Extension of Levinson's Theorem to the Relativistic Case

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A generalization of Levinson's theorem is proved. The proof requires that the elastic partial-wave scattering amplitude satisfy a dispersion relation, and that the N/D integral equation be of Fredholm type with nonzero determinant. Inelastic processes are taken into account fully by means of a complex phase shift. The high-energy behavior of the imaginary part of the phase shift is subject to mild restrictions. For spinless particles the theorem states that $\delta_l(\infty)=(-n_b+n_c)\pi$. The real part δ_l of the phase shift is normalized to zero at threshold. n_b is the number of "particle poles"; i.e., elementary particle poles or bound state poles of the amplitude. n_c is the number of Castillejo-Dalitz-Dyson (CDD) poles of the D function. An unfamiliar aspect of the CDD ambiguity is discussed. For complete generality in computing particle poles from a given left cut discontinuity, a new sort of CDD pole must be admitted at real energies below threshold. This type of pole is to be associated with a stable particle with energy below threshold, whereas an ordinary CDD pole corresponds to an unstable particle above threshold.

1. INTRODUCTION

IN a paper published in 1949, Levinson[1] established the following relation between the number n_b of bound states of angular momentum l and the high-energy limit of the scattering phase shift:

$$\delta_l(\infty)=-n_b\pi. \tag{1.1}$$

(The phase shift is normalized to zero at zero kinetic energy.) Levinson's theorem holds for two particles interacting through any short-range potential of a fairly broad class.[2] Attempts to extend the theorem beyond the domain of potential scattering either have to do with soluble model field theories,[3] or else fail to retain much of the original character of the theorem.[4] One obvious difficulty in a proper relativistic theory is the presence of infinitely many channels for each angular momentum value. The other main problem is how to find a sufficient description of the interaction when neither a wave equation nor a soluble Hamiltonian is available. Lee and Cook[4] deal with both difficulties by means of an infinite system of partial-wave dispersion relations. The latter are solved formally by the matrix ND^{-1} method.[5] A study of the infinite-dimensional matrix D gives a generalization of Eq. (1.1), but $\delta_l(\infty)$ is replaced by $\ln\det S_l(\infty)$, where S_l is the entire S matrix for angular momentum l. Besides the experimental inaccessibility of $\ln\det S_l(\infty)$, another shortcoming of the matrix approach is the use of heuristic mathematical arguments which seem hard to justify. On the other hand, Bosco[6] has suggested that a Levinson theorem might hold for an elastic partial-wave amplitude satisfying a single dispersion relation in which inelastic processes are neglected entirely.

The point of view of the present paper is similar to that of Bosco, except that inelastic processes are taken into account fully by means of a complex phase shift. A generalization of Levinson's theorem is proved. The hypotheses of the theorem include mild restrictions on the asymptotic behavior of the imaginary part of the phase shift, as well as the requirement that the N/D integral equation be of Fredholm type with nonvanishing Fredholm determinant. The equation that replaces (1.1) is

$$\delta_l(\infty)=(-n_b+n_c)\pi, \tag{1.2}$$

where δ_l is the real part of the phase shift and n_c is the number of CDD poles.[7,8] n_b is the number of "particle poles" of the scattering amplitude, i.e., the number of poles at real energies just below threshold that may reasonably be associated with elementary particles *or* bound states.

From time to time, the hope has been raised that a generalized Levinson theorem would contribute to an operational definition of bound state, or would lead to some distinction between elementary particles and bound states.[9] The possibility of such a distinction is not ruled out by the present work, although that is not apparent from Eq. (1.2) alone. In fact, two different types of particle poles are characterized. It is at least conceivable that both types are realized in nature, and that one type is somehow more elementary than the other. The distinction I have in mind depends on whether or not the particle pole comes out of an N/D procedure in which all singularities of the amplitude

[1] N. Levinson, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **25**, No. 9 (1949). Actually Levinson did not rule out the possibility $\delta_l(\infty)=-(n_b+\frac{1}{2})\pi$ under certain circumstances.

[2] The potential $V(x)$ is assumed to be piecewise continuous and to be sufficiently well behaved at $x=0$ and $x=\infty$ so that $\int_0^1 x|V(x)|dx+\int_1^\infty x^2|V(x)|dx<\infty$.

[3] M. T. Vaughn, R. Aaron, and R. D. Amado, Phys. Rev. **124**, 1258 (1961). This paper contains all references on soluble field models that I know of, except for the following recent work: G. C. Ghirardi, M. Pauri, and A. Rimini, Ann. Phys. (N. Y.) **21**, 401 (1963).

[4] L. F. Cook, Jr., and B. W. Lee, Phys. Rev. **127**, 283 (1962).

[5] J. D. Bjorken, Phys. Rev. Letters **4**, 473 (1960).

[6] B. Bosco, Nuovo Cimento **26**, 342 (1962). The proof of Levinson's theorem in this paper is not correct. Bosco's Eq. (6) does not follow from Eq. (11) unless $\bar{D}(0)=0$, because of the pole of D at $s=0$. To assume $\bar{D}(0)=0$ is to assume what is to be proved.

[7] L. Castillejo, R. H. Dalitz, and F. J. Dyson, Phys. Rev. **101**, 453 (1956).

[8] G. Frye and R. L. Warnock, Phys. Rev. **130**, 478 (1963).

[9] M. A. Ruderman and C. M. Sommerfield, Bull. Am. Phys. Soc. **4**, 375 (1959).

except the pole itself are taken as input, the N/D equations being of the normal type in which the D function is either without poles or has only the well-known CDD poles. It is usually assumed that any particle pole will come out of such equations. However, there is the mathematical possibility of particle poles that may be obtained only from equations in which D involves poles other than normal CDD terms. These extra poles of D are called CDD poles of the second kind. They arise when the dispersion relation for D requires subtractions in addition to those associated with normal CDD terms.[10] If a particle pole can be obtained only from equations with second-kind CDD terms, it is sharply distinguished from either the bound-state poles of potential theory[11] or the elementary particle poles arising by a "bootstrap" phenomenon in relativistic theory.[12] It will be called a particle pole of the second kind in the following. If particles of the second kind are to make sense physically, it will be necessary to give a physical interpretation of the second-kind CDD poles. An interpretation along the lines of Dyson's[13] interpretation of ordinary CDD terms may be possible, but the matter is not pursued here.

To decide theoretically on whether or not second-kind particles occur in a specific, nontrivial theory seems to require a more complete description of the scattering amplitude than is available at present. An experimental decision is possible, in principle, if one knows the total number n_b of poles of any kind. For spinless scattered particles, if $n_b > n_d + \epsilon$ there are exactly $n_b - n_d - \epsilon$ particle poles of the second kind. Otherwise, there are none. n_d is the number of times δ_l passes downward through a multiple of π, and ϵ is one or zero depending on whether δ_l is negative or positive, respectively, just above threshold.

Section 2 involves two steps. First, it is proved that under fairly general conditions the real part δ_l of the phase shift approaches an integral multiple of π. The second step is a proof that if $\delta_l(\infty) = n\pi$ and the appropriate N/D equation is of Fredholm type with nonzero determinant, then $\delta_l(\infty) \geq -n_b\pi$.

Section 3 is concerned with the CDD case and the proof of Eq. (1.2). An aspect of the CDD ambiguity which may have escaped notice up to now is elucidated. The point is that the ambiguity involved in calculating a particle pole from a given left cut discontinuity may be more severe than is often supposed. Arbitrary constants may enter which are not directly associated with points at which δ_l rises through an integral multiple of π. These constants are the positions and residues of the second-kind CDD poles. The latter are located at real energies below threshold.

[10] The correspondence between CDD poles and certain subtractions in the dispersion relation for D is explained in Ref. 8, and in Sec. 3 of the present paper.

