Three-Body Effect in Nuclear Matter to All Orders of Perturbation*†

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It is shown that there are three-body clusters in all orders of the Goldstone expansion for the binding energy of nuclear matter and that they converge extremely poorly. A Γ matrix is defined which, on expansion, is shown to generate this infinite sequence of three-nucleon clusters. It is shown that the Γ matrix can be evaluated in terms of a three-body correlation function for which a differential equation is derived. Solving this equation leads to the evaluation of Γ. This demonstrates that a finite sum does exist for these cluster diagrams, and gives a method for finding it. Finally, it suggests that for strong short-range potentials it is better to expand the binding energy in powers of the density rather than in powers of the interaction.

1. INTRODUCTION

THE Brueckner-Goldstone many-body theory gives rise to an expansion for the binding energy of nuclear matter in powers of the reaction matrix G and the single-particle potential U.[1,2] The terms in this expansion can be represented by a series of Goldstone diagrams,[1] of which the lowest order ones give the first approximation to the binding energy. The effect of the higher order diagrams was considered to be small, and methods were suggested to incorporate their effect into the first-order terms by making a suitable choice of the single-particle energies. The best work in this direction done by Bethe *et al.*[3] absorbs certain third-order diagrams (there being no second-order terms in the formalism) into the first-order ones by an appropriate choice of U.

It was then shown by the author[4] that the rest of the third-order diagrams, which are genuine three-body clusters, are large and comparable to the ones considered by Bethe *et al.* and that these could be treated in a similar manner.

This then prompts us to look at the higher order diagrams in order to estimate the rate of convergence of the series. Upon doing so, we find that there are three-body cluster terms even among diagrams of the fourth or higher order in the reaction matrix. These are the diagrams involving only three hole lines, and are larger than the other higher order terms. A simple estimate of their magnitude shows that they are comparable to the third-order diagrams. A fourth-order diagram, as compared to the third-order one, has one extra G/e matrix to be integrated over the one additional intermediate state involved, where e is the energy denominator. With a potential containing a hard core of radius c, G/e has a maximum proportional to c^3 for momenta of the order of $1/c$, and then falls off.[3] Now, $1/c$ is about 2.5F^{-1} compared to a fermi momentum k_F equal to 1.5F^{-1}. When G/e is integrated over particle states, the result is about $(1/c^3)c^3$, as compared to about $k_F{}^3c^3$ on integration over hole states. Thus, when the additional intermediate state in the fourth-order diagram is a particle state, this diagram is of roughly the same magnitude as the third-order term. Similarly, higher order diagrams constructed by adding intermediate particle states only, will also have the same magnitude. This results in an infinite series of diagrams, all with only three hole lines, with an absolute rate of convergence of the order of unity. Therefore, one is not justified in assuming that the fourth- and higher-order diagrams do not contribute much to the binding energy. Owing to the hard-core nature of the internucleon potential, we have three-body clusters of comparable magnitude, in all orders of the reaction matrix.

Thus, it becomes necessary to show that a finite sum exists for this series and to find a method of estimating the same. That is the purpose of this work. It is shown that on making certain approximations similar to the ones used in the reference spectrum method of Bethe *et al.*[3] these three-body clusters of all orders combine to form a "three-body reaction matrix," Γ. This matrix is finite and can be written as an integral of a three-body correlation function. A Schrödinger-type differential equation is derived for this function. A solution of this equation would lead to the evaluation of the Γ matrix, in terms of which we can find the three-nucleon contribution to the binding energy.

Whereas it has been shown[4] that all third-order diagrams in the Goldstone series can be treated as inserts in the single-particle energies, the three-nucleon clusters of higher orders cannot be so absorbed, and their sum will have to be evaluated explicitly.

