},s_-,h_{fi-},t_{fi-},s_{i-})]/2i.
\end{aligned}$$

[11] J. C. Polkinghorne, Nuovo Cimento **23**, 360 (1962); **25**, 901 (1962).
[12] H. Stapp, Phys. Rev. **125**, 2139 (1962).

The last term on the right-hand side of Eq. (19) is the only one containing Dirac delta functions. We may separate from A^c the pole term which is required by the delta function. We set

$$A^c(s_f,s,h,t_\pm,s_i)=A^r(s_f,s,h,s_i)+A^p(s_f,t_\pm,s_i), \quad (20)$$

where $A^p(t_\pm)$, represented diagrammatically in Fig. 4, is the last term of Eq. (19) with the delta function $\delta(t-m^2)$ replaced by $[\pi(m^2-t\mp i\epsilon)]^{-1}$. Since A^p is simply a sum of products of the two-body scattering amplitude divided by $\pi(m^2-t)$, it has no three-body cut in s, and no higher order Landau singularity h which contributes to the imaginary part. On the other hand, near the physical region, A^r has no cross-channel poles. The last term on the right-hand side of Eq. (19) may, thus, be written as

$$\begin{aligned}&[A^p(s_{f+},t_+,s_{i-})-A^p(s_{f+},t_-,s_{i-})]/2i\\&\quad=[A^c(s_{f+},s_+,h_+,t_+,s_{i-})-A^c(s_{f+},s_-,h_-,t_-,s_{i-})]/2i\\&\qquad-[A^c(s_{f+},s_+,h_+,t_\pm,s_{i-})-A^c(s_{f+},s_-,h_-,t_\pm,s_{i-})]/2i.\end{aligned}$$

If we make these substitutions into Eq. (19), we obtain

$$\begin{aligned}&[A^c(s_{f+},s_+,h_{fi+},t_{fi\pm},s_{i-})-A^c(s_{f+},s_-,h_{fi-},t_{fi\pm},s_{i-})]/2i\\&=\int A^c(s_{f+},s_+,h_{fo+},t_{fo+},s_{o-})A^c(s_{o+},s_-,h_{oi-},t_{oi-},s_{i-})d\varphi\\&\qquad+[\sum_d A^c(s_{f+},s_+,h_{fi+},t_{fi+},s_i{}^d)\\&\qquad\qquad-\sum_d A^c(s_f{}^d,s_-,h_{fi-},t_{fi-},s_{i-})]/2i, \quad (21)\end{aligned}$$

where

$$\begin{aligned}\sum_d A^c(\cdots s_i{}^d)\equiv&\,2A^c(\cdots s_{i1+},s_{i2+},s_{i3+})\\&-A^c(\cdots s_{i1+},s_{i2+},s_{i3-})-A^c(\cdots s_{i1+},s_{i2-},s_{i3+})\\&-A^c(\cdots s_{i1-},s_{i2+},s_{i3+})+A^c(\cdots s_{i1-},s_{i2-},s_{i3-}),\end{aligned}$$

and

$$\begin{aligned}\sum_d A^c(s_f{}^d\cdots)\equiv&\,2A^c(s_{f1-},s_{f2-},s_{f3-}\cdots)\\&-A^c(s_{f1+},s_{f2-},s_{f3-}\cdots)-A^c(s_{f1-},s_{f2+},s_{f3-}\cdots)\\&-A^c(s_{f1-},s_{f2-},s_{f3+}\cdots)+A^c(s_{f1+},s_{f2+},s_{f3+}\cdots).\end{aligned}$$

In Eq. (21) we have reversed the signs of the subscripts of s_o, by introducing between the two factors of A^c the identity

$$\begin{aligned}1=&[1-2iA_{2,2}(s_{03-})][1-2iA_{2,2}(s_{02-})][1-2iA_{2,2}(s_{01-})]\\&\times[1+2iA_{2,2}(s_{01+})][1+2iA_{2,2}(s_{02+})][1+2iA_{2,2}(s_{03+})],\end{aligned}$$

and by repeatedly making use of the aforementioned unitarity condition of A^c across its two-body cuts.

What we have been doing is transforming the unitarity condition, which originally gave the imaginary part and, thus, the sum of all the discontinuities into an expression for particular discontinuities. This process could be continued [further transforming the last terms of Eq. (21) so that the two-body energy variables always have the same sign], by identifying the discontinuity across the three-body cut and across the other higher order Landau singularities. The unitarity condition is, in fact, regarded as a prescription for constructing the amplitude; if we can find a function which has such discontinuities that the unitarity condition is satisfied, that function is presumably the amplitude.

