

Chemical Potential and Low-Lying Excitation in the Many-Body Bose System

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For a Bose system interacting through a two-body potential there is a large occupation number in the zero-momentum state with small fluctuations. Thus, we can express the chemical potential and low-lying excitation energy in terms of the density fluctuation only. The formulas derived are shown to coincide with well-established results in the low-density Bose gas. The use of experimental neutron scattering data is suggested to improve the numerical answer. A particular application is made to the He⁴ case. If we utilize the Feynman variational excited state as the (approximate) true state vector, which is correct experimentally for low-momentum excitations, we get an explicit phonon excitation for actual He⁴.

1. FORMULATION OF THE PROBLEM

THE many-body system of Bose particles interacting through a two-body potential has been analyzed rather exactly in the low-density limit¹ and approximately in the intermediate density range.² In all these theories the essential assumptions are that a considerable fraction of the particles is in the zero-momentum state and that the fluctuation in this fraction is small. Consequently, the annihilation and creation operators for zero-momentum state particles can be replaced by a number, as expressed in the equation

$$a_0^*, a_0 \rightarrow \sqrt{N_0} \quad (\text{some } c \text{ number}); \quad (1)$$

this is the Bogoliubov approximation. In the direct calculation of energies and the state vector the "number" N_0 appears explicitly, but the theoretical estimate of N_0 is difficult except in the low-density limit. Some conjectures³ about N_0 have been made using experimental data, but since these estimates range from 50 to 8% of N (the total number of particles) it remains unspecified.

The aim of this paper is to use the facts that the zero-momentum occupation is considerable and that the fluctuation is small, while avoiding the explicit appearance of N_0 in the theoretical quantities as far as possible.

We express the fact the zero-momentum occupation number is large while the fluctuation is small by the approximate equality

$$|Q\rangle_N \approx \frac{1}{\sqrt{N_0}} a_0^* |Q\rangle_{N-1}, \quad (2)$$

where $|Q\rangle_N$ represents the true Schrödinger wave

¹ T. D. Lee, K. Huang, and C. N. Yang, Phys. Rev. **106**, 1135 (1957); N. M. Hugenholtz and D. Pines, *ibid.* **116**, 489 (1959); K. Sawada, *ibid.* **116**, 1344 (1959); S. T. Beliaev, Zh. Eksperim. i. Teor. Fiz. **34**, 417 (1958) [translation: Soviet Phys.—JETP **7**, 289 (1958)]; K. A. Brueckner and K. Sawada, Phys. Rev. **106**, 1117 (1957).

² K. A. Brueckner and K. Sawada, Phys. Rev. **106**, 1128 (1957) [referred to as B.S.(II)].

³ A. Miller, D. Pines, and P. Nozières, Phys. Rev. **127**, 1452 (1962); O. Penrose and L. Onsager, *ibid.* **104**, 576 (1956); W. E. Parry and D. ter Haar, Ann. Phys. (N. Y.) **19**, 496 (1962).

function with total momentum Q and total particle number N . This equality is to be applied in the calculation of energy expectation values only and will be justified in more detail below. We start by assuming that we can write the state vector in the form

$$|Q\rangle_N = C_Q (a_0^* + B_0) |Q\rangle_{N-1}, \quad (3)$$

with B_0 to be a small correction of order of magnitude $(1/N_0)$.