[11] R. Blankenbecler, M. L. Goldberger, N. N. Khuri, and S. B. Treiman, Ann. Phys. (N. Y.) **10**, 62 (1960).

[12] G. Chew, Phys. Rev. Letters **9**, 233 (1962); F. E. Low, Nuovo Cimento **25**, 678 (1962).

[13] F. J. Dyson, Phys. Rev. **106**, 157 (1957).

Section 4 deals with the generalization to spin-0 $-$ spin-$\frac{1}{2}$ scattering; in Secs. 2 and 3 the scattered particles have zero spin.

In Sec. 5 various points are illustrated on a simple model in which the dynamical singularities are represented by one or two poles.

Section 6 contains an application of a uniqueness theorem for solutions of partial-wave dispersion relations.[8] The result is that if an appropriate N/D equation is of Fredholm type and its solution involves no ghosts, then a CDD ambiguity is possible only in low angular momentum states ($l<3$ for spinless particles, and $J<\frac{3}{2}$ in the spin-0 $-$ spin-$\frac{1}{2}$ case). The appropriate N/D equation is one without CDD terms, and with particle poles excluded from the N function.

The hypotheses of the generalized Levinson theorem are collected at the end of Sec. 2. Much of the analysis is based on Ref. 8, which will be called "FW," henceforth.

2. LEVINSON'S RELATION WITHOUT CDD POLES

The scattering of two spin-0 particles of mass μ will be treated in detail. The changes necessary in the spin-0 $-$ spin-$\frac{1}{2}$ case, indicated in Sec. 4, are rather minor. In the notation of Chew and Mandelstam[14] the partial-wave amplitude at physical energies is

$$A(\nu) = \frac{1}{2i}\left[\frac{\nu+1}{\nu}\right]^{1/2}\left[\eta(\nu)\exp 2i\delta(\nu) - 1\right]. \tag{2.1}$$

Indices indicating angular momentum and isotopic spin values have been suppressed. ν is defined by $\nu = q^2/\mu^2$, where q is the momentum in the center-of-mass system. $\delta(\nu)$ is the real part of the complex phase shift $\alpha(\nu)$ and $\eta(\nu) = \exp[-2\,\mathrm{Im}\alpha(\nu)]$. The normalization $\delta(0)=0$ is adopted. The dispersion relation for $A(z) = A^*(z^*)$ is assumed to be[14]

$$A(z) = a + \frac{z-\nu_0}{\pi}\int_{-\infty}^{-1} \frac{d\nu\,\mathrm{Im}A(\nu)}{(\nu-\nu_0)(\nu-z)} + \frac{z-\nu_0}{\pi}\int_0^{\infty} \frac{d\nu\,\mathrm{Im}A(\nu)}{(\nu-\nu_0)(\nu-z)} + (z-\nu_0)\sum_{i=1}^{n_b} \frac{r_i}{\nu_{bi}-z}. \tag{2.2}$$

For convenience ν_0 is chosen to lie in the interval $(-1, 0)$. Possible elementary particle poles (Born terms) as well as bound-state poles are to be included in the sum over n_b "particle poles." The dispersion relation may be replaced, in a certain sense, by a linear, nonsingular, integral equation through the generalized N/D procedure given in FW. Actually, a slight modification of that procedure is required in the present case. Here the particle poles are to correspond to zeros of D, while in FW they could be associated with poles of N. This requires a reclassification of the possible

[14] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

N/D equations which is carried out by introducing the function

$$\mathfrak{D}(z)=\mathfrak{D}(z;\delta)=\exp\left[-\frac{z}{\pi}\int_0^\infty \frac{\delta(\nu)d\nu}{\nu(\nu-z)}\right]. \quad (2.3)$$

If $\delta(\nu)$ satisfies a continuity condition[15] and tends to a finite limit $\delta(\infty)=\pi p$, then according to Appendix A of FW,

$$\mathfrak{D}(z)=z^p e^{\lambda(z)}, \quad (2.4)$$

where $|\lambda(z)|\leq \epsilon \ln|z|$ for any $\epsilon>0$ and all z such that $|z|>r(\epsilon)$. Roughly speaking, $\mathfrak{D}(z)$ behaves as z^p at infinity "except for logarithmic factors." Under the hypotheses of the theorem to be proved, δ does approach a finite limit, so (2.4) will be directly applicable. Now with $\mathfrak{N}=A\mathfrak{D}$ one has a representation $A=\mathfrak{N}/\mathfrak{D}$ which satisfies the requirements imposed in FW; viz., $\mathfrak{D}(z)=\mathfrak{D}^*(z^*)$ is analytic in the z plane with the cut $(0,\infty)$ and has the phase $-\delta(\nu)$ on that cut, while $\mathfrak{N}(z)=\mathfrak{N}^*(z^*)$ is meromorphic in the plane with the cuts $(-\infty,-1)$, (ν_i,∞), ν_i being the threshold for inelastic processes. However, $\mathfrak{N}$ contains particle poles which are conveniently removed through multiplication by the polynomial

$$\Phi(z)=\prod_{i=1}^{n_b}(z-\nu_{bi}). \quad (2.5)$$

Then with $N=\Phi\mathfrak{N}$, $D=\Phi\mathfrak{D}$ one has proved the existence of an N/D decomposition with all required properties. According to FW, it follows that $\mathrm{Im}D$ satisfies a linear integral equation, but the character of this equation depends on the asymptotic behavior of D, and therefore, on p and n_b through (2.4) and (2.5).

Let Class A be the class of all amplitudes $A(\nu)$ for which $p\leq -n_b$. By FW and the remarks above, any Class A amplitude has an N/D representation in which the particle poles appear as zeros of D and for which

$$n(\nu)=-\left[\frac{\nu+1}{\nu}\right]^{1/2}\frac{\mathrm{Im}D(\nu)}{\nu-\nu_0}$$

satisfies the following integral equation,

$$\eta(\nu)n(\nu)=\frac{a+C(\nu)}{\nu-\nu_0}+\frac{1}{\pi}\int_0^\infty d\nu'\left[\frac{\nu'}{\nu'+1}\right]^{1/2}\left[\frac{C(\nu)-C(\nu')}{\nu-\nu'}\right]n(\nu'), \quad (2.6)$$

where

$$C(\nu)=\mathrm{Re}B(\nu+i0)=\mathrm{Re}[B^U(\nu+i0)+B^I(\nu+i0)], \quad (2.7)$$

$$B^U(z)=\frac{z-\nu_0}{\pi}\int_{-\infty}^{-1}\frac{d\nu\,\mathrm{Im}A(\nu)}{(\nu-\nu_0)(\nu-z)},$$
$$B^I(z)=\frac{z-\nu_0}{\pi}\int_{\nu_i}^\infty d\nu\left[\frac{\nu+1}{\nu}\right]^{1/2}\frac{[1-\eta(\nu)]/2}{(\nu-\nu_0)(\nu-z)}. \quad (2.8)$$

If $p>-n_b$ extra terms occur, in general, on the right-hand side of the integral equation because of the necessity of poles in the Cauchy representation of D. This case is deferred to Sec. 3. The substitution $\nu+1=1/s$ puts (2.6) in the form

$$x(s)=y(s)+\int_0^1 K(s,t)x(t)dt, \quad (2.9)$$

where

$$x(s)=(\nu+1)\eta^{1/2}(\nu)n(\nu),$$
$$y(s)=\frac{\nu+1}{\nu-\nu_0}\frac{a+C(\nu)}{\eta^{1/2}(\nu)},$$
$$K(s,t)=\frac{1}{\pi}\left[\frac{\nu'}{\nu'+1}\right]^{1/2}\frac{(\nu+1)(\nu'+1)}{[\eta(\nu)\eta(\nu')]^{1/2}}\left[\frac{C(\nu)-C(\nu')}{\nu-\nu'}\right].$$