However, it is not our purpose here to pursue the analysis of the three-body amplitude, and Eq. (21) is in a form which is convenient for our purposes. Namely, we shall assume that this equation may be continued some distance, however short, clockwise in s. To make this continuation, it is necessary to make a choice of the other independent variables. This choice is somewhat arbitrary, but we may take, for example,

$$A_{fi}{}^c=A^c(s,s_{f1},s_{f2},s_{i1},s_{i2},t_{fi1},t_{fi2},t_{fi3}),$$

where $s=(p_{f1}+p_{f2}+p_{f3})^2$, $s_{f1}=(p_{f2}+p_{f3})^2$, $t_{fi}=(p_{f1}-p_{i1})^2$, and similarly by cyclic permutation and substitution of i for f. This choice of variables has the advantage that as far as possible each normal threshold cut near the physical region is associated with one independent variable which is displayed explicitly. The first variable has the three-body cut, the next four each have a corresponding two-body cut. The only other normal threshold cuts near the physical region are those associated with s_{f3} and s_{i3}, but they are not independent since

$$s=s_{f1}+s_{f2}+s_{f3}-m_1{}^2-m_2{}^2-m_3{}^2; \quad (22a)$$

$$s=s_{i1}+s_{i2}+s_{i3}-m_1{}^2-m_2{}^2-m_3{}^2. \quad (22b)$$

Consequently, all the normal threshold branch points of A^c—and these are the only branch points we assume explicitly—near the physical region are in the space of the five complex energy variables (labeled s) and none are encountered as the three momentum transfer variables (labeled t) are varied. If s_{f1}, s_{f2}, s_{i1}, and s_{i2} are held at fixed values we will, thus, find three cuts in the s variable: the three-body normal threshold cut beginning at $s=(m_1+m_2+m_3)^2$, a two-body cut beginning at $s=s_{f1}+s_{f2}+(m_2+m_3)^2-m_1{}^2-m_2{}^2-m_3{}^2$, and another at $s=s_{i1}+s_{i2}+(m_2+m_3)^2-m_1{}^2-m_2{}^2-m_3{}^2$.

Since there are no cuts in the momentum transfer variables near the physical region, the "continuation in s" that we are proposing may be regarded as a continuation on the Riemann surface of the five independent energy variables. Let us note that our assumption concerning the location of the singularities and our choice of independent variables are consistent with the assumption that A^c is a real analytic function of these independent variables. In Eq. (20), A^p is by construction a real analytic function. If we set

$$s=(\sum_{\lambda=1}^{3}m_\lambda)^2-\sum_{\lambda=1}^{3}\alpha_{f\lambda}, \quad s_{i1}=(m_2+m_3)^2-\alpha_{i1},$$

$$s_{f1}=(m_2+m_3)^2-\alpha_{f1},$$

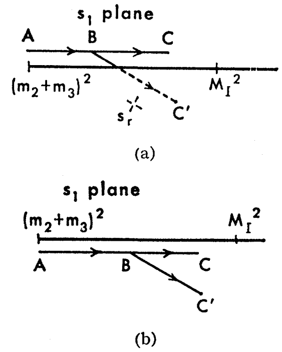


FIG. 5(a). Path of integration over s_{01} on the Riemann surface of $A^c(s_0,s,h_{0i-},t_{0i-},s_i)$. The elastic cut in s_{01} extends from $(m_2+m_3)^2$ to M_I^2. (b). Path of integration over s_{01} on Riemann surface of $A^c(s_f,s,h_{f0+},t_{f0+},s_0)$.

and cyclic permutation, where the α are small real positive numbers satisfying $\sum_{\lambda=1}^3 \alpha_{f\lambda}=\sum_{\lambda=1}^3 \alpha_{i\lambda}$, we obtain points on a portion of the real axis of each of the independent variables, consistent with the restrictions of Eqs. (22) and where A^r is evaluated below all its threshold cuts. On these real points, we take A^r to be real. Consequently, A^r is a real analytic function and by Eq. (20) A^c is also. This consistency is nontrivial since we have considered 13 poles and cuts distributed among 8 independent variables. Furthermore, the existence of the "Symanzik region," a domain which lies below all cuts and where A^c is real, justifies the heretofore cavalier way in which we have been indicating the value of the amplitude by writing its determination with respect to singularities. We take the Symanzik region to be the original domain of definition of the amplitude. The plus and minus subscripts occurring in Eq. (21) now acquire a precise meaning. They indicate the path along which the amplitude is to be continued from the Symanzik region to the point, labeled by values of the independent variables, at which the value of the amplitude is required. This path passes above or below a singularity according as the variable corresponding to the singularity bears a plus or a minus sign.