From

$$H|Q\rangle_N = E_{Q;N}|Q\rangle_N,$$

we have

$$\begin{aligned} E_{Q;N} C_Q (a_0^* + B_0) |Q\rangle_{N-1} \\ = C_Q ([H, a_0^*] + a_0^* H + H B_0) |Q\rangle_{N-1} \\ = C_Q (\sum_q v_q a_q^* \rho_q + a_0^* E_{Q;N-1} + H B_0) |Q\rangle_{N-1}, \end{aligned} \quad (4)$$

where

$$\begin{aligned} H &= \sum_q \frac{q^2}{2m} a_q^* a_q + \frac{1}{2} \sum_{q,k,l} v_q a_{k+q}^* a_l^* a_{l+q} a_k, \\ \rho_q &= \sum_l a_l^* a_{l+q}, \\ v_q &= \frac{1}{\Omega} \int v(\mathbf{r}) e^{i\mathbf{q}\cdot\mathbf{r}} d\mathbf{r}. \end{aligned} \quad (5)$$

Multiplying (4) from the left with ${}_N\langle Q|$ yields an energy equation

$${}_N\langle Q| - (E_{Q;N} - E_{Q;N-1}) a_0^* + \sum_q v_q a_q^* \rho_q |Q\rangle_{N-1} = 0, \quad (6)$$

and ${}_N\langle j|$ ($j \neq Q$) gives ${}_N\langle j| B_0 |Q\rangle_{N-1}$,

$$\begin{aligned} {}_N\langle j| B_0 |Q\rangle_{N-1} &= \frac{1}{E_{Q;N} - E_{j;N}} {}_N\langle j| \left[\sum_q v_q a_q^* \rho_q \right. \\ &\quad \left. - (E_{Q;N} - E_{Q;N-1}) a_0^* \right] |Q\rangle_{N-1}, \end{aligned}$$

Hence,

$$|Q\rangle_N = C_Q \left(a_0^* + \frac{1 - P_Q^N}{E_{Q;N} - H} \left[\sum_q v_q a_q^* \rho_q - (E_{Q;N} - E_{Q;N-1}) a_0^* \right] \right) |Q\rangle_{N-1},$$

with

$$1 = C_Q \cdot {}_N\langle Q| a_0^* |Q\rangle_{N-1} \quad \text{and} \quad P_Q^N = |Q\rangle_N \cdot {}_N\langle Q|. \quad (7)$$

It should be noted that (6) is just an identity because

$$-[H, a_0^*]_- + \sum_q v_q a_q^* \rho_q = 0 \quad (8)$$

is an operator equation. Equation (6) can be generalized in the following way:

$${}_N\langle Q' | - (E_{Q'; N} - E_{Q; N-1}) a_0^* + \sum_q v_q a_q^* \rho_q | Q \rangle_{N-1} = 0, \quad (6')$$

where ${}_N\langle Q' |$ represents the system with total momentum Q' but in a more general state than (7). First, notice

$${}_N\langle Q' | v_0 a_0^* \rho_0 | Q \rangle_{N-1} = {}_N\langle Q' | v_0 (N-1) a_0^* | Q \rangle_{N-1}. \quad (9)$$

Then

$$\begin{aligned} {}_N\langle Q' | [H, a_q^*]_- &= {}_N\langle Q' | a_q^* (E_{Q'; N} - H) \\ &= {}_N\langle Q' | \left(\frac{q^2}{2m} a_q^* + \sum_{q'} v_{q'-q} a_{q'}^* \rho_{q'-q} \right), \end{aligned} \quad (10)$$

$${}_N\langle Q' | a_q^* = {}_N\langle Q' | \sum_{q'} v_{q'-q} a_{q'}^* \rho_{q'-q} \frac{1}{E_{Q'; N} - H - q^2/2m}.$$

Equation (10) holds only when the denominator has no singularity, for which

$$E_{Q'; N} - E_{0; N} + \mu - \frac{q^2}{2m} < 0, \quad (11)$$

$$(\text{chemical potential}) \mu = E_{0; N} - E_{0; N-1},$$

because the energy denominator is

$$\begin{aligned} E_{Q'; N} - H^{N-1} - q^2/2m &= (E_{Q'; N} - E_{0; N}) \\ &+ (E_{0; N} - E_{0; N-1}) - (H^{N-1} - E_{0; N-1}) - q^2/2m \\ &= (E_{Q'; N} - E_{0; N}) + \mu - q^2/2m - (H^{N-1} - E_{0; N-1}), \end{aligned} \quad (12)$$

where H^{N-1} denotes the number of particles is $N-1$. In the actual He^4 system, μ is negative; hence, for low-lying excited states the denominator shows no singularity.