2. THE Γ MATRIX

Let v_{ij} be the two-particle potential between particles i and j. Consider a three-particle operator ω capable of acting on products of three single-particle plane wave states, and given by

$$\langle b_1b_2b_3|\omega|a_1a_2a_3\rangle = v_{12}\delta_{b_3a_3} + v_{23}\delta_{b_1a_1} + v_{31}\delta_{b_2a_2}. \quad (1)$$

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[1] J. Goldstone, Proc. Roy. Soc. (London) **A239**, 267 (1957).
[2] K. A. Brueckner and C. A. Levinson, Phys. Rev. **97**, 1344 (1955); K. A. Brueckner, *ibid.* **97**, 1353 (1955).
[3] H. A. Bethe, B. H. Brandow, and A. G. Petschek, Phys. Rev. **129**, 225 (1963).
[4] R. Rajaraman, Phys. Rev. **129**, 265 (1963).

Let

$$\langle b_1b_2b_3|\Gamma|a_1a_2a_3\rangle=\langle b_1b_2b_3|\omega|a_1a_2a_3\rangle-\sum_{c_1c_2c_3}\frac{\langle b_1b_2b_3|\omega|c_1c_2c_3\rangle Q\langle c_1c_2c_3|\Gamma|a_1a_2a_3\rangle}{e(c_1c_2c_3;\,a_1a_2a_3)}, \tag{2}$$

where the energy denominator e is a function of the states in parentheses and will be defined later on, and Q is an operator which requires that at least two of the three intermediate states should be above the Fermi sea. On expansion

$$\langle b_1b_2b_3|\Gamma|a_1a_2a_3\rangle=\langle b_1b_2b_3|\omega|a_1a_2a_3\rangle-\sum_{c_1c_2c_3}\frac{\langle b_1b_2b_3|\omega|c_1c_2c_3\rangle Q\langle c_1c_2c_3|\omega|a_1a_2a_3\rangle}{e(c_1c_2c_3;\,a_1a_2a_3)}$$

$$+\sum_{c_1c_2c_3}\sum_{d_1d_2d_3}\frac{\langle b_1b_2b_3|\omega|d_1d_2d_3\rangle Q\langle d_1d_2d_3|\omega|c_1c_2c_3\rangle Q\langle c_1c_2c_3|\omega|a_1a_2a_3\rangle}{e(d_1d_2d_3;\,a_1a_2a_3)\times e(c_1c_2c_3;\,a_1a_2a_3)}. \tag{3}$$

The terms in the above expansion can be expressed diagrammatically, bearing in mind Eq. (1) for the operator ω. The dashed horizontal lines represent the potential v and the vertical lines the wave functions. A typical diagram will be as in Fig. 1. Any diagram drawn with an arbitrary combination of the v interactions will belong to the expansion for Γ, with the stipulation that at least two of the intermediate states be above the sea. It can now be seen that all the diagrams in the Goldstone series with three hole lines are contained in our expansion. For example, consider the two diagrams shown in Fig. 2. Figure 2(a) is a typical Goldstone diagram and Fig. 2(b) is the equivalent of that diagram in our Γ matrix. The directions of the arrows in Fig. 2(b) have no significance other than to indicate that the upgoing lines are states above the Fermi sea and the down-going lines are states below the sea. It is, of course, more elegant and convenient to draw the diagram as in the Goldstone theory, but Fig. 2(b) helps us to see that the same diagram satisfies the requirements for belonging to the expansion for $\langle lmn|\Gamma|lmn\rangle$. In this manner, it can be seen that all direct diagrams in all orders of the reaction matrix G involving the three hole states l, m, and n belong to $\langle lmn|\Gamma|lmn\rangle$. The exchange diagrams belong to the corresponding exchange matrix elements of Γ. Thus, the diagram in Fig. 3 belongs to $\langle lnm|\Gamma|lmn\rangle$. Thus, one would expect the sum of all the three-body clusters to all orders of perturbation, i.e., all the diagrams involving three internal hole lines, to be $\langle lmn|\Gamma|lmn\rangle$ plus its exchanges, summed over the holes l, m, and n. Of course, from spin and isospin considerations, the different exchange diagrams have different signs and relative weights. For a pure central potential, Goldstone diagrams with two closed loops have a relative weight of $-\frac{1}{4}$, and ones with a single closed loop have a relative weight of $+\frac{1}{16}$.[4] These correspond to Γ matrix elements with one or two pairs of states exchanged, respectively. Thus, the sum of all the three-body clusters should correspond to

$$\tfrac{1}{6}\sum_{lmn,<k_F}\langle lmn|\Gamma|lmn\rangle$$

$$-\tfrac{1}{4}\{\langle lmn|\Gamma|lnm\rangle+\langle lmn|\Gamma|nml\rangle+\langle lmn|\Gamma|mln\rangle\}$$

$$+\tfrac{1}{16}\{\langle lmn|\Gamma|mnl\rangle+\langle lmn|\Gamma|nlm\rangle\}. \tag{4}$$

