Complex Angular Momentum in Perturbation Theory*

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The leading singularity in the complex angular momentum plane is studied for certain sets of Feynman graphs. Two models are considered: (a) the ladder graphs in the $\lambda\varphi^4$ theory in which bubbles are exchanged, and (b) the ladder graphs for the scattering of two scalar mesons by vector meson exchange. The method used is the summation of the most singular term in every order of perturbation theory. In both models the leading singularity is a branch point on the real l axis to the right of $l=0$. As the coupling constant is decreased, this branch point approaches $l=0$. The nature of the branch point is very similar to that of the branch point (near $l=-\frac{1}{2}$ for weak coupling) in the case of scattering from a potential with a r^{-2} singularity.

I. INTRODUCTION

RECENTLY, there has been considerable interest in the general subject of complex angular momentum in field theory.[1–6] By field theory we mean field-theoretic approximations, e.g., sets of Feynman graphs. In particular, it has been shown that summation of ladder graphs in a theory of the scattering of scalar bosons by the multiple exchanges of scalar bosons leads to a leading Regge trajectory.[1] This Regge trajectory shows a strong similarity to the leading trajectory in the case of scattering from a Yukawa potential.

In potential scattering not all potentials give a Regge pole as the leading singularity (i.e., farthest to the right) in the l plane. A potential with a r^{-2} singularity at the origin gives rise to a branch cut in the l plane, between the points $l=-\frac{1}{2}\pm\sqrt{g}$, where g is the strength of the r^{-2} term. Under some circumstances the right-hand branch point, $l=-\frac{1}{2}+\sqrt{g}$, will be the leading singularity.

We may ask whether similar branch points will be encountered in field theory. Evidently scalar boson exchange gives rise to a force which is not sufficiently singular at $r=0$ to create such difficulties. However, in other field-theoretic models the force might be expected to be more singular at $r=0$ and branch points, therefore, would be anticipated.

It has been shown for one such such case that no branch point in fact arises near $l=-\frac{1}{2}$ (for weak coupling). This is the case of the exchange of a scalar boson with a continuous mass distribution, in which the mass spectral function is designed to simulate a r^{-2} potential at the origin, if used in a superposition of Yukawa potentials,

$$V(r)=\int_a^\infty d\mu\frac{\rho(\mu)e^{-\mu r}}{r}$$

where

$$\rho(\mu)\to\text{const}\quad\text{as}\quad\mu\to\infty\,.$$

It was shown in Ref. 1 that this spectral function (used as a weight function for the masses in the propagator for an exchanged particle) gives rise to a leading *pole* (near $l=-\frac{1}{2}$ for weak coupling) in the field theoretic case.[7]

In the present work some ladder graphs are examined in two cases which are somewhat more singular than the above example. The first model is the sum of the bubble exchange graphs in the $\lambda\varphi^4$ theory. The second model is the sum of vector meson exchange graphs in the scattering of scalar bosons. In both cases we find a leading branch point in the angular momentum plane, now near $l=0$ for small coupling, but in other ways very similar to the branch points for the r^{-2} potential. The method used is the summation of the most singular term in every order of perturbation theory (for the ladder graphs).

In Sec. 2 we discuss in more detail the two models. In Sec. 3 the treatment of theories with leading poles is discussed, as an example of the technique of summation of most singular terms. In Sec. 4 the essential difference between pole producing mechanisms and branch point producing mechanisms is stated. In Sec. 5 the summation technique for the leading cut theories is explained and the results are stated for the two relativistic models and for the r^{-2} potential (which serves as a check on the method). Section 6 is devoted to the $\cos\theta\to\infty$ limits implied by the leading branch points. The evaluation of integrals is outlined in the appendices.

II. THE DIAGRAMS TO BE CONSIDERED

The first model is the sum of the ladder graphs in Fig. 1 for the $\lambda\varphi^4$ theory, in which bubbles are exchanged. The problem is to be formulated in terms of the partial-wave Bethe-Salpeter equation.

