Electromagnetic Correction Effects on the $\pi^+ \to \pi^0 e^+ \nu$ Decay*

NGEE-PONG CHANG
Department of Physics, Columbia University, New York, New York

(Received 25 March 1963)

Recently experiments have been done on the rare decay of the π^+ into the $\pi^0 e^+ \nu$ mode as a third and independent test of the conserved vector current hypothesis. This note reports briefly on a simple perturbation calculation of the radiative corrections. With a cutoff of around a nuclear mass, the radiative correction shortens the lifetime by $\sim -1.1\%$.

I. INTRODUCTION

THE conserved vector current hypothesis[1] has not, until very recently, been well established at all, from an experimental point of view. The $\sim 2\%$ discrepancy[2] between the coupling constant for μ decay, G_μ, and the vector coupling constant for nuclear β decay, G_ν, remains, even today, as a stumbling block to the experimental confirmation of the hypothesis. One generally hopes nowadays for an explanation of this discrepancy to come from a deeper physical understanding of the action of weak interactions, that is to say, from the interactions of the intermediate vector bosons.[3] Such an explanation,[4] however, must necessarily depend on the mass of the boson, an experimental parameter that is yet unknown. Therefore, empirically speaking, the intermediate boson explanation cannot yet be considered as complete.

Meanwhile, the conserved vector current idea has been thoroughly and well verified in the alternate test involving the so-called "weak magnetic" effect on nuclear β-transitions as proposed by Gell-Mann.[5] The latest experimental data,[6] available only very recently, show, in the test case of the B^{12}, C^{12}, N^{12} isotopic triplet, a very close agreement with the prediction of the conserved vector current hypothesis. This experiment, however, does not cast any light on the discrepancy in the coupling constants.

Recently, experiments[7–9] have been carried out on yet another test of the conserved vector current idea, viz., the rare β decay of π^+ into the $\pi^0 e^+ \nu$ mode. On account of the extreme rarity of the decay, present data from such experiments are far from accurate. Nevertheless, the data serve some purpose in corroborating roughly the validity of the conserved current idea in the realm of particle physics, as opposed to the realm of the physics of complex nuclei. With time and effort, future experiments will yield more accurate data, and will thereby help establish a solid picture of the empirical validity of the conserved vector current hypothesis.

For such accurate experiments, of course, the small electromagnetic effect corrections begin to be important. In this brief note, we report on a simple calculation of the radiative corrections to the rare β decay of the π^+.

II. $\pi^+ \to \pi^0 e^+ \nu$

We begin with a few definitions.

The $\pi^+ \to \pi^0 e^+ \nu$ decay matrix element is, in its complete generality, given by (see Fig. 1)

$$(2\pi)^4 i\delta^{(4)}(p'-p-k-k')M/(4p_0p_0')^{1/2}, \tag{1}$$

where M, in turn, is given by

$$M = \frac{1}{2i}[\bar{U}(k')\gamma_\rho(1+\gamma_5)V(k)] \times [(p+p')_\rho W_+ + (p-p')_\rho W_-]. \tag{2}$$

Equation (2) is consistent with the Lorentz invariance of the theory only if $W_\pm$ are invariant functions of the scalars $S \equiv -(p'-k)^2$ and $t \equiv -(p'-p)^2$. Equation (2) also conforms to the two-component neutrino theory.

In the absence of electromagnetic interactions, and to lowest order in weak interactions, the functions $W_\pm$ depend only on t. This is so since, in that case, the $(e\nu)$ interact as a unit in weak interactions. This "locality" of the $(e\nu)$ current in the weak-interaction Lagrangian has been amply substantiated in low-energy weak-interaction physics.

Furthermore, on general grounds (this is the usual dispersion argument) the functions $W_\pm$ are analytic in a cut complex t plane with the cut running along the positive real axis. The cut begins at $t = (m_{\pi^+} + m_{\pi^0})^2$. This is almost as much as one can say in complete generality without any detailed analysis. A description

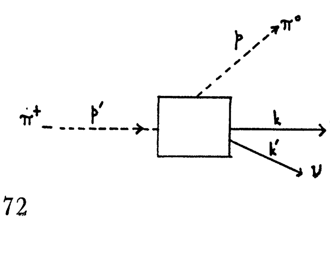

FIG. 1. $\pi^+ \to \pi^0 e^+ \nu$ decay.