Therefore, if a procedure could be found for evaluating the Γ matrix, we might expect to have essentially summed all the three-nucleon contributions to the binding energy. However, two major difficulties have to be cleared up before the above statement becomes true, and herein lie some of the weaknesses of the method. These difficulties are:

(1) In order for the corresponding diagrams from the Goldstone series and the Γ matrix expansion to be equal, we have to ensure that the energy denominators in the two cases are equal for all the intermediate states. The problem is then to pick a suitable function for $e(c_1c_2c_3;\,a_1a_2a_3)$ in Eq. (2) such that it generates the appropriate energy denominators.

(2) Even though all the three-body cluster terms in the Goldstone series are present in the Γ matrix, the latter contains some spurious diagrams, which do not belong to the former set, and have to be corrected for.

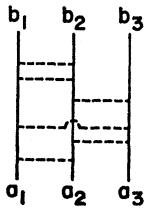

FIG. 1. A typical term in the Γ matrix expansion.

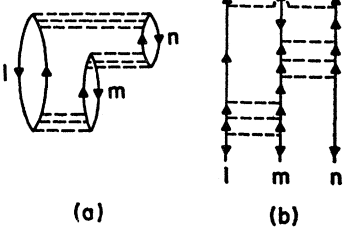

FIG. 2. Corresponding diagrams in the Goldstone and the Γ matrix expansions.

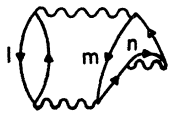

FIG. 3. An "exchange" diagram belonging to $\langle lmn|\Gamma|lnm\rangle$.

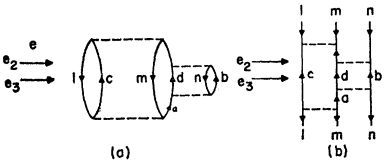

FIG. 4. Energy denominators in the corresponding diagrams.

We will now discuss the first of these problems. The energy denominator required by the Goldstone theory is H_0-E_0, where H_0 is the unperturbed Hamiltonian and E_0, the ground-state energy of the "vacuum." This is equal to the energy of the particle states present minus the energy of the hole states. This quantity clearly depends on the single-particle energies assigned to these states. If we use the recent choice of the reference spectrum,[3] then, for states above the Fermi sea,

$$E(b)=(\hbar^2/2Mm^*)(b^2+2\Delta k_F^2), \tag{5}$$

where

$\hbar b$, the momentum of the state, is $>\hbar k_F$,

$m^*=0.88$,

$\Delta=0.75$,

$\hbar k_F=$ Fermi momentum,

and

$M=$ mass of the nucleon.

For states below the sea, we have

$$E(l)=(\hbar^2/2Mm^*)l^2. \tag{6}$$

Now, a Goldstone diagram with three hole lines, will either have two nucleon loops or three, at any intermediate state. Thus, in Fig. 4(a), the energy denominator e_3 is given by

$$e_3=(\hbar^2/2Mm^*)(c^2+d^2+b^2-l^2-n^2-m^2+6\Delta k_F^2), \tag{7}$$

while e_2 is given by

$$\begin{aligned} e^2&=(\hbar^2/2Mm^*)(c^2+a^2-l^2-m^2+4\Delta k_F^2)\\ &=(\hbar^2/2Mm^*)(c^2+a^2+n^2-l^2-m^2-n^2+4\Delta k_F^2). \end{aligned} \tag{8}$$