The second model is that of the ladder graphs for the exchange of vector bosons by scalar bosons, Fig. 2. We take the two scalar bosons, a and b, to be oppositely charged to make the force attractive. For the vector boson propagator, we shall take merely

$$-\frac{g_{\mu\nu}}{(2\mu)^4}\frac{i}{k^2-\mu^2};$$

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[1] B. W. Lee and R. F. Sawyer, Phys. Rev. **127**, 2266 (1962).
[2] B. W. Lee and R. F. Sawyer, Phys. Rev. **127**, 2274 (1962).
[3] P. G. Federbush and M. T. Grisaru (to be published).
[4] M. Gell-Mann and M. L. Goldberger, Phys. Rev. Letters **9**, 275 (1962).
[5] J. C. Polkinghorne (to be published).
[6] L. Bertocchi, S. Fubini, and M. Tonin, Nuovo Cimento **25**, 626 (1962).

[7] In this paper we deal only with leading singularities for weak coupling.

that is, we consider the interaction to be with a conserved current. In a boson theory this implies the existence of interaction terms of the form, $\varphi^*\varphi A_\mu A_\mu$, as in scalar boson electrodynamics. Graphs of the form shown in Fig. 3 result from this term. These graphs have been omitted from our summation.[8]

In both cases the diagrams under consideration are finite for $l>0$. Our procedure will involve finding an analytic interpolating function for $l=1, 2, 3\cdots$. The Sommerfeld-Watson representation will now be of the form

$$f(s, x=\cos\theta)=f_0(s)-\frac{1}{2i}\int_\Gamma \frac{(2l+1)dlf(s,l)P_l(-x)}{\sin\pi l}, \quad (1)$$

where Γ encloses the points $l=1, 2, 3\cdots$ as shown in Fig. 4. When we finally look at the high-x limit of (1),

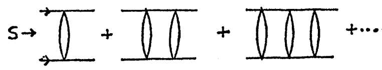

FIG. 1. Diagrams from the $\lambda\varphi^4$ theory.

we will find, of course, a constant background term, $f_0(s)$.

III. MODELS WITH A LEADING POLE

It is instructive to review how, in the case of models with a leading Regge pole, the perturbation trajectory may be found by summing the most singular terms of each order. We have in mind the cases of superpositions of Yukawa potentials (with well-behaved weight functions) and the case of the ladder graphs in the $\lambda\varphi^3$ theory.

In these cases, the scattering amplitude is given by the solution of an integral equation which has been extended to complex l,[1,9]

$$T_l(\xi,\xi',s)=V_l(\xi,\xi',s)+\int d\xi''\frac{V_l(\xi,\xi'',s)T_l(\xi'',\xi',s)}{h(\xi'',s)}. \quad (2)$$

Here s is the total energy (energy^2 in the relativistic problems). The ξ's stand for a set of variables (relative momentum in potential scattering; relative momentum and relative energy for the Bethe-Salpeter case) which are to be finally fixed at values determined from s, when the amplitude is taken on the energy shell.

Now, in general, there is a region, $\mathrm{Re}l>l_0$, in which the kernel of Eq. (2) is analytic. In the cases cited above,

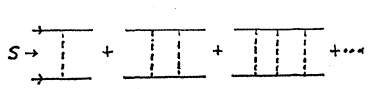

FIG. 2. Vector boson exchange graphs. Solid lines are scalar bosons; dashed lines are vector bosons.

[8] These graphs are apparently as singular in each order as the ones which have been summed. This example, therefore, is of interest mainly in its relevance to Bethe-Salpeter equations; the behavior of the ladder is very likely not characteristic of the field theory. The following has been checked: the diagrams of Fig. 3 do not remove the singularities at $l=0$, order by order.

[9] L. Brown, D. I. Fivel, B. W. Lee, and R. F. Sawyer (to be published).

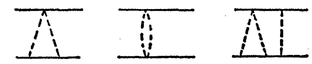

FIG. 3. Some omitted diagrams.

the first singularity of V_l is a simple pole at $l=l_0$. In the Regge pole models, the residue of V_l at this pole is of a particularly simple form:

$$V_l(\xi,\xi')=\frac{gr(\xi)t(\xi')}{l-l_0}+\text{a regular term}. \quad (3)$$

The most singular term in T_l in each order of perturbation theory is given by inserting the first term of (3) into the iteration solution of (2):

$$T_l^{\mathrm{sing}}(s)=\sum_{n=0}^{\infty} V_l^s[h(s)^{-1}V_l^s]^n, \quad (4)$$

where V_l^s is the first term on the right-hand side of (3). We have

$$T_l^{\mathrm{sing}}(\xi,\xi',s)=\sum_{n=0}^{\infty}\frac{g^{n+1}}{(l-l_0)^{n+1}}r(\xi)t(\xi')\times\left(\int\frac{r(\xi'')t(\xi'')}{h(\xi'',s)}d\xi''\right)^n$$
$$=gr(\xi)t(\xi')\left(l-l_0-g\int\frac{r(\xi'')t(\xi'')}{h(\xi'',s)}d\xi''\right)^{-1}. \quad (5)$$