For convenience we may define a "physical sheet" in the space of the five complex variables s, s_{i1}, s_{i2}, s_{f1}, s_{i2}. It is the surface which may be reached by analytic continuation from the Symanzik region and which is bounded by s real, $s\geqslant(m_1+m_2+m_3)^2$, s_{i1} real, $s_{i1}\geqslant(m_2+m_3)^2$, s_{i2} real, $\geqslant(m_1+m_3)^2$, $s-s_{i1}-s_{i2}+(m_1+m_2+m_3)^2$ real, $\geqslant(m_1+m_2)^2$ and similarly for the s_f. The amplitude is single valued on this surface. We, of course, do not wish to imply that the correct amplitude has an analytic structure of this sort. However, the general method is to begin with a simple structure and see what further singularities are required. In this paper we are concerned only with the additional singularities due to resonance poles.

Following these formal remarks, we may now discuss the analytic continuation of the unitarity condition expressed by Eq. (21). Our purpose is to discover what further singularities are required by the presence of resonance poles and this unitarity condition. Let us suppose that particles 2 and 3 have a resonance pole near the physical region at a square mass s_r. Then, as established in the preceding section, there will be a pole in the s_1 variable of A^c at $s_1=s_r$. The integral over the three-particle phase space which is represented in Eq. (15) contains an integration with respect to s_1, which has as upper limit $(s^{1/2}-m_1)^2$. Consequently, as s varies we may expect an end-point singularity in the right-hand side of Eq. (21) at $(s^{1/2}-m_1)^2=s_r$ or $s^{1/2}=s_r^{1/2}+m_1$, i.e., at a center-of-mass energy given by the sum of the masses of the stable particle m_1 and the unstable particle $m_r=s_r^{1/2}$. This does, in fact, occur. We shall see in detail how this comes about, determine on what sheet of the s variable the singularity lies, and obtain a formula for the corresponding discontinuity.

Heretofore, we have considered the unitarity condition for real values of the independent variables. We now propose to continue in s clockwise, that is, to let s have a negative imaginary part. Then the continuation from the points s_- leads onto the physical sheet, whereas from s_+ onto a second sheet adjacent to the physical points. We reflect this in our notation by relabelling the paths s^{I} and s^{II} instead of s_- and s_+, respectively. By Eqs. (22) which also hold for the s_0, at least one of the initial, intermediate and final two-body variables s_i, s_o, s_f of Eq. (21) also then have negative imaginary parts. To be definite, we let all of them have negative imaginary parts, as we may do without violating the constraints of Eqs. (22), and we likewise relabel the two-body energy variables. The unitarity condition then reads

$$[A^c(s_f^{\mathrm{II}},s^{\mathrm{II}},h_{fi+},t_{fi\pm},s_i^{\mathrm{I}})-A^c(s_f^{\mathrm{II}},s^{\mathrm{I}},h_{fi-},t_{fi\pm},s_i^{\mathrm{I}})]/2^i$$

$$=\int A^c(s_f^{\mathrm{II}},s^{\mathrm{II}},h_{f0+},t_{f0+},s_0^{\mathrm{I}})A^c(s_0^{\mathrm{II}},s^{\mathrm{I}},h_{0i-},t_{0i-},s_i^{\mathrm{I}})d\varphi$$
$$+[\sum_d A^c(s_f^{\mathrm{II}},s^{\mathrm{II}},h_{fi+},t_{fi+},s_i^d)$$
$$-\sum_d A^c(s_f^d,s^{\mathrm{I}},h_{fi-},t_{fi-},s_i^{\mathrm{I}})]/2i. \quad (23)$$

Since the poles terms represented by $t_\pm$ and the discontinuity across the two-body cuts are known, this equation is, in fact, an integral equation for $A^c(\cdots s^{\mathrm{II}}\cdots)$ if $A^c(\cdots s^{\mathrm{I}}\cdots)$ is given. It is the means for effecting the continuation across the three-body cut, just as the two-body unitarity condition studied in the preceding section effects the continuation across the two-body cut. The phase-space integration, given by Eq. (15) contains an integration with respect to s_{01} from $s_{01}=(m_2+m_3)^2$ to $s_{01}=(s^{1/2}-m_1)^2$. When s is physical, as in Eq. (21), the path of integration lies, respectively, on the lower and upper lips of the two-body s_{01} cut of $A^c(\cdots s_{0-})$ and $A^c(s_{0+}\cdots)$, as shown by the path ABC in Figs. 5(a) and (b). As s is continued clockwise the upper limit of integration moves continuously from C to C', and the path of integration becomes ABC' in Figs. 5(a) and (b). We see that the new path ABC' retreats away from the s_{01} cut of $A^c(\cdots s_0)$, but, as indicated by the notation, passes through the s_{01} cut of $A^c(s_0\cdots)$ onto the second sheet where resonance poles lie. If no other singularities prevent continuing in s to the point $s=(s_r^{1/2}+m_1)^2$ (which will lie arbitrarily close to the physical region for a sufficiently sharp resonance s_r), then at this value of s, an end-point singularity will occur since $C'=(s^{1/2}-m_1)^2=s_r$. The pole of the amplitude at $s_{01}=s_r$ is fixed, just below the physical points, and, consequently, remains fixed there for any continuation whatsoever in the other independent variables. This is a direct application of the continuity theorem. We note that the end-point singularity in s has been unambiguously located near the physical points on the adjacent Riemann sheet.