If the diagram in Fig. 4(b) is to be equal to the one in Fig. 4(a), then the corresponding energy denominators in the two should be equal. Remembering that the energy denominators in Fig. 4(b) are generated as in Eq. (2), we find that the function

$$\begin{aligned} &e(c_1c_2c_3;a_1a_2a_3)\\ &\quad=(\hbar^2/2Mm^*)(c_1^2+c_2^2+c_3^3-a_1^2-a_2^2-a_3^2+r\Delta k_F^2) \end{aligned}$$

will give the correct denominator for e_3 if $r=6$ and for e_2 if $r=4$. However, one must have a unique value of e as a function of the intermediate state momenta, in order that it may be used in Eq. (2) for the definition of Γ. Thus, a definite value of r has to be chosen. An appropriate value clearly lies between 4 and 6, and has to be a suitably averaged number. This average should incorporate the relative effects of the two-particle-hole-pair intermediate states and the three-pair states. This is very difficult to do exactly. We will use for simplicity the value $r=6$. This is not so unreasonable as we will be subtracting later on the effect of the leading terms which involve two-particle-hole pairs, leaving behind predominantly three-loop intermediate states. With greater effort, perhaps, a more accurate choice could be made.

3. THE THREE-BODY REFERENCE EQUATION

With the above definition for the energy denominator, we then have

$$\langle b_1b_2b_3|\Gamma|lmn\rangle=\langle b_1b_2b_3|\omega|lmn\rangle-\sum_{c_1c_2c_3}\frac{\langle b_1b_2b_3|\omega|c_1c_2c_3\rangle Q\langle c_1c_2c_3|\Gamma|lmn\rangle}{(\hbar^2/2Mm^*)(c_1^2+c_2^2+c_3^2-l^2-m^2-n^2+6\Delta k_F^2)}. \tag{9}$$

The operator Q restricts the summation over c_1, c_2, and c_3 by requiring that at least two of them be above the Fermi sea. However, if we drop this restriction, and include in our summation momenta below the sea as well, the error made is not large. This is not only because the phase space occupied by hole states is small compared to that occupied by particle states, but also because, owing to our choice of the energy spectrum, the energy denominators remain fairly large even for intermediate state momenta below k_F. It is to be noted that we are only extending the range of momenta of the intermediate states, and that their energies are still given by Eq. (5), as if they were particle states. Therefore, the Q operator, which has a complicated form in coordinate space, can be dropped as a reasonable approximation. This is analogous to the procedure used in the reference spectrum method.[3]

Putting $Q=1$ in Eq. (9) and summing both sides over the complete set of states $b_1b_2b_3$, we get

$$\Gamma|lmn\rangle=\omega|lmn\rangle-\sum_{c_1c_2c_3}\omega|c_1c_2c_3\rangle\frac{1}{e}\langle c_1c_2c_3|\Gamma|lmn\rangle, \tag{10}$$

where e stands for the energy denominator.

Let us define a three-body wave function ψ_0 given by

$$|\psi_0\rangle=\left(1-\frac{1}{e}\omega+\frac{1}{e}\frac{1}{e}\omega\omega-\cdots\right)|\Phi_0\rangle, \tag{11}$$

where

$$|\Phi_0\rangle=|lmn\rangle=\exp[i(\mathbf{l}\cdot\mathbf{r}_1+\mathbf{m}\cdot\mathbf{r}_2+\mathbf{n}\cdot\mathbf{r}_3)].$$

Using (10),

$$|\psi_0\rangle=\left(1-\frac{1}{e}\Gamma\right)\Big|\Phi_0\Big\rangle;$$

therefore,

$$\frac{\Gamma}{e}\Big|\Phi_0\Big\rangle=\Phi_0-\psi_0=\chi \text{ (say)}. \tag{12}$$