Assuming condition (11) is satisfied for state Q' , we get for (6')

$$\begin{aligned} (6') &= {}_N\langle Q' | - (E_{Q'; N} - E_{Q; N-1}) a_0^* + v_0 (N-1) a_0^* \\ &+ \sum_{q(\neq 0)} \sum_{q'} v_{q'-q} a_{q'}^* \rho_{q'-q} \frac{1}{E_{Q'; N} - H - q^2/2m} v_q \rho_q | Q \rangle_{N-1}. \end{aligned}$$

Separating out a_0^* and repeating the application of (10), we finally get

$$\begin{aligned} (6') = 0 &= {}_N\langle Q' | a_0^* \left\{ - (E_{Q'; N} - E_{Q; N-1}) + v_0 (N-1) \right. \\ &+ \sum_q' v_{-q} \rho_{-q} \frac{1}{E_{Q'; N} - H - q^2/2m} v_q \rho_q \\ &+ \sum_q' \sum_{q'} v_{-q'} \rho_{-q'} \frac{1}{E_{Q'; N} - H - q'^2/2m} v_{q'-q} \rho_{q'-q} \\ &\left. \times \frac{1}{E_{Q'; N} - H - q^2/2m} v_q \rho_q + \dots \right\} | Q \rangle_{N-1}, \\ &(\sum_{q'} = \sum_{q(\neq 0)}). \end{aligned} \quad (13)$$

Now for the state ${}_N\langle Q' |$ let us take the state ${}_N\langle Q |$ given by (7), where

$${}_N\langle Q | a_0^* \approx \sqrt{N} {}_{N-1}\langle Q |$$

holds. Omitting upper primes, we get

$$\begin{aligned} {}_{N-1}\langle Q | - (E_{Q; N} - E_{Q; N-1}) + v_0 (N-1) \\ + \sum_q' v_{-q} \rho_{-q} \frac{1}{E_{Q; N} - H - q^2/2m} v_q \rho_q \\ + \sum_q' \sum_{q'} v_{-q'} \rho_{-q'} \frac{1}{E_{Q; N} - H - q'^2/2m} v_{q'-q} \rho_{q'-q} \\ \times \frac{1}{E_{Q; N} - H - q^2/2m} v_q \rho_q + \dots | Q \rangle_{N-1} = 0. \end{aligned} \quad (14)$$

The remaining terms from (7) give rise to corrections to $(E_{Q; N} - E_{Q; N-1})$ which vanish in the thermodynamic limit ($N, \Omega \rightarrow \infty, N/\Omega = \text{const}$). Equation (14) is the fundamental equation for the discussion which follows.

Before going into the detailed applications of (14) we can give a rather convincing argument that the series in (14) should converge. Expression (14) written in the form

$$\begin{aligned} E_{Q; N} = E_{Q; N-1} + v_0 (N-1) + {}_{N-1}\langle Q | \sum_q' v_{-q} \rho_{-q} \\ \times \frac{1}{E_{Q; N} - H - q^2/2m} v_q \rho_q + \dots | Q \rangle_{N-1} \end{aligned} \quad (15)$$

is the energy of a system with $N-1$ particles in state Q with one distinguishable particle with momentum zero, written as a Brillouin-Wigner series. In fact, writing particle 1 as the distinguishable one, we have

$$\begin{aligned} E_{Q; N} \Psi_Q &= \left(H^{N-1} + \sum_{j=2}^N v(\mathbf{r}_j - \mathbf{r}_1) + \frac{p_1^2}{2m} \right) \Psi_Q, \\ \Psi_Q &= | Q \rangle_{N-1} \frac{1}{\sqrt{\Omega}} + \dots, \end{aligned} \quad (15')$$