Equation (2.9) is subject to the standard Fredholm theory provided y and K are square-integrable:

$$\int_0^1 y^2(s)ds<\infty, \quad \int_0^1\int_0^1 K^2(s,t)ds\,dt<\infty. \quad (2.10)$$

If the integrals (2.10) exist and the Fredholm determinant of K is not zero, (2.9) has a unique solution in the class of square-integrable solutions. But since

$$\int_0^1 x^2(s)ds=\int_0^\infty d\nu\left[\frac{\nu+1}{\nu}\right]\eta(\nu)\left[\frac{\mathrm{Im}D(\nu)}{\nu-\nu_0}\right]^2,$$

Eqs. (2.4) and (2.5) show that any amplitude in Class A corresponds to a square-integrable solution of (2.9). It is certainly a reasonable guess that such solutions come from an equation in which y and K are square-integrable. Conditions (2.10) will, in fact, be assumed in the following. The reasons for the assumption are, first, simplicity, and second, the circumstance that these conditions have been met in several specific calculations.[16]

A useful condition on the function $C(\nu)$, necessary for (2.10), may now be obtained. By returning to the variable ν, one sees that (2.10) implies

$$\int_0^\infty\int_0^\infty d\nu d\nu'\frac{1}{\eta(\nu)\eta(\nu')}\left[\frac{C(\nu)-C(\nu')}{\nu-\nu'}\right]^2=\int_0^\infty\int_0^\infty f(\nu,\nu')d\nu d\nu'<\infty. \quad (2.11)$$

[15] Both the real and the imaginary parts of the phase shift are supposed to satisfy a Hölder condition in any finite interval; cf. Ref. 16 of FW.

[16] See, for example, J. L. Uretsky, Phys. Rev. **123**, 1459 (1961), in which further references are given.

By (2.7), Ref. 15, and Appendix B of FW, (2.11) can fail only as a result of bad behavior at infinity; the line $\nu=\nu'$ causes no trouble. Since the integrand $f(\nu,\nu')$ is positive, its integral over a finite region is certainly less than its integral over the entire quadrant. Moreover, the integral over an appropriate finite region may be expressed in polar coordinates. Hence,

$$\int_{\theta_1}^{\theta_2} d\theta \int_\rho^R rdr\, f(r\cos\theta, r\sin\theta) < \int_0^\infty \int_0^\infty d\nu d\nu' f(\nu,\nu') < \infty\,, \quad (2.12)$$

where the constant limits are chosen so that $R>\rho>0$, $0<\theta_1<\theta_2<\pi/4$. Assume that

$$\eta(\nu)=O(\ln^{-\epsilon}\nu)\,, \quad \epsilon\geq 0\,, \quad (2.13)$$

for large ν. This is no restriction since $\epsilon=0$ is allowed and $\eta\leq 1$ by unitarity. Let $\alpha=\cos\theta$, $\beta=\sin\theta$, and $g(r)=r\ln^{1/2}r$. Then by the mean value theorem[17] and (2.13),

$$\int_\rho^R rdr\, f(r\alpha,r\beta)=\int_\rho^R \frac{rdr}{g^2(r\gamma)}[C'(r\gamma)g(r\gamma)]^2\frac{1}{\eta(r\alpha)\eta(r\beta)}$$
$$\geq M\int_\rho^R \frac{dr}{r\gamma\ln r\gamma}[C'(r\gamma)g(r\gamma)]^2\ln^\epsilon r\alpha \ln^\epsilon r\beta$$
$$\geq M'\int_\rho^R \frac{dr}{r\ln r}[C'(r\gamma)r\gamma\ln^{\frac{1}{2}+\epsilon}r\gamma]^2 \quad (2.14)$$

for sufficiently large ρ. $\gamma=\gamma(r)$ satisfies $\beta<\gamma<\alpha$, and M and M' are constants. The last inequality in (2.14) follows from the bounds on α, β, γ which are set by the bounds on θ_1, θ_2. For example, one can say that $\ln r\alpha=\ln r\gamma(1+\ln(\alpha/\gamma)/\ln r\gamma)=\ln r\gamma(1+\zeta)$, where $|\zeta|<1$ for all $r>\rho$. Thus, by (2.12) and (2.14),

$$\int_\rho^\infty \frac{dr}{r\ln r}[C'(r\gamma)r\gamma\ln^{\frac{1}{2}+\epsilon}r\gamma]^2<\infty\,. \quad (2.15)$$

The condition (2.15) can now be turned to advantage by noting that the dispersion relation (2.2) may be rearranged so as to display the function $C(\nu)$. By taking the real parts of (2.1) and (2.2) one finds

$$C(\nu)=\frac{1}{2}\left[\frac{\nu+1}{\nu}\right]^{1/2}\eta(\nu)\sin 2\delta(\nu)-a-(\nu-\nu_0)\sum_{i=1}^{n_b}\frac{r_i}{\nu_{bi}-\nu}$$
$$-\frac{\nu-\nu_0}{\pi}P\int_0^\infty d\nu'\left[\frac{\nu'+1}{\nu'}\right]^{1/2}\frac{\eta(\nu')\sin^2\delta(\nu')}{(\nu'-\nu_0)(\nu'-\nu)}\,. \quad (2.16)$$

[17] By Appendix B of FW, C has a continuous first derivative in any region where $d\eta/d\nu$ satisfies a Hölder condition. It is permissible to assume that $d\eta/d\nu$ meets this condition for large ν, since η is expected to have bad behavior only at s wave, two-body thresholds.

The asymptotic behavior of $C(\nu)$ and $C'(\nu)$ is investigated by the following lemma which is proved in the Appendix: If when x is large $\varphi(x)=O(x^{-1}\ln^\alpha x)$ and $\varphi(x)\in C'$, then for large t

$$tP\int_{x_0}^\infty \frac{\varphi(x)dx}{t-x}=\int_{x_0}^t \varphi(x)dx+O(\ln^\alpha t)\,, \quad (2.17)$$

where $x_0>0$. Now consider first the case $\eta(\nu)=O(\ln^{-\epsilon}\nu)$, $\epsilon>0$. By (2.16) and (2.17) we have

$$C(\nu)=-a-(\nu-\nu_0)\sum_{i=1}^{n_b}\frac{r_i}{\nu_{bi}-\nu}$$
$$+\frac{1}{\pi}\int_0^\nu d\nu'\left[\frac{\nu'+1}{\nu'}\right]^{1/2}\frac{\eta(\nu')\sin^2\delta(\nu')}{\nu'-\nu_0}+R(\nu)\,, \quad (2.18)$$

where the remainder $R(\nu)=O(\ln^{-\epsilon}\nu)$ is a function with a continuous derivative for large ν.[17] By Ref. 26 of FW, $R'(\nu)=O(\nu^{-1}\ln^{-\epsilon-1}\nu)$. Therefore,

$$C'(\nu)=\frac{1}{\pi}\left[\frac{\nu+1}{\nu}\right]^{1/2}\frac{\eta(\nu)\sin^2\delta(\nu)}{\nu-\nu_0}+O(\nu^{-1}\ln^{-\epsilon-1}\nu)\,. \quad (2.19)$$

Unless $\sin^2\delta(\nu)$ approaches zero, it is clear that (2.15) cannot be satisfied if $\eta(\nu)\geq A\ln^{-\epsilon-\frac{1}{2}}\nu$ for large ν, where A is a positive constant. This follows by the comparison test with the integral $\int^\infty (x\ln x)^{-1}dx=\infty$ if $\sin\delta$ has a finite number of zeros at most. (The latter condition on $\sin\delta$ is much stronger than necessary, but since it is essential in Sec. 3 it will be assumed.) Thus, (2.15) and the following condition (2.20) are together sufficient to ensure that the phase shift approaches an integral multiple of π, provided $\epsilon>0$.