Thus, we see how the sum of the most singular term in each order gives a Regge pole form, $\beta(s)[l-\alpha(s)]^{-1}$,

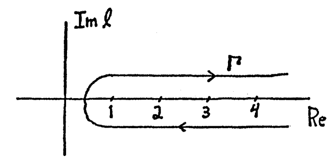

FIG. 4. Sommerfeld-Watson contour.

where both β and α are determined to first order,

$$\beta(s)=gr(s)t(s),$$
$$\alpha(s)=l_0+g\int\frac{r(\xi'')t(\xi'')}{h(\xi'',s)}d\xi''. \quad (6)$$

Models in which the singular part of the kernel is not separable as in (3), but still is expressible as the sum of a finite number of separable terms, are essentially no different. The single perturbation trajectory is replaced by the sum of several trajectories, all approaching l_0 for zero coupling strength.

IV. BRANCH POINTS

In all such models there is some real number, l_1, such that for $\mathrm{Re}l\leq l_1$ the individual integrals no longer converge in the Born series solution to (2). In all theories with known leading Regge poles, the leading singularity of V_l at l_0 is to the right of the line $\mathrm{Re}\,l=l_1$. If, on the other hand, $l_1\geq l_0$ we may anticipate that the leading singularities in each term come from the continuation of the integrals, rather than from the singularity of V_l. We shall argue on the basis of our examples that when $l_1\geq l_0$ the leading singularity will be a branch point (at least for weak coupling).

We list l_0 and l_1 for several cases:

(a) Yukawa potential	$l_0=-1$,	$l_1=-\frac{3}{2}$
(b) Ladders for $\lambda\varphi^3$ theory	$l_0=-1$,	$l_1=-\frac{3}{2}$
(c) Scalar bosons exchanging vector bosons	$l_0=-1$,	$l_1=0$
(d) Bubble exchange in $\lambda\varphi^4$ theory	$l_0=0$,	$l_1=0$
(e) r^{-2} potential	$l_0=-\frac{1}{2}$,	$l_1=-\frac{1}{2}$.

V. SUMMATION TECHNIQUE

Now concentrating on those theories [(c), (d), (e) above] in which $l_1\geq l_0$ we examine the nature of the term by term singularities at the point l_1. We find in the $(n+1)$th Born term in the various cases (keeping only the highest order pole term in each order of perturbation theory):

$$\text{(c)}\qquad c_{n+1}(s)\frac{g^{2n+2}}{l^n};$$

$$\text{(d)}\qquad d_{n+1}(s)\frac{g^{2n+2}}{l^{2n+1}};$$

$$\text{(e)}\qquad e_{n+1}(s)\frac{g^{n+1}}{(l+\frac{1}{2})^{2n+1}}.$$

We note the similarity to the expansion (5). If the coefficients, c, were of the form $c_{n+1}(s)=\beta(s)a^n(s)$ we could again sum the most singular terms into a Regge pole form; they are not, however. In the Appendix the coefficients c, d, and e are worked out to all orders and it is shown that these terms are generated by the series expansions of[10]

$$\text{(c)}\quad f_l^{\text{sing.}}(s)=-\frac{\pi}{4}\left(\frac{s-4m^2}{s}\right)^{1/2}\left[l\left(l-\frac{g^2}{16\pi^2}\right)\right]^{1/2}\quad\text{(vector meson exchange)}$$

$$\text{(d)}\quad f_l^{\text{sing.}}(s)=-\frac{\pi}{4}\left(\frac{s-4m^2}{s}\right)^{1/2}\left(l^2-\frac{g^2}{8(2\pi)^4}\right)^{1/2}\quad(\varphi^4\text{ theory})\qquad(7)$$

$$\text{(e)}\quad f_l^{\text{sing.}}(s)=-\tfrac{1}{2}[(l+\tfrac{1}{2})^2-g]^{1/2}\quad(r^{-2}\text{ potential}).$$

[10] Here $f_l(s)$ stands for $\exp i\delta_l\sin\delta_l$. Note that the singular terms depend only kinematically upon s. The variable s is the energy (Ref. 2) in the relativistic cases, the energy in the nonrelativistic case.