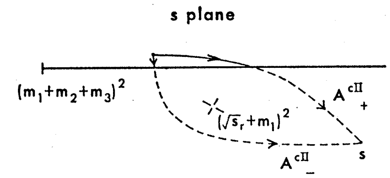
The coincidence of a singularity of the integrand with the end point of integration actually only establishes a necessary condition for a singularity in the integral, but is not a sufficient condition. To establish that the singularity is, in fact, present in $A^c(\cdots s^{\text{II}}\cdots)$ —it is not present in $A^c(\cdots s^{\text{I}}\cdots)$ by assumption—we shall use Eq. (23) to evaluate $A^c(\cdots s^{\text{II}}\cdots)$ along two paths which differ by a circuit around $s=(s_r^{1/2}+m_1)^2$. Denoting by $A^c(\cdots s_+^{\text{II}}\cdots)$ and $A^c(\cdots s_-^{\text{II}}\cdots)$ the values of $A^c(\cdots s^{\text{II}}\cdots)$ obtained by continuing in s along the paths shown in Fig. 6, we have from Eq. (23)

$$\begin{aligned}
&[A^c(s_f^{\text{II}},s_+^{\text{II}},h_{fi+},t_{fi\pm},s_i^{\text{I}})-A^c(s_f^{\text{II}},s_-^{\text{II}},h_{fi+},t_{fi\pm},s_i^{\text{I}})]/2i\\
&=\int_{C_+}A^c(s_f^{\text{II}},s_+^{\text{II}},h_{f0+},t_{f0+},s_0^{\text{I}})A^c(s_0^{\text{II}},s^{\text{I}},h_{0i-},t_{0i-},s_i^{\text{I}})d\varphi\\
&\quad-\int_{C_-}A^c(s_f^{\text{II}},s_-^{\text{II}},h_{f0+},t_{f0+},s_0^{\text{I}})A^c(s_0^{\text{II}},s^{\text{I}},h_{0i-},t_{0i-},s_i^{\text{I}})d\varphi\\
&\quad+\sum_d\,[A^c(s_f^{\text{II}},s_+^{\text{II}},h_{fi+},t_{fi+},s_i^d)\\
&\qquad\qquad-A^c(s_f^{\text{II}},s_-^{\text{II}},h_{fi+},t_{fi+},s_i^d)]/2i,\qquad(24)
\end{aligned}$$

where the contours C_+ and C_- of the s_{01} integration are shown in Fig. 7. Denoting by O the path C_+-C_-, we obtain

$$\begin{aligned}
&[A^c(s_f^{\text{II}},s_+^{\text{II}},h_{fi+},t_{fi\pm},s_1^{\text{I}})-A^c(s_f^{\text{II}},s_-^{\text{II}},h_{fi+},t_{fi\pm},s_i^{\text{I}})]/2i\\
&=\int_{C_-}[A^c(s_f^{\text{II}},s_+^{\text{II}},h_{f0+},t_{f0+},s_0^{\text{I}})\\
&\qquad-A^c(s_f^{\text{II}},s_-^{\text{II}},h_{f0+},t_{f0+},s_0^{\text{I}})]\\
&\qquad\times A^c(s_0^{\text{II}},s^{\text{I}},h_{0i-},t_{0i-},s_i^{\text{I}})d\varphi\\
&\qquad+\sum_d\,[A^c(s_f^{\text{II}},s_+^{\text{II}},h_{fi+},t_{fi+},s_i^d)\\
&\qquad-A^c(s_f^{\text{II}},s_-^{\text{II}},h_{fi+},t_{fi+},s_i^d)]/2i\\
&\qquad+\int_O A^c(s_f^{\text{II}},s_+^{\text{II}},h_{f0+},t_{f0+},s_0^{\text{I}})\\
&\qquad\qquad\times A^c(s_0^{\text{II}},s^{\text{I}},h_{0i-},t_{0i-},s_i^{\text{I}})d\varphi.\qquad(25)
\end{aligned}$$

FIG. 6. Paths of the analytic continuation in s across the three-body cut beginning at $s=(m_1+m_2+m_3)^2$ which lead to the values $A^c(\cdots s_+^{\text{II}}\cdots)$ and $A^c(\cdots s_-^{\text{II}}\cdots)$.