$$A\ln^{-\epsilon-\frac{1}{2}}\nu\leq\eta(\nu)\leq B\ln^{-\epsilon}\nu\,. \quad (2.20)$$

The same statement holds for $\epsilon=0$, but the proof is slightly altered since the behavior of $R'(\nu)$ cannot be estimated by the method used above. Suppose (2.20) holds with $\epsilon=0$. Then $R(\nu)=0(1)$, and if $\sin^2\delta(\nu)$ does not vanish at infinity the third term [call it $T(\nu)$] of (2.18) tends to infinity with increasing ν. In that case l'Hospital's rule may be applied as follows[18]:

$$\lim_{\nu\to\infty}\frac{C(\nu)}{\ln^{1/2}\nu}=\lim_{\nu\to\infty}\frac{T(\nu)}{\ln^{1/2}\nu}=\lim_{\nu\to\infty}2C'(\nu)\nu\ln^{1/2}\nu$$
$$=\lim_{\nu\to\infty}\frac{2}{\pi}\eta(\nu)\sin^2\delta(\nu)\ln^{1/2}\nu\,. \quad (2.21)$$

But now since $\sin^2\delta(\nu)$ has a finite number of zeros,

[18] It is not necessary to assume that the limits exist. By the usual proof of l'Hospital's theorem (or, equivalently, by Ref. 26 of FW), if $C(\nu)\ln^{-1/2}\nu$ approaches no limit, then neither does $2C'(\nu)\nu\ln^{1/2}\nu$. Moreover, the latter quantity takes on only values that are also taken on by $C(\nu)\ln^{-1/2}\nu$. Since $C(\nu)\ln^{-1/2}\nu$ clearly has no zeros for large ν, the same contradiction between (2.15) and $\sin^2\delta(\infty)\neq 0$ is obtained.

there is a contradiction of Eqs. (2.20) and (2.15) if $\sin^2\delta$ does not vanish at infinity.

The condition (2.20) with $\epsilon\geq 0$ will be adopted. This does not seem unduly restrictive in view of indications in favor of a picture in which a single Regge trajectory dominates the scattering at infinity. If the partial-wave amplitude is calculated approximately by integrating over just the diffraction peak of the Regge term $\beta(t)s^{\alpha(t)}$, one estimates that $\eta\cos 2\delta-1$ goes to zero as $1/\ln s$. Thus, the limits $\eta\rightarrow 1$ and $\delta\rightarrow n\pi$ are strongly suggested by the Regge pole hypothesis, even if one has not quite proved that they follow from it. Since the result $\eta\rightarrow 1$ is extremely nonclassical, Frye[19] has attempted to reinstate the old presumption that η vanishes. As a model, he introduces a second-order Regge pole. He finds that a logarithmic decrease in η (but no faster decrease) can be accommodated, but at the expense of having total cross sections increase logarithmically. Furthermore, some work of Sugawara and Tubis[20] at least suggests that any deviation from the single Regge pole behavior could be expected to have only logarithmic character. Thus, (2.20) is expected to hold quite generally.

Now assume that conditions (2.10) are fulfilled, that the Fredholm determinant of K is not zero, and that the phase shift goes to an integral multiple of π [through satisfaction of (2.15) and (2.20), or otherwise]. Under these circumstances the integral equation (2.9) has only one square-integrable solution. This solution cannot correspond to an amplitude for which $p<-n_b$, because if it did another square-integrable solution could be found. Let $D(\nu)$ be the denominator function $\Phi(\nu)\mathfrak{D}(\nu)$ for an amplitude with $p<-n_b$, and $x(s)$ the associated solution of (2.9). Then by (2.4) and (2.5), $\bar{D}(\nu)=(1+\nu-\nu_0)D(\nu)$ has an asymptotic behavior such that the corresponding function $\bar{x}(s)=(1+\nu-\nu_0)x(s)$ still satisfies (2.9) and is square-integrable. Thus, under the conditions laid down any amplitude satisfies

$$\delta(\infty)\geq -n_b\pi, \qquad (2.22)$$

and any amplitude of Class A satisfies

$$\delta(\infty)=-n_b\pi. \qquad (2.23)$$

It may be worthwhile to collect the hypotheses of the theorems (2.22) and (2.23). They are as follows:

(i) The partial-wave dispersion relation (2.2) holds.
(ii) The N/D integral equation (2.9) is of Fredholm type with nonzero Fredholm determinant.
(iii) At high energies the absorption factor $\eta(\nu)$ satisfies condition (2.20) with $\epsilon\geq 0$.
(iv) $\sin\delta(\nu)$ has at most a finite number of zeros.

Weak continuity conditions on δ and η have also been assumed, but they are probably met in any physical theory; cf. Refs. 15 and 17. Conditions (i) through (iv) are also sufficient for the Levinson theorem including CDD poles.

[19] G. Frye, Phys. Rev. **129**, 1453 (1963).
[20] M. Sugawara and A. Tubis, Phys. Rev. Letters **9**, 355 (1962).

3. LEVINSON'S RELATION INCLUDING CDD POLES

If p is an integer greater than $-n_b$, the function $D=\Phi\mathfrak{D}$ of the preceding section does not satisfy a dispersion relation with one subtraction; i.e., D/z does not vanish at infinity. In order to derive an integral equation analogous to (2.6) in this case one must introduce further subtractions by considering $D=R\mathfrak{D}$, where R is a rational function. The positions of the poles of R may be chosen arbitrarily, but it turns out that the integral equation has an especially nice form if the poles are at physical points ν_u where $\sin\delta(\nu_u)=0$. If also $(d\delta/d\nu)_{\nu_u}>0$, such poles correspond exactly to the poles of the D function first discussed by Castillejo, Dalitz, and Dyson.[7] They have been given a physical interpretation by Dyson,[13] van Kampen,[21] and others. If D is not required to have a zero for each particle pole (the particle poles being put into N), then Sec. VI of FW shows that one always has sufficiently many CDD points ν_u so that no poles of R other than CDD poles are necessary to give D the required asymptotic behavior. In the present case D is required to have the particle zeros, and, in general, there may or may not be sufficiently many CDD points. Amplitudes for which there are (are not) sufficiently many points will be called Class B (Class C) amplitudes; these classes are defined more precisely below.

An integral representation of $D=R\mathfrak{D}$ is needed, but now it is not necessarily provided by Cauchy's integral theorem. Instead, one uses a theorem of Herglotz, as explained in Secs. VI and VII of FW. A function $H(z)$ is called a Herglotz function if it is analytic in the half-plane $\mathrm{Im}z>0$ and has the property $\mathrm{Im}H(z)\geq 0$, $\mathrm{Im}z>0$. An adaptation of Herglotz's theorem[22] states that any such function has the representation

$$H(z)=Az+c+\int_{-\infty}^{\infty}d\alpha(\nu)(1+\nu z)(\nu-z)^{-1}, \qquad (3.1)$$

where $\alpha(\nu)$ is a bounded, nondecreasing function, and A and c are real and $A\geq 0$. In directions not parallel to the real axis, $\lim_{z\rightarrow\infty} z^{-1}H=A$. If $A\neq 0$, a representation like (3.1) cannot be deduced directly from Cauchy's integral theorem.