Note that the right-hand sides of Eq. (7) can be multiplied by a function $g(l,s)$ analytic at $l=0$, 0, $-\frac{1}{2}$, respectively, in the three cases, and such that $g_c(0,s)$, $g_d(0,s)$, $g_e(-\frac{1}{2},s)$ are unity. This will not change the leading singular terms of the perturbation expansion. In particular, the functions g may be chosen to give the correct threshold behaviors

$$g_c(l,s)=g_d(l,s)=\left(\frac{s-4m^2}{a}\right)^{l/2},$$

$$g_e(l,s)=\left(\frac{s}{a}\right)^{(l+1)/2}.$$

Here a is an arbitrary number.

Though the procedure of summation of only the most singular terms cannot be justified, the result for case (e), giving the correct branch points at $l=-\frac{1}{2}\pm\sqrt{g}$ for the r^{-2} potential gives us some confidence in its validity.

Our conjecture that the branch point in cases (c) and (d) is indeed independent of s and will not begin to move in the next order of approximation is supported only by the analogy to the r^{-2} case.

These conclusions for the $\lambda\varphi^4$ theory differ from the previous results of Lee and the author.[2] In this previous work, Regge pole behavior of the same graphs was extracted using a limiting process with a parameter, η, in which the genuine Feynman graphs corresponded to the limit $\eta\to 0$. This η was introduced to convert the relevant Bethe-Salpeter equation into a Fredholm equation; the method of attack was the Fredholm method. For small η it was concluded that there existed a leading pole near $l=0$. In the present work it has been seen that when $\eta=0$, this pole turns into a branch point. The previously published conclusions on the asymptotic dominance of the crossed bubble-exchange (or multiperipheral) graphs, over certain other sets of graphs, are maintained in our new result.

VI. SOMMERFELD-WATSON CONTOUR AND HIGH COSθ LIMIT

We begin with the representation

$$f(s,x)=f_0(s)-\frac{1}{2i}\int_\Gamma dl\frac{(2l+1)P_l(-x)}{\sin\pi l}f_l(s),\qquad(8)$$

where Γ is the contour of Fig. 4. In both of our field-theoretic examples the leading singularity is on the real axis somewhere to the right of $l=0$. For the weak coupling case, this branch point will be near $l=0$. When the Sommerfeld-Watson contour is opened up and moved to the left, the leading dependence for high x will be provided by the integral around the leading end of the cut as shown in Fig. 5. Let Λ be that segment of this contour which is drawn with a heavy line in Fig. 5. We now examine the high-x limit in the two cases, (7c) and (7d):

$$\lim_{x\to\infty} -\frac{1}{2i}\int_{\mathrm{A}} dl(2l+1)(\sin\pi l)^{-1}P_l(-x)f_l^{\mathrm{sing}}(s). \quad (9)$$

In case (c) we obtain, in the weak coupling limit,

$$\lim_{x\to\infty} f(s,x) = -\frac{\pi^{3/2}}{2g}\left(\frac{s-4m^2}{s}\right)^{1/2}(-x)^{\alpha_c}[\ln(-x)]^{-3/2}, \quad (10)$$

where

$$\alpha_c = g^2/16\pi^2.$$

In case (d) we obtain, in the weak coupling limit,

$$\lim_{x\to\infty} f(s,x) = -2^{-3/4}\pi^{3/2}g^{-1/2}\left(\frac{s-4m^2}{s}\right)^{1/2} \times(-x)^{\alpha_d}[\ln(-x)]^{-3/2}, \quad (11)$$

with

$$\alpha_d = g(2\pi)^{-2}8^{-1/2}.$$

These limits provide the high energy behavior of the crossed graphs of Fig. 6 [with $x=1+2t/(s-4m^2)$]. Note that the difference between the expressions, (10) and (11), and the prediction of a fixed Regge pole is the logarithmic factor alone.

VII. CONCLUSIONS

We have seen that in the perturbation treatment of a class of models one may easily distinguish between the mechanisms that produce a leading pole and those that produce a leading branch point in the l plane. These models are: superpositions of Yukawa potentials; ladder graphs with scalar meson exchange, vector meson exchange, and bubble exchange. In each of these examples the scattering amplitude obeys an integral equation, the kernel of which is an analytic function of l with certain singularities, in particular with a leading simple pole at $l=l_0$. In each case there is also a line in the l plane, $\mathrm{Re}\,l=l_1$, to the left of which the individual integrals in the perturbation series diverge.