The last term on the right has the explicit form

$$\begin{aligned}
&\oint ds_{01}\int\frac{d\hat{\Omega}_{01}}{8s}\{[s-(s_{01}^{1/2}+m_1)^2][s-(s_{01}^{1/2}-m_1)^2]\}^{1/2}\\
&\quad\times\int\frac{d\hat{n}_{01}}{8s_{01}}\{[s_{01}-(m_2+m_3)^2][s_{01}-(m_2-m_3)^2]\}^{1/2}\\
&\quad\times A^c(s_f^{\text{II}},s_+^{\text{II}},h_{f0+},t_{f0+},s_0^{\text{I}})A^c(s_0^{\text{II}},s^{\text{I}},h_{0i-},t_{0i-},s_i^{\text{I}}),\quad(26)
\end{aligned}$$

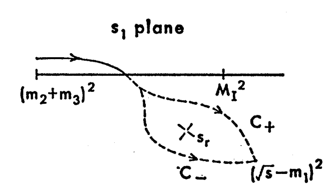
in which use is made of the three-body phase space, Eq. (15). The contour of the s_{01} integration surrounds the pole of $A^c(s_0^{\text{II}}\cdots)$ at $s_{01}=s_r$. Recalling that $\hat{n}_{01}$ is the direction of p_{02} in the c.m. frame of particles 2 and 3, and that the pole at s_r is a resonance in the 2–3 system, the principal part of $A^c(s_0^{\text{II}},s^{\text{I}},h_{0i-},t_{0i-},s_i^{\text{I}})$ is given by Eq. (8) of the preceding section. It is

$$[\pi(s_r-s_1)]^{-1}\sum_\mu\left[g\left(\frac{4\pi}{2l+1}\right)^{1/2}Y_{l,\mu}(\hat{n}_{01})\right]\times A_u{}^\mu(s^{\text{I}},h_{0i-},t_{0i-},s_i^{\text{I}})$$

in which $A_u{}^\mu(s^{\text{I}},h_{0i-},t_{0i-},s_i^{\text{I}})=A_i{}^\mu(\hat{\Omega}_{01},s,s_{i1},\hat{\Omega}_{i1},\hat{n}_{i1})$ designates the unphysical amplitude for the process $1+r\leftrightarrow 1+2+3$ and is independent of $\hat{n}_{01}$. Consequently, expression (26), evaluated by means of the Cauchy integral theorem, equals

$$\begin{aligned}
&\int\frac{d\hat{\Omega}_{01}}{8s}\{[s-(s_r^{1/2}+m_1)^2][s-(s_r^{1/2}-m_1)^2]\}^{1/2}\\
&\quad\times\sum_\mu A_u{}^\mu(s^{\text{I}},h_{0i-},t_{0i-},s_i^{\text{I}})\\
&\quad\times\int\frac{d\hat{n}_{01}}{8s_r}\{[s_r-(m_2+m_3)^2][s_r-(m_2-m_3)^2]\}^{1/2}\\
&\quad\times A^c(s_f^{\text{II}},\ s_+^{\text{II}},\ h_{f0+},\ t_{f0+},\ s_{01}^{\text{I}}=s_r,\ s_{02}^{\text{I}},\ s_{03}^{\text{I}})\\
&\qquad\qquad\times 2i\left[g\left(\frac{4\pi}{2l+1}\right)^{1/2}Y_{l,\mu}(\hat{n}_{01})\right].
\end{aligned}$$

FIG. 7. The contours of integration C_+ and C_- pass, respectively, above and below the resonance pole at $s_1=s_r$ on the second sheet.



But by Eq. (7), the integration with respect to $\hat{n}_{01}$ is the projection which yields the amplitude

$$A_{u,\mu}(s_f^{\rm II},s_+^{\rm II},h_{f0+},t_{f0+})=A_{u,\mu}(s_{f1},\hat{\Omega}_{f1},\hat{n}_{f1},s,\hat{\Omega}_{01})$$

and, consequently, expression (26) takes the form

$$\int\frac{d\hat{\Omega}_{01}}{8s}\{[s-(s_r^{1/2}+m_1)^2][s-(s_r^{1/2}-m_1)^2]\}^{1/2}$$
$$\times\sum_\mu A_{u,\mu}(s_f^{\rm II},s_+^{\rm II},h_{f0+},t_{f0+})A_u{}^\mu(s^{\rm I},h_{0i-},t_{0i-},s_i^{\rm I}).\quad(27)$$

We abbreviate this by introducing the two-body unphysical phase space

$$\int d\varphi_u\equiv\int\frac{d\hat{\Omega}}{8s}\{[s-(s_r^{1/2}+m_1)^2][s-(s_r^{1/2}-m_1)^2]\}^{1/2}.$$