Let ν_{ui}, $i=1,\cdots,n_u$, be all points at which the phase δ goes up through an integral multiple of π, and let ν_{di}, $i=1,\cdots,n_d$ be all points at which δ goes down through an integral multiple of π. Let

$$\hat{\delta}(\nu)=\delta(\nu)+\epsilon\pi-\pi\sum_i\Theta(\nu-\nu_{ui})+\pi\sum_i\Theta(\nu-\nu_{di}), \qquad (3.2)$$

[21] N. G. van Kampen, Physica **23**, 157 (1957).
[22] J. A. Shohat and J. D. Tamarkin, *The Problem of Moments* (American Mathematical Society, New York, 1943), see especially p. 23.

where $\Theta(x)$ is the unit step, zero for $x<0$ and one for $x>0$. ϵ is equal to one if $\delta(\nu)$ is negative just above threshold, and equal to zero otherwise. With the definition (2.3) of $\mathfrak{D}$, one can show that

$$H(z)=-\mathfrak{D}(z;\hat{\delta}) \tag{3.3}$$

is Herglotz. The proof is obtained by computing Im H from (2.3) and (3.3) and using the bounds on $\hat{\delta}$ implied in its definition: $0\leq\hat{\delta}(\nu)\leq\pi$. It is easy to check that the jumps in $\hat{\delta}(\nu)$ at the points ν_{ui} and ν_{di} lead to poles and zeros, respectively, of $H(z)$. In fact, by evaluation of the integrals in the exponent of $\mathfrak{D}(z;\hat{\delta})$,

$$H(z)=az^{\epsilon}\Big[\prod_{i=1}^{n_d}(z-\nu_{di})\Big/\prod_{i=1}^{n_u}(z-\nu_{ui})\Big]\mathfrak{D}(z;\delta), \tag{3.4}$$

where a is a constant. Of course, $\mathfrak{D}(z;\delta)$ has no zeros or poles, except possibly at infinity. Application of the theorem (3.1) to (3.4) gives

$$H(z)=H(\nu_0)+(z-\nu_0)\Bigg[A+\sum_{i=1}^{n_u}\frac{c_{ui}}{\nu_{ui}-z}+\frac{1}{\pi}\int_0^{\infty}\frac{d\nu\,\mathrm{Im}H(\nu)}{(\nu-\nu_0)(\nu-z)}\Bigg], \tag{3.5}$$

where $H(\nu_0)$ is real, and A, c_{ui}, and $\mathrm{Im}H(\nu)$ are all greater than or equal to zero. $\alpha(\nu)$ in (3.1) clearly contains jumps which produce the poles, while its continuously differentiable part $\alpha_c(\nu)$ is related to $\mathrm{Im}H(\nu)$ by $\mathrm{Im}H(\nu)d\nu=(1+\nu^2)d\alpha_c(\nu)$.

Although the integral representation (3.5) is the sort needed in finding the integral equation, $H(z)$ as it stands is not a suitable denominator function because it does not have particle zeros. But now consider Class B amplitudes, which are defined as those for which $n_d+\epsilon\geq n_b$. For such amplitudes, zeros at the points ν_{di} can be traded for particle zeros; i.e., $H(z)$ can be multiplied by the rational function

$$S(z)=\Big[\prod_{i=1}^{n_b}(z-\nu_{bi})\Big/z^{\epsilon}\prod_{i=1}^{n_b-\epsilon}(z-\nu_{di})\Big], \tag{3.6}$$

and the product has the particle zeros but no new poles. Moreover, $S(z)H(z)$ has a representation like (3.5), as will be proved presently. In general, $S(z)H(z)$ is still not quite a suitable D function, since it may involve a greater number of poles than is necessary to allow a representation of the type of (3.5). This comes about if $n_d+\epsilon>n_b$, in which case a number $n_d+\epsilon-n_b$ of poles may be removed without changing the asymptotic behavior. One multiplies by a second rational function

$$T(z)=\Big[\prod_{i=1}^{n_d+\epsilon-n_b}(z-\nu_{ui})\Big/\prod_{i=n_b+1-\epsilon}^{n_d}(z-\nu_{di})\Big]. \tag{3.7}$$

Now let

$$D(z)=bS(z)T(z)H(z)=U(z)H(z), \tag{3.8}$$

where b is a constant chosen to make $D(\nu_0)=1$. The rational function $U(z)$ has the asymptotic behavior

$$U(z)=b+O(|z|^{-1}), \tag{3.9}$$

uniformly in direction. Equation (3.9) allows one to prove that $D(z)$ has an integral form

$$D(z)=1+(z-\nu_0)\Bigg[A'+\sum_{i=1}^{n}\frac{c_{ui}'}{\nu_{ui}-z}+\frac{1}{\pi}\int_0^{\infty}\frac{d\nu\,\mathrm{Im}D(\nu)}{(\nu-\nu_0)(\nu-z)}\Bigg]. \tag{3.10}$$

The constants A' and c_{ui}' are no longer positive semidefinite, and neither is $\mathrm{Im}D(\nu)$. The locations ν_{ui} of the poles have been relabeled, and their number is $n=n_u-n_d-\epsilon+n_b$. To prove (3.10), apply Cauchy's integral theorem to $U(z)H(z)/(z-\nu_0)$, and take note of (3.5). By (3.9), integration of the A term over the circle at infinity gives the A' term of (3.10) with $A'=bA$. The fourth term of (3.5), call it $\psi(z)$, contributes nothing to the integral over the infinite circle, since an application of Cauchy's theorem to (3.5) by itself shows that

$$\int_{\gamma}\frac{dz'\,\psi(z')}{(z'-\nu_0)(z'-z)} \tag{3.11}$$

tends to zero as the radius of the circle γ increases. The same is true of

$$\int_{\gamma}\frac{dz'\,U(z')\psi(z')}{(z'-\nu_0)(z'-z)}, \tag{3.12}$$

because it follows from (3.4) and (2.4) that $\psi(z)/(z-\nu_0)=O(|z|^{\epsilon})$, any $\epsilon>0$, uniformly in direction. Thus, the part of $U(z)$ which is $O(|z|^{-1})$ gives a vanishing contribution to (3.12), while the other part gives a constant times (3.11). Finally, the first and third terms of (3.10) appear without difficulty, and so does the fourth term, since (3.5) and the bound $\mathrm{Im}H/(\nu-\nu_0)=O(\nu^{\epsilon})$, any $\epsilon>0$, imply the convergence of the integral.

The number n of poles in (3.10) can be related to the value of $p=\pi^{-1}\delta(\infty)$. Because of the continuity of the phase shift $\delta(\nu)$ it is easy to see that

$$p=n_u-n_d-\epsilon+n_{\infty}, \tag{3.13}$$

where $n_{\infty}=1$ if $\delta(\nu)$ approaches its limit from below, and $n_{\infty}=0$ otherwise. Therefore,

$$\delta(\infty)=(-n_b+n_c)\pi, \tag{3.14}$$

where $n_c=n+n_{\infty}$. n_c will be called the "number of CDD poles." n_{∞} represents the number of CDD poles at infinity, since the A' term of (3.10) is what is meant by a CDD pole at infinity, and according to (3.4), (3.1), (3.13), and (2.4), the A' term is present or absent according as n_∞ is one or zero. Equation (3.14) is the form of Levinson's theorem for Class B amplitudes.