When $l_0>l_1$, we find a leading pole; when $l_1\geq l_0$, a leading branch point. That is, in perturbation theory, the leading moving pole arises from the repeated iterations of the fixed pole of the kernel. Leading cuts arise from analytic continuations of integrals.

The method of summing leading terms in each order of perturbation theory is clearly not adequate to find any properties of subsidiary singularities in the l plane, for example, Regge trajectories beginning at $l=-1$, or some such place. A better approach would seem to be

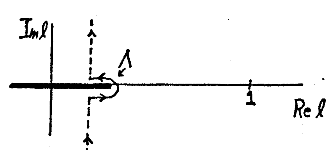

FIG. 5. Contour for high $\cos\theta$ limit.

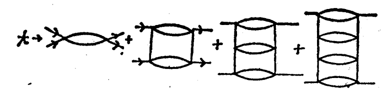

FIG. 6. Multiperipheral graphs.

from the Bethe-Salpeter equation itself, by solving in some noniterative approach. The Fredholm method used in Ref. 1 is not applicable, however. The models with cuts are in each case characterized by a singular integral equation to which the Fredholm method may not be applied.

The results are discouraging to the program of finding Regge behaviors from the simple dispersion theoretic approximations. For example the simple ND^{-1} method (with a subtraction in D and with N computed from a Born term) is designed to be a cheap way of approximately summing a chain such as that of Fig. 1 or Fig. 2. However, we may see that this technique completely misses the branch points in the l plane and predicts a totally different $\cos\theta\to\infty$ limit.[11]

The results are hopeful in one respect; it is always nice to be able to sum an infinite set of orders of perturbation theory even when these terms are a small fraction of all the terms which need to be considered. There is some slight hope that general criteria may be found for deciding which terms will be the most singular, and then perhaps these terms may be summed (these criteria more or less exist for the $\lambda\varphi^3$ theory; the dominant graphs are the ladders, see Ref. 3). In any event it would seem that much more work on simple models is required.

VIII. ACKNOWLEDGMENT

The author is indebted to Professor C. Goebel for his solution to a difference equation.

APPENDIX A: r^{-2} POTENTIAL

We treat the r^{-2} potential in some detail as an illustration of our method, which will become somewhat more complicated in field theory. Consider a superposition of Yukawa potentials

$$V(r) = -\frac{1}{r}\int_{m_0^2}^{\infty} dy\, y^{-1/2}\rho(y)\exp(-y^{1/2}r). \quad (\mathrm{A}1)$$

Here y is a (mass)^2 variable.

The partial wave projection of the Lippman-Schwinger integral equation for the scattering amplitude in this potential is[9]

$$T_l(\xi,\xi',s) = V_l(\xi,\xi') + \frac{1}{\pi}\int_0^{\infty}\frac{d\xi''}{\xi''-s-i\epsilon}V_l(\xi,\xi'')T_l(\xi'',\xi',s), \quad (\mathrm{A}2)$$

where

$$V_l(\xi,\xi') = \frac{(\xi\xi')^{-1/4}}{2}\int_{m_0^2}^{\infty} dy\, y^{-1/2}\rho(y)Q_l\left(\frac{\xi+\xi'+y}{2(\xi\xi')^{1/2}}\right), \quad (\mathrm{A}3)$$

[11] P. G. O. Freund, Nuovo Cimento 28, 263 (1963).

and s is the energy. The mass of the particle which is scattered has been taken to be $\frac{1}{2}$. The energy shell is defined by $\xi=\xi'=s$. In this limit, $T_l(\xi,\xi',s)=(\exp i\delta_l)\times\sin\delta_l$.

If the weight function, $\rho(y)$, behaves at infinity as

$$\rho(y)=\tfrac{1}{2}g+O(y^{-b}),\quad b>0 \tag{A4}$$

the potential will have a r^{-2} singularity at the origin. We shall take $\rho(y)=g/2$; only the r^{-2} singularity is of importance in our calculation. Thus we have

$$V(r)=-(g/r^2)\exp(-m_0r), \tag{A5}$$

where the value of m_0, the minimum mass exchanged, is of no significance to us.