When expression (27) is substituted into Eq. (25) we obtain

$$\text{disc}_uA^c(s_f^{\rm II},s^{\rm II},h_{fi+},t_{fi\pm},s_i^{\rm I})$$
$$=2i\int_{C_-}\text{disc}_uA^c(s_f^{\rm II},s^{\rm II},h_{f0+},t_{f0+},s_0^{\rm I})$$
$$\times A^c(s_0^{\rm II},s^{\rm I},h_{0i-},t_{0i-},s_i^{\rm I})d\varphi$$
$$+\sum_d\text{disc}_uA^c(s_f^{\rm II},s^{\rm II},h_{fi+},t_{fi+},s_i^d)$$
$$+\int d\varphi_u\sum_\mu A_{u,\mu}(s_f^{\rm II},s_+^{\rm II},h_{f0+},t_{f0+})$$
$$\times A_u{}^\mu(s^{\rm I},h_{0i-},t_{0i-},s_i^{\rm I}),\quad(28)$$

where

$$\text{disc}_uA^c(\cdots s^{\rm II}\cdots)\equiv[A^c(\cdots s_+^{\rm II}\cdots)-A^c(\cdots s_-^{\rm II}\cdots)]/2i$$

is the discontinuity across the unphysical cut. This last equation establishes the result to be proved, namely, $\text{disc}_uA^c\neq0$, since the last term on the right is nonzero.

In the preceding section, unphysical amplitudes were defined as the factors of the residue at a resonance pole. In expression (27) the three-body integral has been reduced to a form which resembles a sum over states with a two-body intermediate state. However, the intermediate state is unphysical, being formed of a stable particle of mass m_1 and an unstable particle of mass $m_r=s_r^{1/2}$. The phase space is complex.

Equation (28) constitutes an integral equation for disc_uA^c. This integral equation may be solved by the ansatz

$$\text{disc}_uA^c(s_f^{\rm II},s^{\rm II},h_{fi+},t_{fi\pm},s_i^{\rm I})$$
$$=\int\sum_\mu A_{u,\mu}(s_f^{\rm II},s_+^{\rm II},h_{f0+},t_{f0+})X^\mu(s,h_{0i},t_{0i},s_i)d\varphi_u,\quad(29)$$

where $X^\mu(s,h_{0i},t_{0i},s_i)\equiv X^\mu(\hat{\Omega}_{01},s,s_1,\hat{\Omega}_{i1},\hat{n}_{i1})$ is a quantity which depends on the same variables as the unphysical amplitude A_u for the process $1+r\leftrightarrow1+2+3$. For if X is chosen to be the solution of

$$X^\mu(s,h_{fi},t_{fi},s_i)-A_u{}^\mu(s^{\rm I},h_{fi-},t_{fi-},s_i^{\rm I})$$
$$=\int X^\mu(s,h_{f0},t_{f0},s_0)A^c(s_0^{\rm II},s^{\rm I},h_{0i-},t_{0i-},s_i^{\rm I})d\varphi$$
$$+\sum_d X(s,h_{fi},t_{fi},s_i^d),\quad(30)$$

we find, upon substitution of Eq. (29) into Eq. (28) that this latter is satisfied identically. We have made some progress in exchanging the integral equation (28) for the integral equation (30) since the unknown X has the variables of a five-legged diagram, instead of six. Furthermore, upon comparing Eqs. (30) and (23) we find that Eq. (30) is identical with the equation obtained by equating the pole term at $s_{f1}{}^{\rm II}=s_r$ on left- and right-hand sides of Eq. (23), taking the minus subscript on t_{fi} and the contour C_- there. [The last term on the right-hand side of Eq. (23), $\sum_d A^c(s_f{}^d\cdots)$, has no pole at $s_{f1}=s_r$. This quantity is defined just below Eq. (21) and its continuation is represented by replacing subscripts + and − by superscripts II and I. There is no s_{f1} pole in $A^c(s_{f1}{}^{\rm I})$, so according to its definition the only pole of $\sum_d A^c(s_f{}^d\cdots)$ is in

$$A^c(s_{f1}{}^{\rm II},s_{f2}{}^{\rm II},s_{f3}{}^{\rm II}\cdots)-A^c(s_{f1}{}^{\rm II},s_{f2}{}^{\rm I},s_{f3}{}^{\rm I}\cdots).$$