Then

$$\langle lmn|\Gamma|lmn\rangle=\frac{\hbar^2}{2Mm^*}(l^2+m^2+n^2-l^2-m^2-n^2+6\Delta k_F^2)\times\left\langle lmn\Big|\frac{\Gamma}{e}\Big|lmn\right\rangle$$

$$=\frac{\hbar^2}{Mm^*}3\Delta k_F^2\left\langle\Phi_0\Big|\frac{\Gamma}{e}\Big|\Phi_0\right\rangle$$

$$=\frac{\hbar^2}{Mm^*}3\Delta k_F^2\langle\Phi_0|\chi\rangle. \tag{13}$$

Therefore, once we know the "three-body correlation function" χ, we can find the Γ matrix element using Eq. (13). In order to find an equation for χ, we note from Eq. (11) that

$$\chi=\Phi_0-\psi_0=\left(\frac{1}{e}(-\omega)-\frac{1}{e}\frac{1}{e}(-\omega)(-\omega)+\cdots\right)\Big|\Phi_0\Big\rangle$$

$$=\frac{1}{e}(-\omega)\Big|\psi_0\Big\rangle.$$

Therefore,

$$e|\chi\rangle=\omega|\psi_0\rangle. \tag{14}$$

If the above equation were to be written in coordinate space, the operator e would take the form

$$e=\frac{\hbar^2}{2Mm^*}(-\nabla_1^2-\nabla_2^2-\nabla_3^2-l^2-m^2-n^2+6\Delta k_F^2),$$

where the ∇^2 operators are with respect to the coordinates of the three bodies. This gives us

$$-\frac{\hbar^2}{2Mm^*}(\nabla_1^2+\nabla_2^2+\nabla_3^2+l^2+m^2+n^2-6\Delta k_F^2)\chi=\omega(\Phi_0-\chi)$$

or

$$(\nabla_1^2+\nabla_2^2+\nabla_3^2-2\gamma^2)\chi=\frac{-2Mm^*}{\hbar^2}(v_{12}+v_{23}+v_{13})(\Phi_0-\chi), \tag{15}$$

where $2\gamma^2=6\Delta k_F^2-l^2-m^2-n^2$.

This equation, which may be called the three-body reference equation, has to be solved for χ under the appropriate boundary conditions. The function χ has to vanish as any of the relative coordinates r_{12}, r_{23}, or r_{13} goes to infinity. If the potential v_{ij} contains a hard core, χ is equal to Φ_0 inside this core and is continuous across the core surface. A three-body Schrödinger-type equation like the one we have is hard to solve even approximately for the simplest potentials. Numerical methods will have to be employed with their complexity depending on the nature of the potential used.

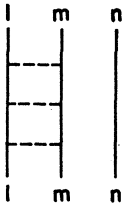

FIG. 5. A spurious diagram of the first kind.

Once χ is found, however, the evaluation of $\langle lmn|\Gamma|lmn\rangle$ is relatively easy with the help of Eq. (13). The exchange matrix elements, which are off diagonal, may be found in a manner analogous to the one described above for the diagonal element. Thus,

$$\langle nlm|\Gamma|lmn\rangle=\frac{\hbar^2}{Mm^*}3\Delta k_F^2\langle\Phi'|\chi\rangle,$$

where

$$|\Phi'\rangle=|nlm\rangle.$$

We may then employ Eq. (4) to add these terms and get the total three-nucleon contribution to the binding energy, provided the spurious diagrams mentioned before are corrected for.

4. SPURIOUS DIAGRAMS

Among the diagrams generated from the expansion in Eq. (3) there are some that do not belong to the class of three-nucleon Goldstone diagrams. The most important of these are the terms where one of the three particles does not interact at all. Thus, Fig. 5, which will arise in the Γ matrix expansion, is not a three-nucleon cluster, but belongs instead to the first-order term. The sum of all such diagrams is clearly the sum of the two-body G matrices between all pairs of nucleons, evaluated consistent with all our approximations.