Now we examine the contribution to (A3) from the leading term in the expansion of $Q_l(z)$ in powers of z^{-2},[12]

$$Q_l(z)=2^{-1-l}z^{-1-l}\frac{\pi^{1/2}\Gamma(1+l)}{\Gamma(l+\frac{3}{2})}+O(z^{-l-3}). \tag{A6}$$

From (A3), (A4), and (A6) we see that $V_l(\xi,\xi')$ has a simple pole at $l=-\frac{1}{2}$, coming from the integral of the first term of (A6). Separating out this first term we have, defining $\lambda=l+\frac{1}{2}$,

$$V_\lambda(\xi,\xi')=\frac{g}{4\lambda}(\xi\xi')^{\lambda/2}(\xi+\xi'+m_0^2)^{-\lambda}+R_\lambda(\xi,\xi'). \tag{A7}$$

Now, for simplicity we fix s at some negative real value in (A2). The terms which will eventually be summed are independent of s, but it is convenient to be able to ignore the singularities of $(\xi''-s)^{-1}$ in (A2) in deriving them.

Now defining

$$G(\xi,\xi')=\frac{1}{\pi}\frac{\delta(\xi-\xi')}{\xi'-s-i\epsilon}, \tag{A8}$$

we write the iteration solution to (A2) as

$$T=\sum_{n=0}^{\infty}V(GV)^n. \tag{A9}$$

We assert that the most singular term at $\lambda=0$ in each order arises from the first term in the expression for V_λ, (A7). The proof consists of noting that $R_\lambda(\xi,\xi')$ in (A7) is not singular at $\lambda=0$ and that

$$|R_\lambda(\xi,\xi')|<M(\xi+\xi'+m_0^2)^{-\lambda}(\xi\xi')^{\lambda/2}, \tag{A10}$$

where M is some number. Note that singularities at $\lambda=0$ come about in two ways: (1) From the $1/\lambda$ in (A7), and (2) from the integrals implied in (A9). For $\lambda>0$ it is easily seen that these integrals converge. At $\lambda=0$ they diverge.

Continuation from the convergent region, $\lambda>0$, will give additional singularities at $\lambda=0$. Now because of the bound, (A10), the singularities coming from the integrals of terms containing $R_\lambda(\xi,\xi')$ are no worse than those from the integrals of the first term on the right of Eq. (A7). These latter terms, however, are more singular because of the $1/\lambda$ dependence. Using only the "most singular part" of $V_\lambda(\xi,\xi')$,

$$\bar{V}_\lambda(\xi,\xi')=\frac{g}{4\lambda}(\xi\xi')^{\lambda/2}(\xi+\xi'+m_0^2)^{-\lambda}, \tag{A11}$$

we write the "most singular part" of the scattering amplitude, $\bar{T}_\lambda(\xi,\xi')$, as

$$\bar{T}_\lambda=\sum_{n=0}^{\infty}\bar{V}_\lambda(G\bar{V}_\lambda)^n=\sum_{n=0}^{\infty}\bar{T}_\lambda^{(n)}, \tag{A12}$$

where

$$\bar{T}_\lambda^{(n)}=\left(\frac{g}{4\lambda}\right)^n\int_0^\infty d\xi_1d\xi_2\cdots d\xi_{n-1}\times\frac{(\xi+\xi_1+m_0^2)^{-\lambda}\xi_1^\lambda(\xi_1+\xi_2+m_0^2)^{-\lambda}\xi_2^\lambda(\xi_2+\xi_3+m_0^2)^{-\lambda}\cdots\xi_{n-1}^\lambda(\xi_{n-1}+\xi'+m_0^2)^{-\lambda}}{(\xi_1-s)(\xi_2-s)\cdots(\xi_{n-1}-s)}. \tag{A13}$$

Now we examine the singularities of the *integrals* in (A13), again multiple poles at $\lambda=0$. It can be seen that the most singular term of (A13) is independent of s. If we replace one of the factors $(\xi_i-s)^{-1}$ by $(\xi_i+1)^{-1}$, the extra term involving the difference, $(\xi_i-s)^{-1}-(\xi_i+1)^{-1}$, goes to zero sufficiently fast as ξ_i goes to infinity to eliminate at least one factor of λ^{-1} from the result. Since the singularities will come from the high ξ_i parts of the integrals, we may also replace ξ_i^λ by $(1+\xi_i)^\lambda$. A new "most singular term" is then defined by

$$\tilde{T}_\lambda^{(n)}=\left(\frac{g}{4\lambda}\right)^n\xi^{\lambda/2}(\xi')^{\lambda/2}\times\int_0^\infty d\xi_1\cdots d\xi_{n-1}(\xi+\xi_1)^{-\lambda}(1+\xi_1)^{-1+\lambda}(\xi_1+\xi_2)^{-\lambda}\times(1+\xi_2)^{-1+\lambda}\cdots(1+\xi_{n-1})^{-1+\lambda}(\xi_{n-1}+\xi')^{-\lambda}. \tag{A14}$$