But this difference has no pole in s_{f1} because such a pole term has no s_{f2} or s_{f3} cut.] Consequently, we conclude that

$$X^\mu(s,h_{fi},t_{fi},s_i)=A_u{}^\mu(s_-^{\rm II},h_{fi+},t_{fi-},s_i^{\rm I}),$$

and Eq. (29) takes the form

$$\text{disc}_uA^c(s_f^{\rm II},s^{\rm II},h_{fi+},t_{fi\pm},s^{\rm I})$$
$$=\int\sum_\mu A_{u,\mu}(s_f^{\rm II},s_+^{\rm II},h_{f0+},t_{f0+})$$
$$\times A_u{}^\mu(s_-^{\rm II},h_{0i+},t_{0i-},s_i^{\rm I})d\varphi_u.\quad(31)$$

This equation gives the main result of this section. It states that corresponding to a two-body resonance pole m_r near the physical region, there is a cut in a three-body amplitude near the physical region beginning at a normal threshold branch point $s=(m_r+m_1)^2$. Furthermore, the discontinuity across the cut is expressible in the form of a unitarity condition with a two-body intermediate state. However, the phase space is specified by a complex mass m_r and the amplitudes do not correspond to the scattering of stable particles.

As a simple application of these results we can easily obtain a two-parameter formula for the effect on an elastic S-wave two-body amplitude of the onset of unstable particle production, i.e., a resonance between a pair of particles in a competing three-particle final state. As we have seen, there is a square-root-normal-threshold branch point at c.m. energy $E=(m+m_r)$, if m and m_r are the masses of the stable and unstable final particles. (The preceding results are easily generalized from a pair theory to coupled two- and three-body channels. The branch point is then found across the three-body cut in both the elastic and production amplitudes.) Let $E_0 \equiv m+\mathrm{Re}m_r$ and assume that $\mathrm{Im}m_r \equiv -\Gamma/2<0$ is very small. In the neighborhood of E_0 we represent the S-matrix element by

$$S=a+b[E-(m+m_r)]^{1/2}, \tag{32}$$

in which a and b are complex constants, on the assumption that in the neighborhood of the branch point the variation of the matrix element is due to the singularity. If the inelastic cross section is small, as it will be if E_0 does not lie too far above the three-body threshold, then $a=\exp(2i\delta)$, where δ is the real phase shift at $E=E_0$. Since we require $|S|^2\approx 1$ for $E<E_0$ and $|S|^2 \lesssim 1$ for $E>E_0$, we must choose $b=-\eta\exp(2i\delta)$, where η is a real positive constant so that

$$S=e^{2i\delta}-\eta e^{2i\delta}(E-E_0+i\Gamma/2)^{1/2},$$

$$S=e^{2i\delta}-\eta e^{2i\delta}(X+iY),$$

$$X\equiv\frac{1}{\sqrt{2}}\{[(E-E_0)^2+\Gamma^2/4]^{1/2}+(E-E_0)\}^{1/2}, \tag{33}$$

$$Y\equiv\frac{1}{\sqrt{2}}\{[(E-E_0)^2+\Gamma^2/4]^{1/2}-(E-E_0)\}^{1/2}.$$

The S-wave cross section is given by

$$\sigma_0=\frac{4\pi}{k^2}\left|\frac{S-1}{2i}\right|^2=\frac{4\pi}{k^2}[\sin^2\delta-\eta\sin\delta(X\sin\delta+Y\cos\delta)]. \tag{34}$$

Results equivalent to Eq. (34) have been obtained by Nauenberg and Pais[2] and by Baz'.[3]

IV. EXTENSION OF LANDAU EQUATIONS AND CUTKOSKY FORMULAS TO UNSTABLE PARTICLES

The discussion of the preceding sections suffers from two defects, one of them serious. The serious defect is that assumptions were made about the analyticity properties of production amplitudes. These assumptions are not known to be true, and are quite possibly false, although, as we shall see, only the details of the results would be invalidated. The other defect is that the discussion was limited to specific reactions, and it is clear that there are corresponding results which have general validity.

In this section we suggest how these defects may be remedied. We obtain the equations of the surface of singularity of any amplitude and the generalized unitarity relations without making unjustified assumptions. On the other hand, the precision of the previous results are not obtained here. The equations of the surfaces of singularity do not indicate on which Riemann sheet of the amplitude the singularities lie, nor is it known, in general, on which sheet of their Riemann surfaces the amplitudes appearing in the generalized unitarity relation are to be evaluated.

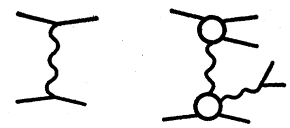


FIG. 8. Unstable particles are represented by wavy lines.