* This research supported in part by the U. S. Atomic Energy Commission.

[1] R. P. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958).
[2] For an up-to-date experimental survey see J. W. Butler and R. Bondelid, Phys. Rev. **121**, 1770 (1961).
[3] T. D. Lee and C. N. Yang, Phys. Rev. **119**, 1410 (1960).
[4] T. D. Lee, Phys. Rev. **128**, 899 (1962).
[5] M. Gell-Mann, Phys. Rev. **111**, 362 (1958).
[6] C. S. Wu, Y. K. Lee, and L. W. Mo (to be published).
[7] R. Bacastow, T. Elioff, R. Larsen, C. Wiegand, and T. Ypsilantis, Phys. Rev. Letters **9**, 400 (1962).
[8] A. Dunaister, V. Petrukhin, Yu. Prokoshkin, and V. Rykalin, Phys. Letters **1**, 138 (1962).
[9] P. Depommier, J. Heintze, A. Mukhin, C. Rubbia, V. Soergel, and K. Winter, Phys. Letters **2**, 23 (1962).

of the actual dependence of the functions $W_\pm$ on t, for instance, requires an analysis which cannot, within the scope of present methods of calculation, be exactly carried out.

We examine now what the conserved vector current hypothesis has to say about these general functions. For this analysis, it is convenient to define

$$(g/\sqrt{2})\langle\pi^0|I_\rho^{(-)}|\pi^+\rangle\equiv\tfrac{1}{2}[(p+p')_\rho W_+ + (p-p')_\rho W_-], \quad (3)$$

where $I_\rho^{(-)}$ is the full current generated from the bare current given by

$$-i\sqrt{2}(\phi^\dagger\partial_\rho\phi_0-\partial_\rho\phi^\dagger\phi_0). \quad (4)$$

The conserved vector current hypothesis relates this matrix element to the corresponding matrix element in the electromagnetic case, viz.,

$$\langle\pi^+|I_\rho^{(3)}|\pi^+\rangle=(p'+p)_\rho F_\pi(t), \quad (5)$$

in the limit of zero mass differences among the charge multiplets. That is to say, neglecting the $\pi^+-\pi^0$, etc. mass differences, the conserved vector current hypothesis predicts that

$$W_+(t)=2gF_\pi(t), \quad W_-(t)=0. \quad (6)$$

$F_\pi(t)$ is the electromagnetic form factor for the pion, normalized to unity at $t=0$. Equation (6) is true to all orders in strong coupling, since we believe strong interactions to be charge independent.

In the presence of the $\pi^+-\pi^0$, etc., mass differences, of course, the relations (6) will be modified by terms depending explicitly on these mass differences. The structure of these additional terms depends on the dynamical origin of the multiplet mass splittings. A commonly accepted cause of the multiplet mass splittings is the presence of electromagnetic interactions. This electromagnetic origin of mass splittings leads to an s dependence in the additional terms referred to above. Indeed, in principle, a complete calculation of the lowest order radiative corrections would then include the effects of these induced mass differences.

Unfortunately, no calculation of the lowest order radiative effects with inclusion of all orders of strong interactions can yet be meaningfully done. Only the simple perturbative diagrams can be handled. Thus, we are not able to compute the change in the strong interaction renormalization of g due to radiative effects. That is to say, whereas previously the whole set of strong interactions graphs which renormalize g summed to unity, now, in the presence of radiative corrections to these strong interaction renormalization graphs, no longer sums to unity. Rather, we would expect that

$$g_r=g[1+o(\alpha)]. \quad (7)$$

The same is true, of course, in nuclear β decay also. Behrends and Sirlin[10] have shown that the effect of the induced mass differences on the g_r is at least of second order in the magnitude of the relative mass differences. In their work, however, they essentially considered the effect of the insertion of all electromagnetic self-energy parts in the set of strong interaction renormalization graphs. Electromagnetism has more effects than just self-energy change of the internal charge lines, and, therefore, the Behrends and Sirlin work does not imply that $g_r-g\sim go(\alpha^2)$. A complete calculation of the $o(\alpha)$ term in g_r is not feasible with present tools of calculation. Recently, an attempt[11] was made in this direction with encouraging though ambiguous results.