Then there are spurious diagrams of a second type, arising because of the neglect of the Q operator. As we argued earlier, this approximation of letting all intermediate states occupy all of momentum space, does

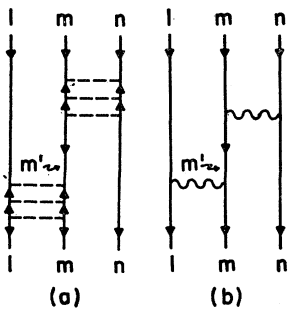

FIG. 6. Spurious diagrams of the second kind.

not change the value of most diagrams substantially. However, it introduces some diagrams in the second order which would otherwise not exist and certainly don't belong to the Goldstone series. A typical diagram of this class is shown in Fig. 6(a) and the sum of some such ladders is shown in Fig. 6(b), where the wiggly lines are G matrices. The intermediate state m' above has the same momentum as m, but has the reference energy given by Eq. (5) rather than Eq. (6). Figure 6(b) is essentially equal to

$$\langle mn|G|mn\rangle \frac{1}{(\hbar^2/Mm^*)\Delta k_F{}^2} \langle lm|G|lm\rangle .$$

Therefore, if we directly calculate the two-body reaction matrix G, then we can evaluate these spurious diagrams of both kinds and subtract them from the Γ matrix element to get the pure three-nucleon contribution.[5]

However, it would be more desirable to do the subtraction in the wave function χ rather than in the final result for the energy. This subtraction is relatively easy for diagrams of the type in Fig. 5. One merely subtracts from the χ resulting from Eq. (15) the two-body correlation functions of the three pairs of particles involved. These two-body correlation functions can be found[4] from a two-body reference equation similar to Eq. (15). The corrected χ function, when used in Eq. (13), will automatically exclude the contribution of spurious diagrams of this type. It is not quite as easy to subtract the effect of the diagrams of the type shown in Fig. 6 directly from the χ function. These may be considered to correspond to one of the particles being in the range of interaction of the other two, which do not interact with each other. A corresponding correlation function will have to be found in coordinate space, and will have to be subtracted from χ. If both these corrections could be made directly in χ, then the resulting function could be used to get the three-nucleon contribution.

5. CONCLUSION

An attempt was made by the author[6] to evaluate the χ function and the three-nucleon contribution to the energy using a pure hard core potential. Even for such a simple potential, the differential equation (15) could be solved only very approximately. The main problem, typical of three-body interactions, arises from the fact that, while the boundary conditions and the potentials depend on the relative coordinates, the differential operator does not have a simple form in these coordinates. A partly numerical and partly analytical method was used to give a result of about 10 MeV per particle for the hard core part of the three-nucleon contribution to the binding energy. The 10-MeV energy was the difference between -30 MeV for the Γ matrix and -40 MeV for the spurious terms. This result, while not unreasonable, does not have much significance, owing to the various approximations that were used in its evaluation.

However, with more effort, and moderate use of computers, a much better solution can be found for the χ function, not only for the hard core, but for more realistic potentials as well. This would lead to a meaningful value for the three-body term in the binding energy. This term provides the largest correction to the binding energy after the first order term, which is just the two-body reaction matrix. From Eq. (4), it is clear that the three-body contribution per particle is going to be proportional to ρ^2, where ρ is the density. This is to be expected from simple physical considerations.

Whether the above method is convenient for computation or not, it clearly shows that a finite closed sum exists for these three-nucleon diagrams. Hitherto, not much attention was given to higher order diagrams, let alone to the convergence of such terms. We have shown that there are terms in all orders of the Goldstone expansion, which are large and comparable. Thus, unless a closed sum is found for these, it is not of much value to use lower order terms as an approximation for the binding energy. We have seen that these clusters can be summed in coordinate space, yielding a result proportional to ρ^2. The four-nucleon cluster terms, which would again appear in all orders of Goldstone theory would give a sum proportional to ρ^3 and so on. This shows that owing to the strong- and short-range nature of the internucleon potential, the binding energy can be expanded more naturally in powers of the density,[7] than in powers of the interaction. This expansion in density should lead to a series which, although the successive terms become harder and harder to evaluate, at least gives better convergence and, therefore, more justification for stopping after the first few terms.

ACKNOWLEDGMENT

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[5] For detailed discussions of two-body reaction matrices, see Refs. 3 and 4.

[6] R. Rajaraman, Ph.D. thesis, Cornell University, 1963 (unpublished).

[7] The behavior of the convergence of the expansion as a function of the density is discussed by N. M. Hugenholtz, Physica **23**, 533 (1957).