$$E_{Q; N} = E_{Q; N-1} + \dots \quad [\text{same as (15)}],$$

where the perturbing potential represents the change caused by the presence of one "impurity." Since we know that the He^3 - He^4 and He^4 - He^4 potentials are nearly the same, and that the He^3 impurities do not cause any phase change in actual He^4 systems (as long as the impurity density is an order of magnitude smaller than the density of He^4), the interaction $\sum_{j=2}^N v(\mathbf{r}_j - \mathbf{r}_1)$ may be considered as a completely local small perturbation. This strongly supports the assertion that the series in (14) [or (13)] converges.

Another way of looking at (15) may be that for the N -particle system, a Born-Oppenheimer type of calculation starting from the state given in (15') can yield accurate energy values.

2. CHEMICAL POTENTIAL OF He⁴

The first application is to calculation of the chemical potential $\mu = E_{0,N} - E_{0,N-1}$ of He⁴. We take $Q=0$ in (15) [or (14)]:

$$E_{0,N} = E_{0,N-1} + v_0(N-1) + {}_{N-1}\langle 0 | \sum'_q \frac{1}{E_{0,N} - H - q^2/2m} v_q \rho_q + \sum'_q \sum'_p \frac{1}{E_{0,N} - H - p^2/2m} v_{p-q} \rho_{p-q} \times \frac{1}{E_{0,N} - H - q^2/2m} v_q \rho_q + \dots | 0 \rangle_{N-1},$$

where $|0\rangle_{N-1}$ now represents the ground state; condition (11) is obviously satisfied. If we expand $E_{0,N}$ in the denominator, we get the Rayleigh-Schrödinger perturbation series:

$$\begin{aligned} \mu = E_{0,N} - E_{0,N-1} = & v_0(N-1) + {}_{N-1}\langle 0 | \sum'_q \frac{1}{E_{0,N-1} - H - q^2/2m} v_q \rho_q \\ & + \sum'_q \sum'_p \frac{1}{E_{0,N-1} - H - p^2/2m} v_{p-q} \rho_{p-q} \\ & \times (v_{p-q} \rho_{p-q} - v_0(N-1)) \frac{1}{E_{0,N-1} - H - q^2/2m} v_q \rho_q \\ & + \dots | 0 \rangle_{N-1}. \end{aligned} \quad (16)$$

As we stated at the end of the last paragraph, the energy (16) $E_{0,N}$ can be obtained as a change in the energy of the ground state of an $N-1$ particle system with one "impurity" atom added. The interaction due to the "impurity" can be assumed to be local. Hence, (16) represents the forward scattering amplitude of the "impurity" atom in the He⁴ medium and can be assumed to be represented by the diagonal element of some scattering matrix.

We shall not attempt the general treatment here but shall be satisfied by a second-order estimate based on a requirement of self-consistency. Equation (16) can be rewritten using Van Hove's⁴ correlation function $S(q, \omega)$:

$$S(q, \omega) = \sum_\beta |\langle \beta | \rho_q | 0 \rangle|^2 \delta(\omega_{\beta 0} - \omega), \quad (17)$$

$$\omega_{\beta 0} = E_\beta - E_0,$$

$$\begin{aligned} \mu = & v_0(N-1) + \sum'_q v_q v_{-q} \\ & \times \int_0^\infty \frac{S(q, \omega)}{-\omega - q^2/2m} d\omega + O(v^3). \end{aligned} \quad (18)$$

⁴L. Van Hove, Phys. Rev. **95**, 249 (1954).

We show in the Appendix that, using Bogoliubov states for the states $\langle \beta |$ and $|0\rangle$, we can reproduce the well-known formula for the energy in the low-density limit by integrating (18) over N .