In order to motivate the derivation of (3.10), let us now find the integral equation satisfied by $\operatorname{Im}D(\nu)$. The method is a slight improvement of the method of Sec. VI of FW.[23] Define the function $\Lambda(z)=\Lambda^*(z^*)$ as follows:

$$\Lambda(z)=\frac{N(z)}{z-\nu_0}-\frac{B(z)D(z)}{z-\nu_0}+\frac{z-\nu_0}{\pi}\int_0^\infty \frac{d\nu\,\operatorname{Im}D(\nu)}{(\nu-\nu_0)(\nu-z)}\frac{\operatorname{Re}B(z)}{\nu-\nu_0}. \quad (3.15)$$

$B=B^U+B^I$ is defined in (2.7). By direct computation using (2.1) it follows that Λ has zero discontinuity over both right and left cuts. Then since Λ is uniformly bounded by a polynomial, it must be a rational function. Moreover, it behaves at worst as a constant at infinity. This follows because the bounds $B(\nu)=O(\ln\nu)$ [obtained from (2.16)] and $D(\nu)=O(\nu^{1+\epsilon})$, any $\epsilon>0$, allow one to conclude that $\Lambda(\nu)=O(\nu^\epsilon)$, any $\epsilon>0$. By reading off the positions and residues of the poles of Λ from (3.15) one finds that

$$\Lambda(\nu)=\frac{a}{\nu-\nu_0}-\sum_{i=1}^{n} C(\nu_u)\frac{c_{ui}'}{\nu_{ui}-\nu}+\lim_{\nu\to\infty}\Lambda(\nu). \quad (3.16)$$

After taking the real part of (3.15) and introducing (3.16), some rearrangement leads to the equation

$$\eta(\nu)n(\nu)=\frac{a+C(\nu)}{\nu-\nu_0}+A[C(\nu)-\lim_{\nu\to\infty}C(\nu)]-\sum_{u=1}^{n}c_{ui}\frac{C(\nu_{ui})-C(\nu)}{\nu_{ui}-\nu}+\lambda+\frac{\nu-\nu_0}{\pi}\int_0^\infty d\nu'\left[\frac{\nu'}{\nu'+1}\right]^{1/2}n(\nu')\frac{1}{\nu'-\nu}\times\left[\frac{C(\nu')}{\nu'-\nu_0}-\frac{C(\nu)}{\nu-\nu_0}\right], \quad (3.17)$$

provided $\lim_{\nu\to\infty}C(\nu)$ exists and is finite. The primes of c_{ui}', A' have been dropped, and λ is defined by

$$\lambda=\lim_{\nu\to\infty}\left\{\eta(\nu)n(\nu)-\frac{\nu-\nu_0}{\pi}\int_0^\infty d\nu'\left[\frac{\nu'}{\nu'+1}\right]^{1/2}n(\nu')\frac{1}{\nu'-\nu}\times\left(\frac{C(\nu')}{\nu'-\nu_0}-\frac{C(\nu)}{\nu-\nu_0}\right)\right\}.$$

[23] The improvement amounts to eliminating conditions such as [VI. 5(a), (b)] of FW. The integral in (3.15) certainly converges, while extra conditions had to be imposed on ReB to guarantee convergence of the analogous integral of FW; cf. Eq. (3.18) ff.

Equation (3.17) cannot be of Fredholm type unless $\lambda=0$ and the coefficient of A vanishes sufficiently rapidly at infinity. It will be noticed that the kernel of (3.17) is slightly different from the kernel of (2.6). However, the demands on $C(\nu)$ to satisfy the square-integrability requirements are nearly the same for both kernels. An equation like (3.17) but with the same kernel as (2.6) can be derived by beginning with

$$\Sigma(z)=N(z)-B(z)D(z)+\frac{1}{\pi}\int_0^\infty \frac{d\nu\,\operatorname{Im}D(\nu)\operatorname{Re}B(\nu)}{(\nu-\nu_0)(\nu-z)}. \quad (3.18)$$

Equation (3.18) seems to place stronger requirements on ReB than are necessary for (3.15), since one must guarantee the convergence of the integral.

Let Class C consist of all amplitudes for which $n_d+\epsilon<n_b$. In Class C, passing from the Herglotz function $H(z)$ to a proper $D(z)$ having all particle zeros involves the introduction of poles not present in $H(z)$. If $m=n_b-n_d-\epsilon$, the function (3.6) is to be replaced by

$$S(z)=\left[\prod_{i=1}^{n_b}(z-\nu_{bi})/z^\epsilon\prod_{i=1}^{n_d}(z-\nu_{di})\prod_{i=1}^{m}(z-\nu_{vi})\right]. \quad (3.19)$$

The positions ν_{vi} of the new poles are most conveniently taken to be real points in the interval $(-1,0)$.[24] Furthermore, the integral equation will have the simplest form if the poles occur at points where the amplitude $A(\nu)$ vanishes. Appropriate zeros of A are available if, as one usually assumes, the residues of the particle poles have a definite sign: $A(\nu)\approx R_i/(\nu_{bi}-\nu)$, $R_i>0$. In this case $A(\nu)$ has a zero between any two bound state poles, or n_b-1 in all. At most $n_b-\epsilon$ zeros are needed. If $\epsilon=0$, there is also a zero between the highest bound-state pole and a point just above threshold, since Re$A(\nu+0)>0$. Thus, $n_b-\epsilon$ zeros are always present.

The D function is now

$$D(z)=bS(z)H(z), \quad (3.20)$$

since the zeros at the ν_{di} are all used up and, therefore, $T(z)$ does not enter. $D(z)$ has a representation like (3.10), but now the pole term is

$$\sum_{i=1}^{n_u}\frac{c_{ui}'}{\nu_{ui}-z}+\sum_{i=1}^{m}\frac{c_{vi}'}{\nu_{vi}-z}. \quad (3.21)$$

Since $N(z)=A(z)D(z)$ does not have poles at the point ν_{vi}, the deduction of the integral equation is the same as before. The only change in Eq. (3.17) is the replacement of the third term on the right by

$$-\sum_{i=1}^{n_u}c_{ui}\frac{C(\nu_{ui})-C(\nu)}{\nu_{ui}-\nu}-\sum_{i=1}^{m}c_{vi}\frac{C(\nu_{vi})-C(\nu)}{\nu_{vi}-\nu}. \quad (3.22)$$

[24] For convenience, all particle poles are assumed to lie in this region also.

If the definition of CDD pole is broadened to include all of the poles (3.21), then the relation

$$\delta(\infty)=(-n_b+n_c)\pi \tag{3.14a}$$

still holds, with $n_c=n_u+m+n_\infty$ being the number of CDD poles.

The extra poles at the points ν_{vi} will be called CDD poles of the second kind. Whether they can be interpreted physically is a question that calls for further work. One might inquire as to whether these poles could be associated with discrete levels of an unperturbed Hamiltonian, along lines suggested by Dyson's work.[13] Class C amplitudes are certainly a mathematical possibility in the sense that examples can be constructed in which there are n_b poles with negative residues in the region $(-1, 0)$ and for which $n_d+\epsilon<n_b$ holds. One such example is given in Sec. 5.

A case in which a Class C amplitude might arise is that in which there is a Born pole. For simplicity, suppose that the Born pole is the only pole of the amplitude, and that there are no first-kind CDD terms. If the amplitude is such that $n_d+\epsilon\geq 1$, the Born term can be "bootstrapped,"[11] i.e., the amplitude can be obtained from the Eq. (2.6) without CDD poles, and the Born pole appears as a zero of D. On the other hand, it seems not out of the question that $n_d+\epsilon<1$, in which case the Born pole cannot be bootstrapped in the usual way. It is then a particle pole of the second kind and a second-kind CDD pole at a point ν_{vi} would enter. Although analysis of simple models has led to the opinion that Born poles should be produced in the bootstrap manner, no general reason for this to be true is given.

4. THE SPIN-0 – SPIN-$\frac{1}{2}$ CASE

With very little elaboration the preceding analysis carries over to the scattering of a spin-0 particle of mass μ by a spin-$\frac{1}{2}$ particle of mass m. Every step of the previous argument has its analog provided one works in the complex plane of w, the energy in the center-of-mass frame. According to Frazer and Fulco,[25] this means working simultaneously with both orbital angular momentum states associated with a given total angular momentum. The necessary N/D formalism is detailed in FW. The Levinson theorem turns out to be

$$\delta_{l+}(\infty)+\delta_{(l+1)-}(\infty)=(-n_b+n_c)\pi, \tag{4.1}$$

where $\delta_{l\pm}(w)$ is the phase shift for $J=l\pm\frac{1}{2}$. n_b (resp. n_c) is the number of particle poles (resp. CDD poles) in the state of total angular momentum $J=l+\frac{1}{2}$. Equation (4.1) holds under conditions parallel to those outlined at the end of Sec. 2.