For orientation purposes, before we proceed with the actual calculation of the radiative effects, we examine briefly some aspects of the nonradiative corrections.

The bare lifetime τ_0, i.e., with $W_+=2g$, $W_-=0$ and neglecting all other effects, is given by

$$\frac{1}{\tau_0}=\left(\frac{\eta\mu}{2}\right)^5\frac{g^2}{30\pi^3}C_R, \quad (8)$$

where

$$C_R=\frac{1}{\eta^5}\left[-30\eta+45\eta^2-10\eta^3-\frac{5}{2}\eta^4-30(1-\eta)^2\ln(1-\eta)\right]$$

$$=\text{recoil correction factor},$$

$$\begin{array}{l}\to\\\to\end{array}\left\{\begin{array}{l}1 \text{ as } \to 0,\\ 2.5 \text{ as } \to 1;\end{array}\right.$$

$\eta\equiv1-(\mu_0/\mu)^2$; μ, μ_0 being the π^+ and π^0 masses, respectively.

We turn now to an examination of the possible effects due to nonlocality or to strong interactions in the strict absence of electromagnetic interactions. As was already pointed out above, the functions, in this case, depend on t only. Such a t dependence can lead to corrections to the bare lifetime at most of order η^2.

For the kinematics of the decay is such that, over the region of physical decay, t ranges over

$$m^2\le t\le(\mu-u_0)^2,$$

m being the electron mass. This range of t is only of order η^2.

To illustrate this kinematic inhibition, we write down the electron spectrum in the nonlocal intermediate vector boson theory (without radiative corrections):

$$\frac{1}{\tau}=\left(\frac{\eta\mu}{2}\right)^5\frac{g^2}{\pi^3}\int_0^1dx\left\{x^2(1-x)^2+\eta x^3(1-x)^2+\eta^2\left[x^4(1-x)^2+\frac{2}{3}\left(\frac{\mu}{m_w}\right)^2x^3(1-x)^3\right]+O(\eta^3)\right\}. \quad (9)$$

The W-boson effect begins to manifest itself only in the η^2 term. Numerically, the change in the lifetime of

[10] R. Behrends and A. Sirlin, Phys. Rev. Letters 4, 186 (1960).

[11] C. R. Schumacher (to be published).

$\pi^+ \to \pi^0 e^+ \nu$ decay due to the W boson is

$$\left.\frac{\tau-\tau_0}{\tau_0}\right]_{W\text{ boson}} = -\frac{\eta^2}{14}\left(\frac{\mu}{m_w}\right)^2 + O(\eta^4) \qquad (10)$$

and, therefore, turns out to be numerically much smaller than order $\eta^2(\eta \sim 1/15$ for $\pi^+ \to \pi^0 e^+ \nu$ decay).

Similarly, the purely strong interaction effects, in the absence of nonlocality, would give rise to a t dependence in $W_\pm$. For small t, we may write

$$W_+(t) = a_0 + a_1 \frac{t}{\mu^2} + \cdots .$$

Current ideas in strong interaction physics tend toward giving $a_1/a_0\mu^2$ the order of magnitude $1/t_r$, t_r being some "resonance mass" squared, and necessarily $t_r > 4_\mu{}^2$. The corresponding change in the lifetime is

$$\left.\frac{\tau-\tau_0}{\tau}\right]_{\text{strong}} = -\frac{\eta^2}{14}\frac{a_1}{a_0} + O(\eta^+)$$

$$\sim -\frac{\eta^2}{14}\frac{\mu^2}{t_r},$$

and is, again, quite small. a_0 is, of course, just $2g$.