Since we assume that μ can be represented by some scattering matrix, the form directly suggested by (18) is

$$t_q = v_q + \sum_p \frac{1}{v_{q-p} - W_p} t_p, \quad (19)$$

$$\frac{1}{-W_p} = \int_0^\infty \frac{1}{(N-1)} \frac{S(p, \omega)}{-\omega - p^2/2m} d\omega.$$

Hence, expanding v 's in terms of t 's by using (19),⁵

$$\mu = t_0(N-1) + O(t^3). \quad (20)$$

We can check whether or not this simple approximation which discards all higher orders in t is valid by comparing μ with the experimental value for Nt_0 . We can obtain t_0 by using the experimental neutron scattering data for $S(q, \omega)$ and solving (19).

We may also remark at this point that since experimentally⁶ we know that the small q behavior of $S(q, \omega)$ is

$$S(q, \omega) = (N-1) \frac{q^2/2m}{\bar{\omega}_q} \delta(\omega - \bar{\omega}_q), \quad (21)$$

we have

$$\begin{aligned} \frac{1}{-W_{q \rightarrow 0}} &= \frac{q^2/2m}{\bar{\omega}_q(-\bar{\omega}_q - q^2/2m)} = -\frac{1}{2mc^2} \\ \left(\frac{1}{-W_{q \rightarrow \infty}} \right) &= \frac{1}{-2(q^2/2m)} \end{aligned} \quad (22)$$

(where $\bar{\omega}_q$ is the phonon energy:

$$\bar{\omega}_{q \rightarrow 0} = c \cdot q, \quad c: \text{sound velocity}).$$

This indicates that the energy denominator of t (19) behaves just as given in B.S.(II)² (where $c^2 = N\alpha/m$; $1/-W_{q \rightarrow 0} = 1/-2N\alpha$).

 3. LOW-LYING EXCITED STATES OF He⁴

Exact treatment of the excited states is, of course, not possible, but we shall exhibit how the excitation spectrum can be derived by using *one assumption: the Feynman state* $\rho_{-q}|0\rangle$ is approximately the true excited state for a system obeying Bose statistics.

The energy of an excited state of momentum Q is given by (15), so that writing

$$E_{Q,N} = \omega_{Q,N} + \mu + E_{0,N-1}, \quad E_{Q,N-1} = \omega_{Q,N-1} + E_{0,N-1},$$

⁵ Similar expansions have been used by K. Sawada, Phys. Rev. **116**, 1344 (1959) in the Bose case; and by Yih Pwu, thesis, University of California, Berkeley, 1961 (unpublished) in the Fermi case.

⁶ D. G. Henshaw and A. D. B. Woods, Phys. Rev. **121**, 1266 (1961).

⁷ R. P. Feynman, Phys. Rev. **94**, 262 (1954); also see Ref. 6.

and expanding the $E_{Q,N}$ in the denominator, we get the Rayleigh-Schrödinger series

$$\frac{d\omega_{Q,N}}{dN} + \mu = v_0(N-1) + \sum'_q v_q v_{-q} \sum_\beta \frac{|\langle \beta | \rho_{-q} | Q \rangle|^2}{E_{Q,N-1} - E_{\beta,N-1} - q^2/2m} + \dots \quad (23)$$

[Condition (11) is satisfied as long as $\omega_Q < -\mu$.]

To reduce this we write our assumption explicitly. The assumption that the Feynman state is approximately the true state is written

$$\rho_{-q} | 0 \rangle = N_q | q \rangle, \quad (24)$$

where the state $|q\rangle$ has an, as yet unknown, energy $\bar{\omega}_q$. From the sum rule

$$\sum_\beta |\langle \beta | \rho_{-q} | 0 \rangle|^2 \omega_{\beta 0} = (q^2/2m)(N-1),$$

we get

$$N_q^2 = (N-1) \frac{q^2}{2m \bar{\omega}_q}. \quad (25)$$

Next, we note that the excitation obeys Bose statistics:

$$\left. \begin{aligned} \langle \beta | \rho_{-q} | Q \rangle &= N_q \delta_{\beta, q+Q}, & q \neq Q \\ &= \sqrt{2} N_q \delta_{\beta, q+Q}, & q = Q \\ &= N_q \delta_{\beta, 0}, & q = -Q \end{aligned} \right\} \beta: \text{two-quantum state;} \\ \beta: \text{ground state.} \quad (26)$$