(i) The partial wave dispersion relation (II.7) of FW holds.

(ii) The N/D integral equation (II.9) of FW is of Fredholm type with nonzero Fredholm determinant.

(iii) At high energies the absorption factors $\eta_{l+}(w)$ and $\eta_{(l+1)-}(w)$ satisfy the following conditions:

$$A_+ \ln^{-\epsilon_+-\frac{1}{2}}w\leq\eta_{l+}(w)\leq B_+ \ln^{-\epsilon_+}w,$$
$$A_- \ln^{-\epsilon_- -\frac{1}{2}}w\leq\eta_{(l+1)-}(w)\leq B_- \ln^{-\epsilon_-}w,$$
$$\epsilon_+,\quad \epsilon_-\geq 0.$$

(iv) $\sin\delta_{l+}(w)$ and $\sin\delta_{(l+1)-}(w)$ have at most a finite number of zeros.

In the complex plane of $s=w^2$ it is possible to write a dispersion relation for each orbital state separately. Does this mean that it is possible to prove a Levinson theorem for separate orbital states? In the s plane formulation, part of the method of Sec. 2 does not work; viz., the proof by means of (2.16) that the phase shift approaches an integral multiple of π. The powers of s that enter are such as to prevent an asymptotic analysis by methods similar to those used in (2.18), (2.19), and (2.21). However, if it is known that the phase shifts do approach integral multiples of π (say by satisfaction of (i) through (iv) above), one can then assume the analogs of conditions (i), (ii), and (iii) in the s-plane formulation and arrive at the statement

$$\delta_{l\pm}(\infty)=(-n_b+n_c)\pi. \tag{4.2}$$

Now the numbers on the right refer to the orbital state l with $J=l\pm\frac{1}{2}$. Requirements for the validity of (4.2) may very well be essentially stronger than those for (4.1). The jump over the unphysical cuts in the s plane dispersion relation is a function different from the corresponding function in the w plane setup. Therefore, the s-plane analogs of conditions (i) and (ii) may represent stronger restrictions than (i) and (ii) themselves.

5. ANALYSIS OF A MODEL

It is informative to see how things work out in a model in which the contribution of the left cut in (2.2) is represented by a few poles, and inelastic effects are neglected. As is well known, the integral equations are then trivially soluble. It is easiest to treat the spin-0 case and the following dispersion relation:

$$A(z)=\frac{1}{\pi}\int_{-\infty}^{-1}\frac{\mathrm{Im}A(\nu)d\nu}{\nu-z}+\frac{1}{\pi}\int_{0}^{\infty}\frac{\mathrm{Im}A(\nu)d\nu}{\nu-z}+\sum_i\frac{r_i}{\nu_{bi}-z}. \tag{5.1}$$

The subtraction introduced in Eq. (2.2) will not be necessary here, since with poles substituted for the first term on the right side, (5.1) has solutions which illustrate the main points of interest. Consider the class of amplitudes $A(\nu)$ for which there is an N/D

[25] W. R. Frazer and J. R. Fulco, Phys. Rev. **119**, 1420 (1960).

representation such that N is free of particle poles and N and $D-1$ satisfy dispersion relations without subtractions:

$$N(z)=\frac{1}{\pi}\int_{-\infty}^{-1}\frac{\mathrm{Im}A(\nu)D(\nu)d\nu}{\nu-z}, \tag{5.2}$$

$$D(z)=1-\frac{1}{\pi}\int_0^\infty d\nu\left[\frac{\nu}{\nu+1}\right]^{1/2}\frac{N(\nu)}{\nu-z}. \tag{5.3}$$

Substitution of (5.3) in (5.2) yields an equation similar to (2.6).

$$N(\nu)=G(\nu)+\frac{1}{\pi}\int_0^\infty d\nu'\left[\frac{\nu'}{\nu'+1}\right]^{1/2}\times\left(\frac{G(\nu)-G(\nu')}{\nu-\nu'}\right)N(\nu'), \tag{5.4}$$

where

$$G(z)=\frac{1}{\pi}\int_{-\infty}^{-1}\frac{\mathrm{Im}A(\nu)d\nu}{\nu-z}. \tag{5.5}$$

With the simplest model of (5.5), viz.,

$$G(z)=\alpha/(\bar{\nu}-z),\quad \alpha \text{ real},\quad \bar{\nu}<0, \tag{5.6}$$

the integral Eq. (5.4) shows immediately that $N(\nu)=\beta/(\bar{\nu}-\nu)$, where β is a real constant. This is also apparent from (5.2), since on the left cut $\mathrm{Im}A(\nu)=\alpha\delta(\bar{\nu}-\nu)$. Suppose β is chosen to produce a "bound-state" zero of D at a point ν_b. By (5.3), that requires $\beta<0$. α is then determined by (5.4). Since the integral in D is monotonic decreasing for $\nu<0$, D has only one zero. To verify Levinson's relation, note that $\mathrm{Re}D<0$ just above threshold, and that $N>0$ at all points above threshold. Moreover, $\mathrm{Re}D\to 1$ as $\nu\to\infty$. Thus, $\mathrm{Re}D$ has at least one zero for $\nu>0$, and one may verify that it has no more than one. Now

$$\tan\delta(\nu)=-\frac{\mathrm{Im}D(\nu)}{\mathrm{Re}D(\nu)}=\left[\frac{\nu}{\nu+1}\right]^{1/2}\frac{N(\nu)}{\mathrm{Re}D(\nu)}. \tag{5.7}$$

Therefore, $\delta(\nu)$ is negative just above threshold, it goes through $-\pi/2$ where $\mathrm{Re}D=0$, and it eventually approaches $-\pi$. Thus, $\delta(\infty)=-n_b\pi$ as expected. There are no ghost zeros of D, since by (2.4) and (2.5) that would be incompatible with $\delta(\infty)=-\pi$ and $D=\Phi\mathfrak{D}\to 1$. It is easy to check that the residue $r=-N(n_b)/D'(\nu_b)$ of the bound-state pole is positive provided the "interaction" pole at $\bar{\nu}$ is to the left of ν_b.

The integral Eq. (5.4) has nonzero Fredholm determinant, since the corresponding homogeneous equation has no nontrivial solution. The condition for a nontrivial solution of the homogeneous equation is

$$\frac{1}{\alpha}=\frac{1}{\pi}\int_0^\infty d\nu\left[\frac{\nu}{\nu+1}\right]^{1/2}\frac{1}{(\bar{\nu}-\nu)^2},$$

or $1/\beta=0$. This is incompatible with $D(\nu_b)=0$. In Sec. 2, the Fredholm determinant of the slightly different Eq. (2.6) was assumed not to vanish. However, in the present example the Eqs. (5.4) and (2.6) actually have the same kernel. Since

$$\begin{aligned}G(\nu)&=\frac{1}{\pi}\int_{-\infty}^{-1}\frac{\mathrm{Im}A(\nu')d\nu'}{\nu'-\nu}\\&=G(\nu_0)+\frac{(\nu-\nu_0)}{\pi}\int_{-\infty}^{-1}\frac{\mathrm{Im}A(\nu')d\nu'}{(\nu'-\nu_0)(\nu'-\nu)}\\&=G(\nu_0)+C(\nu),\end{aligned}$$

one has $G(\nu')-G(\nu)=C(\nu')-C(\nu)$.