In other words, for the $\pi \to \pi e \nu$ decay, nonlocality and strong interactions are not the primary source of corrections to the actual decay. Rather, the effects due to the electromagnetic interactions become of primary importance.

III. RADIATIVE CORRECTION

The radiative effects in the $\pi \to \pi e \nu$ decay have been calculated[12] using perturbation theory. We shall quote here in brief the main results of that calculation. An earlier calculation[13] of the radiative effects by others contained some errors.

The diagrams included in the lowest order calculations are shown in Fig. 2. The electromagnetic insertions in the strong interaction renormalization graphs have been omitted as already pointed out earlier.

The evaluation of the diagrams in Fig. 2 is entirely straightforward and not worthy of note. We might just mention, for reference sake, that the renormalization factor for π^+ is (purely electrodynamics)

$$\frac{\alpha}{4\pi}\left(2\ln\frac{\Lambda}{\lambda} - \frac{3}{4}\right), \qquad (11)$$

where the renormalization factor has been defined in exactly the same manner as for the electron which is

$$-\frac{\alpha}{4\pi}\left(\ln\frac{\Lambda}{m} + \frac{9}{4} - 2\ln\frac{m}{\lambda}\right),$$

[12] N. P. Chang, Doctoral thesis, Columbia University, 1963 (unpublished).

[13] G. DaPrat and G. Putzolu, Nuovo Cimento **21**, 541 (1961).

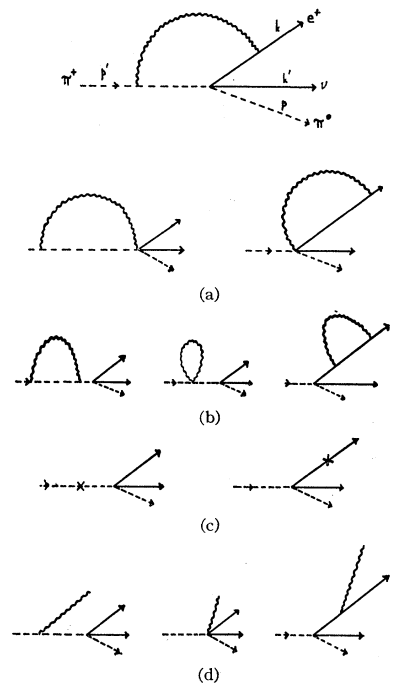

FIG. 2. Perturbation diagrams for the lowest order radiative corrections to the $\pi^+ \to \pi^0 e^+ \nu$ decay.

where Λ is the cutoff mass, used in the Feynman regulator. λ is the "fictitious" photon mass.

The result for the virtual diagram gives $(m^2/\mu^2 \ll 1)$.

$$W_+ = 2g\left\{1 + \frac{\alpha}{2\pi}\left(\frac{3}{2}\ln\frac{\Lambda}{m} + (\xi + \ln\eta x - 1)\left(2\ln\frac{\lambda}{m} + 1\right)\right.\right.$$

$$\left.\left. - (\xi + \ln\eta x)^2 + 2\ln\eta x \ln(1-\eta x) - 2L(\eta x)\right)\right\}, \qquad (12)$$

where the notation is $\xi \equiv \ln(\mu/m)$, $\eta \equiv 1 - (\mu_0/\mu)^2$,

$$L(x) = \int_0^x \frac{\ln(1-t)}{t} dt,$$

$x =$ (electron momentum/max electron mom.) in π^+ rest frame.

In deriving the above result, the Feynman regulator has been used to regulate the photon propagator.

The inner bremsstrahlung processes are also straight forward though tedious to evaluate. These processes have been computed here with integration over all compatible photon energies. In view of the extreme rarity of these events, and in view of the experimental difficulties of the detection of the decay events, energy discrimination against the inner bremsstrahlung photons (maximum energy ~ 5 MeV) hardly seems practical.