[$S(q, \omega)$ behaves as given in (21).] Putting (26) into (23) and expanding v in terms of t (19), we have ($\omega_{Q,N} = \bar{\omega}_Q$)

$$\begin{aligned} \frac{d\bar{\omega}_Q}{dN} &= t_Q t_Q \left(\frac{N_Q^2}{\bar{\omega}_Q - Q^2/2m} + \frac{N_Q^2}{-\bar{\omega}_Q - Q^2/2m} \right) \\ &= t_Q^2 \frac{N_Q^2/2m}{\bar{\omega}_Q} \frac{-Q^2/m}{-\bar{\omega}_Q^2 + (Q^2/2m)^2}, \end{aligned} \quad (27)$$

which reduces to

$$\frac{d}{dN} \left[-\frac{1}{4} \bar{\omega}_Q^4 + \frac{1}{2} \bar{\omega}_Q^2 \left(\frac{Q^2}{2m} \right)^2 \right] = -2N t_Q^2 \left(\frac{Q^2}{2m} \right)^2. \quad (28)$$

Integrating with the condition $\bar{\omega}_Q|_{N \rightarrow 1} = Q^2/2m$ gives

$$\begin{aligned} \bar{\omega}_Q^4 - 2\bar{\omega}_Q^2 \left(\frac{Q^2}{2m} \right)^2 + \left(\frac{Q^2}{2m} \right)^4 \\ - \int_1^N 2N' t_Q^2 (N') dN' \cdot \left(\frac{Q^2}{m} \right)^2 = 0, \\ \bar{\omega}_Q^2 = \left(\frac{Q^2}{2m} \right)^2 \pm \frac{Q^2}{m} \left(\int_1^N 2N' t_Q^2 (N') dN' \right)^{1/2}. \end{aligned} \quad (29)$$

The assumption (24) (26) is experimentally well verified for small q . In view of this and $\bar{\omega}_Q^2 > 0$, we take the plus sign in (29), assuming Q is small.

Hence ($\bar{\omega}_Q$ real positive),

$$\bar{\omega}_Q = \left\{ \left(\frac{Q^2}{2m} \right)^2 + \frac{Q^2}{m} \left(\int_1^N 2N' t_Q^2 (N') dN' \right)^{1/2} \right\}^{1/2}. \quad (30)$$

Neglecting the N' dependence of $t_Q(N')$ we get

$$\bar{\omega}_Q \approx \left\{ \left(\frac{Q^2}{2m} \right)^2 + \frac{Q^2}{m} |t_Q| N \right\}^{1/2}. \quad (31)$$

The absolute value $|t_Q|$ in (31) which appears in this treatment results from the assumption that $\rho_{-q}|0\rangle$ is the excited state; this approach may be used even for $t_0 < 0$. Obviously, if we use Bogoliubov states for the state vectors $|0\rangle$, $|Q\rangle$, and $\langle \beta |$, this equation follows in exactly the same way and reproduces the well-known excitation formula.

Finally, we note that since the chemical potential μ is known from (20), the compressibility can be obtained by using

$$dp/dn = n d\mu/dn \quad (n = N/\Omega; p: \text{pressure}).$$

The classical relation

$$(\text{phonon velocity})^2 = (1/m) n d\mu/dn$$

does not agree formally with that derived from (30),

$$c^2 = \frac{1}{m} \left\{ \int_0^N 2N' t_Q^2 (N') dN' \right\}^{1/2}.$$

Of course, the classical relation only holds if there is an undamped phonon excitation which exhausts the sum rule. Since in deriving (30) we assumed that the Feynman state has this property, these two velocity expressions should have nearly the same (numerical) value, at least with respect to expansions in the t matrix. (We can check for low orders in t that both expressions coincide, at least where $t_0 > 0$.)