A CDD pole may be added to the amplitude just constructed. Add the term $c/(\nu_u-\nu)$, $\nu_u>0$, $c<0$, to the right-hand side of (5.3). If $|c|$ is taken to be small and ν_u large, the effect of the CDD pole in the bound-state region and not too far above threshold will be negligible. The initial behavior of the phase shift will also be the same as before; viz., δ falls from zero through $-\pi/2$ into the region $-\pi<\delta<-\pi/2$. But then, as the CDD pole is approached, another zero of $\mathrm{Re}D$ occurs. This resonance is the "unstable particle" associated with the CDD pole. Thus, the phase swings back through $-\pi/2$, and it then has a zero at the position of the CDD pole. Finally, it approaches zero from above, so that Levinson's relation is verified: $\delta(\infty)=(-n_b+n_c)\pi=(-1+1)\pi=0$. If ν_u is taken to be smaller, the pole may be approached before the remainder of $\mathrm{Re}D$ has a zero. In that case there is no zero of $\mathrm{Re}D$ at all. δ is initially negative, it has a zero at the pole, and it eventually approaches zero from above. If the residue c is taken to be positive, a pair of complex ghosts appears. δ tends to -2π. Since $D=\Phi\mathfrak{D}/(\nu_c-\nu)\to 1$, Eq. (2.4) shows that the polynomial Φ has three zeros, two of them being ghosts. The correct sign of c makes $-D$ a Herglotz function and, therefore, disallows complex zeros. The incorrect sign of c not only gives ghosts, it also makes the phase go down rather than up through a multiple of π.

To construct a Class C amplitude, take $A=N/D$, where

$$N(z)=\frac{\beta_1}{\nu_{b1}-z}+\frac{\beta_2}{\nu_{b2}-z},$$

$$D(z)=1-\frac{1}{\pi}\int_0^\infty d\nu\left[\frac{\nu}{\nu+1}\right]^{1/2}\frac{N(\nu)}{\nu-z},$$

and $\beta_1,\beta_2>0$, $\nu_{b1},\nu_{b2}<0$. D obviously has no real zeros; and since it is Herglotz, it also has no complex zeros. The residues of the two poles are negative, so the poles may be reasonably associated with particles. Since D has no zeros or poles, it is a constant times $\mathfrak{D}$. Equation (2.4) then implies $\delta(\infty)=0$, because $D\to 1$. The phase shift is negative just above threshold, so the quantity ϵ of Sec. 3 is equal to one. $\tan\delta$ has no zeros for $0<\nu<\infty$, so $n_d=0$. Since $n_b=2$, the condition $n_d+\epsilon<n_b$ for a

Class C amplitude is fulfilled. It should be noted that for simplicity no interaction poles were included in the example above. Such poles may be introduced without altering the Class C property of the amplitude. For example, a term $\beta/(\bar{\nu}-z)$, $\beta>0$, $\bar{\nu}<-1$, may be added to N and interpreted as an interaction pole.

6. UNIQUENESS OF SOLUTION OF THE DISPERSION RELATION FOR HIGH l

When the argument of Sec. V of FW is adapted to the spin-0 case, the following theorem is obtained. Theorem: If there are two distinct solutions $A^{(1)}(\nu)$, $A^{(2)}(\nu)$ of the dispersion relation (2.2), then[26]

$$\pi^{-1}\delta^{(i)}(\infty)+n_b^{(i)}\geq\tfrac{1}{2}l-1\,,\quad i=1,2\,. \tag{4.1}$$

Thus, if there is any solution which contradicts the inequality (4.1), it is the only solution. For amplitudes of Class A, $\pi^{-1}\delta^{(i)}(\infty)+n_b^{(i)}=0$. Thus, whenever a Class A solution of (2.2) with $l\geq3$ exists, it is the only solution of any class with angular momentum l. Moreover, one can easily show that any solution of (2.2) obtained from the integral Eq. (2.6) is of Class A. That follows from the asymptotic behavior of D implied by the square-integrability of solutions of (2.9). The only way that uniqueness can fail for $l\geq3$ is for amplitudes constructed from (2.6) to involve ghost poles.

The analogous result for spin-0 — spin-$\frac{1}{2}$ scattering is that if the amplitude constructed from a solution of Eq. (III.7) of FW has no ghosts, it is a unique solution of the dispersion relation if $J\geq\frac{3}{2}$.

A statement can be made about uniqueness within the class of solutions having some fixed number of particle poles. If there is a (spin-0) amplitude with n_b particle poles which contradicts

$$\pi^{-1}\delta(\infty)\geq\tfrac{1}{2}l-\tfrac{1}{2}-n_b\,,$$

it is the only amplitude with n_b particle poles.

ACKNOWLEDGMENT

I enjoyed a conversation with Dr. J. L. Uretsky in which I learned that he had also noticed an "extra" ambiguity in computing bound states from given unphysical singularities.

[26] This theorem and the following remarks apply only to the class of solutions for which the following conditions hold: (i) $\delta(\nu)$ tends to a constant $\delta(\infty)$; (ii) $A(\nu)$ has the proper threshold behavior ($A\propto\nu^l$ at $\nu=0$); (iii) the residues of the particle poles are all negative ($A=R_i/(\nu_{bi}-\nu)$, $R_i>0$ for ν near ν_{bi}).

APPENDIX

The lemma of Eq. (2.17) was proved in Appendix D of FW under the restriction $\alpha<0$. For general α one may begin the proof in the same way with the decomposition

$$t\int_{x_0}^{\infty}\frac{\varphi(x)dx}{t-x}=\int_{x_0}^{t(1+\epsilon)}\varphi(x)dx+\int_{x_0}^{t(1-\epsilon)}\frac{x\varphi(x)dx}{t-x}+\int_{t(1-\epsilon)}^{t(1+\epsilon)}\frac{x\varphi(x)dx}{t-x}+t\int_{t(1+\epsilon)}^{\infty}\frac{\varphi(x)dx}{t-x}=I_1+I_2+I_3+I_4.$$

The principal value symbol P is suppressed, and $0<\epsilon<1$, $x_0<t(1-\epsilon)$. Clearly,

$$I_1=\int_{x_0}^{t}\varphi(x)dx+O(\ln^\alpha t).$$

As in FW,

$$|I_2|\leq Mt^{-1}\int_{x_0}^{t(1-\epsilon)}\ln^\alpha x dx=Mt^{-1}J_2\,,$$

and an application of l'Hospital's rule to $J_2/t\ln^\alpha t$ shows that $I_2=O(\ln^\alpha t)$. For I_3 note that

$$I_2=\int_{t(1-\epsilon)}^{t(1+\epsilon)}\left(\frac{\varphi(x)-\varphi(t)}{t-x}\right)xdx+\varphi(t)P\int_{t(1-\epsilon)}^{t(1+\epsilon)}\frac{xdx}{t-x}\,.$$

Since $\varphi'(x)=O(x^{-2}\ln^\alpha x)$ (cf. Ref. 26 of FW), the mean-value theorem shows that the first term is $O(\ln^\alpha t)$. The second term is equal to $-2t\epsilon\varphi(t)=O(\ln^\alpha t)$. I_4 is handled by means of the substitution $x=tu$ which gives

$$|I_4|\leq M\int_{1+\epsilon}^{\infty}\frac{\ln^\alpha tu}{u(u-1)}du$$
$$=M\left[\int_{1+\epsilon}^{t}\frac{du}{u(u-1)}\left(1+\frac{\ln u}{\ln t}\right)^\alpha\ln^\alpha t+\int_{t}^{\infty}\frac{du}{u(u-1)}\left(1+\frac{\ln t}{\ln u}\right)^\alpha\ln^\alpha u\right].$$

For large t, in the first integral $\frac{1}{2}<1+\ln u/\ln t<2$ and in the second integral $1<1+\ln t/\ln u<2$. Therefore, $I_4=O(\ln^\alpha t)$, and the proof is complete.