The total result for the election spectrum is $(\eta\to 0)$

$$\frac{1}{\tau}=\left(\frac{\eta\mu}{2}\right)^5\frac{g^2}{\pi^3}\int_0^1 dx\,x^2(1-x)^2\Bigg\{1+\frac{\alpha}{\pi}\Bigg[h(x)+2+(\xi+\ln\eta x)\frac{(1-x)^2}{12x^2}+(\xi+\ln\eta x-1)\frac{(2-11x)}{3x}\Bigg]$$
$$+\eta\Bigg(x+\frac{\alpha}{\pi}\Bigg[xh(x)+x-x\ln\eta x+\frac{(1-x)^2}{24x}+(\xi+\ln\eta x)\frac{(1-x)^2}{x^2}\left(\frac{1}{30}+\frac{x}{20}\right)$$
$$+(\xi+\ln\eta x-1)\frac{1}{x}\left(\frac{1}{6}+x-\frac{25}{6}x^2\right)\Bigg]\Bigg)+O(\eta^2)\Bigg\}\,, \quad (14)$$

where

$$h(x)=-1+\tfrac{3}{2}\ln\frac{\Lambda}{m}-\frac{\pi^2}{3}-2(\xi+\ln\eta x-1)\ln\frac{x}{1-x}\,.$$

The total change in the lifetime due to the radiative effects may be summarized by

$$\left.\frac{\tau-\tau_0}{\tau_0}\right|_{\text{rad.}}=-\frac{\alpha}{\pi}\left(-2+\tfrac{3}{2}\ln\frac{\Lambda}{m}\right)$$
$$-\frac{\alpha}{4\pi}\frac{30}{C_R}\Bigg\{-\frac{\xi}{15}-\frac{1}{15}\ln\eta-\frac{4\pi^2}{45}+\frac{257}{300}$$
$$+\eta\left[-\tfrac{1}{10}\xi-\tfrac{1}{6}\ln\eta-\frac{2\pi^2}{45}+\frac{1063}{1080}\right]$$
$$+\eta^2\left[-\frac{2}{35}\xi-\frac{4}{35}\ln\eta-\frac{8\pi^2}{315}+\frac{1627}{4900}\right]+\cdots. \quad (15)$$

It is well worth noticing that the radiative corrections to the electron spectrum occur in the lowest order in η, in contrast to the other strong and nonlocal effects which occur in the higher order in η terms. This is due to the electromagnetic renormalization of g which is present even when η is set equal to zero. Photons being massless, there can be renormalization effects even when there is no recoil.

The results are evidently cutoff-dependent. This is a manifestation of the nonrenormalizability of the conventional theory of weak interactions. It casts serious doubt, as a matter of principle, on the usefulness of the result. Nevertheless, in line with usual practice, we shall estimate the radiative correction numerically by using a 2π resonance mass for Λ. This is in accord with the current ideas about the structure of the π^+. With this value for Λ, we find

$$\left.\frac{\tau-\tau_0}{\tau_0}\right|_{\text{rad.}}\cong -1.12\%(\sim m_p)\,. \quad (16)$$

To illustrate the insensitivity, if Λ is taken to be twice the nuclear mass, the correction to the lifetime becomes -1.14%. That is to say, for reasonable values of Λ around a nucleon mass, the correction is very close to -1.1%.

IV. CONCLUSION

We have studied the various corrections to the $\pi\to\pi e\nu$ decay and found that the radiative correction is the most important correction to the decay rate. A simple perturbation calculation of the radiative effects is reported in this paper. The result is dependent logarithmically on a cutoff. For a cutoff around a nucleon mass, the correction to the lifetime is to decrease it by $\sim 1.1\%$. In carrying through this simple calculation, however, the radiative corrections to the strong interaction renormalization graphs have been omitted. This omission may be crucial in the future when experimental data are accurate enough to permit a meaningful comparison between the coupling constants for μ decay and for $\pi\to\pi e\nu$ decay. The same omission was made in the nuclear β-decay radiative correction calculation. It may well be the cause of the $\sim 2\%$ discrepancy, or, at least, the partial cause in a nonlocal theory.

ACKNOWLEDGMENT

It is always a pleasure to acknowledge the kind help and advice of Professor Gerald Feinberg throughout the course of this work.