4. CONCLUSION

The assumption of large occupation and small fluctuation in the zero-momentum state enables us to reduce the calculation of the chemical potential to the calculation of the forward scattering amplitude of a zero-momentum "impurity" atom. This physical picture also applies to part of the energy difference between the excited states of N and $N-1$ particle systems. We can now incorporate experimental scattering data into the theoretical framework. These data exist at present for neutron scattering only. Use of the neutron scattering data (which might serve to check the consistency of our approximation) has been suggested in connection with the calculation of the chemical potential. In the same approximation, the t matrix has been shown to have the characteristics given in B.S.(II).² The difficulty which arises in the usual Bogoliubov calculation in the case of a negative scattering length at zero

momentum is avoided by assuming that the Feynman variational state is approximately the true state for small q . This assumption seems to be well borne out by experimental results.

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APPENDIX

Let us take as the state vector $|0\rangle$ or $\langle\beta|$ the Bogoliubov state, which is the state obtained by applying Bogoliubov transformation. Then

$$\begin{aligned} \langle\beta|\rho_{-k}|0\rangle &= \langle\beta|\sum_i a_{i+k}^* a_i|0\rangle \approx \sqrt{N}\langle\beta|a_k^* + a_{-k}|0\rangle \\ &= \sqrt{N}\langle\beta|U_k d_k^* - V_k d_{-k} + U_{-k} d_{-k} - V_{-k} d_k^*|0\rangle, \end{aligned}$$

where a^* 's are creation operators of boson and d^* 's are Bogoliubov operators related by the following: ($\epsilon_k = k^2/2m$)

$$\begin{aligned} a_k^* &= U_k d_k^* - V_k d_{-k}, \\ U_k^2 &= \frac{1}{2} \left(\frac{\epsilon_k + N v_k}{\omega_k} + 1 \right), \\ V_k^2 &= \frac{1}{2} \left(\frac{\epsilon_k + N v_k}{\omega_k} - 1 \right), \\ \omega_k &= \left\{ \left(\frac{k^2}{2m} \right)^2 + 2N v_k \frac{k^2}{2m} \right\}^{1/2}, \end{aligned}$$

representing a Bogoliubov transformation. We have

$$\begin{aligned} S(k, \omega) &= \sum_\beta |\langle\beta|\rho_k|0\rangle|^2 \delta(\omega_{\beta 0} - \omega) \\ &= N(U_k - V_k)^2 \delta(\omega_k - \omega) = N \frac{k^2/2m}{\omega_k} \delta(\omega_k - \omega). \end{aligned}$$

Hence, Eq. (18) for μ becomes

$$\begin{aligned} \mu &= N v_0 + \sum_k v_k^2 N \frac{k^2/2m}{\omega_k} \frac{1}{-\omega_k - k^2/2m} \\ &= N v_0 + \frac{1}{2} \sum_k v_k \left(\frac{k^2/2m}{\omega_k} - 1 \right). \end{aligned}$$

Integration over N gives

$$\begin{aligned} E &= \int_0^N \mu(N') dN' = \int_0^N dN' \left\{ N' v_0 \right. \\ &\quad \left. + \frac{1}{2} \sum_k v_k \left(\frac{k^2/2m}{[(k^2/2m)^2 + N' v_k k^2/m]^{1/2}} - 1 \right) \right\} \\ &= \frac{1}{2} N^2 v_0 + \frac{1}{2} \sum_k \left(\omega_k - \frac{k^2}{2m} - v_k N \right). \end{aligned}$$

Comparison with B.S.(I) (26) shows that this is identical (where the v 's were replaced by α).