The methods developed by Polkinghorne[11] for stable particles apply equally well to unstable particles and, consequently, the results he has obtained apply indiscriminately to stable and unstable particles. Polkinghorne has shown that—given the masses of the stable interacting particles—in consequence of analyticity and crossing, and particularly of unitarity conditions, scattering amplitudes have, in addition to other possible singularities, the set of singularities specified by the Landau equations. Polkinghorne's method consists in demonstrating that the set of singularities given by the Landau equations, but no subset, is closed under the unitarity integration

$$\mathrm{disc}A(p_i,p_j)=\sum\int A^{\mathrm{II}}(p_i,q_j)A^{\mathrm{I}}(q_j,p_k)$$
$$\times\delta^4(\sum p-\sum q)\delta(q_j^2-m_j^2)d^4q_j. \tag{35}$$

The set is "closed" under the unitarity integration means that if A^{I} and A^{II} have singularities specified by the Landau equations, then so will disc A. Let us suppose that A^{I} and A^{II} are singular on the Landau curves corresponding to stable particles and also contain resonance poles at complex masses m_r. Then A^{I} and A^{II} will, in general, be singular on the Landau curves given by perturbation theory and also on the Landau curves corresponding to diagrams in which groups of external lines are joined to the graph through only one internal line, corresponding to a complex mass m_r, as shown in Fig. 8. The argument remains valid, and disc A will, in general, be singular on the Landau curves corresponding to the Feynman diagrams obtained by joining A^{I} and A^{II} as shown in Fig. 9. This argument does not show that the contour is actually pinched so that the singularity is, indeed, present in disc A, although this is expected to occur on some sheet. However, it does show that a consistent set of singularities is obtained by assuming that A is singular on the Landau curves corresponding to all possible Feynman diagrams in which lines represent stable or unstable particles indiscriminantly. The Landau equa-

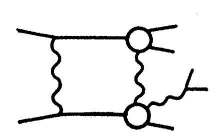


FIG. 9. The Feynman graph formed by joining the two graphs of Fig. 8.

tions remain unaltered, and yield both first- and second-type[13] singularities as solutions, but they are extended to include all masses, corresponding to stable and unstable particles. As the number of resonance poles, or more generally, any poles on unphysical sheets, may well be infinite, as in the case in potential scattering for each partial wave, this is a considerable extension.

We should like to call the poles "dynamical" singularities, since the problem of locating them is the dynamical problem of finding the masses, in contradistinction to the singularities specified by the Landau equations, once the masses are given, and which may be called "kinematical." What other dynamical singularities may be present is not known. If there are any —and we may hope that there are none—they presumably give rise to further sets of kinematical singularities.

The result of the preceding section suggests that the discontinuities across the first type singularities given by the extended Landau equations are specified by the corresponding extension of the generalized unitarity formulas first derived by Cutkosky[8] in perturbation theory. Polkinghorne[11] has outlined a program for deriving generalized unitarity, in the case of real masses, from the physical unitarity relation. The results of this analysis are applicable to the present case in which the masses are complex. Consider the physical unitarity relation

$$\operatorname{disc}_s A(s_f,\Omega_f,s,s_i,\Omega_i) = \sum \int A^{\mathrm{I}}(s_f,\Omega_f,s,s_0,\Omega_0)A^{\mathrm{II}}(s_0,\Omega_0,s,s_i,\Omega_i)ds_0 d\Omega_0, \quad (36)$$

in which s represents the c.m. energy variable and s_p, Ω_p represent partial energy and angular variables. Let us suppose that there is a resonance pole in one of the partial energies, say s_i. By equating residues on left- and right-hand sides, we obtain a unitarity condition for the unphysical amplitude.

These considerations raise various questions and possibilities. If a calculational scheme is possible which makes use only of quantities defined on the physical sheet and its boundaries, it is important to ascertain whether or not any singularities due to unstable particles extend into the physical sheet. On the other hand, one would like to be able to construct amplitudes that possess the singularities which lie near the physical points, even though off the physical sheet. The unphysical amplitudes and their crossing and pseudo-unitarity conditions appear to be likely tools for such a construction. As a practical matter, calculations are, in fact, performed at present[14] in which unstable particles are exchanged and scattered. These calculations may now be systemized and related to formal S-matrix theory where they find a natural place. It may even be worthwhile to reformulate the dynamical problem in a way that makes no *a priori* distinction between stable and unstable particles, rather than as at present, stating it in terms of stable particles and introducing the unstable particles as auxiliary devices. This would be more logical and, in practice, allow greater flexibility, since it is not known until after the detailed dynamical problem has been solved whether a given particle, teetering on the edge of stability, is actually stable or not. The symmetry that exists between stable and unstable particles appears to be a fundamental feature of S-matrix theory, and we may, consequently, expect that the theory should be stated in a way that makes this symmetry apparent.

ACKNOWLEDGMENTS

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[13] D. B. Fairlie, P. V. Landshoff, J. Nuttall, and J. C. Polkinghorne, J. Math. Phys. **3**, 594 (1962).

[14] See, for example, F. Zachariasen and C. Zemach, Phys. Rev. **128**, 849 (1962).