[12] *Higher Transcendental Functions*, edited by A. Erdelyi (McGraw-Hill Book Company, Inc., New York, 1953), Vol. 1, Eq. (3.2.41).

We begin by integrating from the left. We define

$$I_1(\xi,\xi_2)=\int d\xi_1(\xi+\xi_1)^{-\lambda}(1+\xi_1)^{-1+\lambda}(\xi_1+\xi_2)^{-\lambda},$$

$$I_k(\xi,\xi_{k+1})=\int d\xi_k I_{k-1}(\xi,\xi_k)(1+\xi_k)^{-1+\lambda}\times(\xi_k+\xi_{k+1})^{-\lambda} \tag{A15}$$

and obtain

$$I_1(\xi,\xi_2)=\lambda^{-1}\xi_2^{-\lambda}+\xi_2^{-\lambda}\ln\xi_2+R_1(\xi,\xi_2,\lambda), \tag{A16}$$

where $R_1(\xi,\xi_2,\lambda)$ is regular at $\lambda=0$ and is bounded by $\xi_2^{-\lambda}M$. Again the term, R_1, will not contribute to the leading singularity. The logarithmic term in (A16), however, is relevant, though regular at $\lambda=0$; in the next integration it will increase the order of the pole at $\lambda=0$.

In a similar way one may verify that the integral, I_k, is given by

$$I_k=\xi_{k+1}^{-\lambda}\sum_{j=0}^{k}C_k{}^j(\ln\xi_{k+1})^j\lambda^{j-k}+\text{a less singular remainder}. \tag{A17}$$

By less singular terms we mean terms of the form

$$(\xi_{k+1})^{-\lambda}\lambda^{-m_1}(\ln\xi_{k+1})^{m_2}M(\xi_{k+1},\lambda),$$

where $m_1+m_2<k$ and M is bounded near $\lambda=0$ for all ξ_{k+1}. We note the relation

$$\int d\xi(1+\xi)^{-1}(\xi+\xi')^{-\lambda}\ln^n\xi=(\xi')^{-\lambda}\left[\frac{\ln^{n+1}\xi'}{n+1}+\left(-\frac{\partial}{\partial\lambda}+\ln\xi'\right)^n\lambda^{-1}\right]+\text{less singular terms}. \tag{A18}$$

Substituting (A17) into (A15) and using (A18) yields a recursion relation for the $C_k{}^j$:

$$C_{n+1}{}^k=\frac{\epsilon_k C_n{}^{k-1}}{k}+\sum_{m=k}^{n}\frac{m!}{k!}C_n{}^m, \tag{A19}$$

with $\epsilon_k=1$ for $k\geq 1$, $\epsilon_0=0$.

The boundary condition comes from (A16);

$$C_1{}^0=C_1{}^1=1.$$

By induction one may verify the solution

$$\begin{aligned}k\geq 1\quad C_n{}^k&=\frac{(k+1)(2n-k)!}{(n-k)!(n+1)!k!},\\ k=0\quad C_n{}^0&=\frac{2(2n-1)!}{(n-1)!(n+1)!}.\end{aligned} \tag{A20}$$

Proceeding now to the end of the chain of integrals, (A14), we find terms of the form

$$(\xi\xi')^{\lambda/2}\left(\frac{g}{4\lambda}\right)^{n+1}C_n{}^k\lambda^{-n+k}(\ln\xi')^k.$$

Since ξ and ξ' are now to be set equal to s, the most singular contribution to the $(n+1)$st order is

$$T_\lambda{}^{(n)\text{sing}}=C_n{}^0(g/4)^{n+1}\lambda^{-2n-1},$$

with the definition (A20) for $C_n{}^0$, $n>0$ and with $C_0{}^0=1$. The summation of most singular terms is

$$T_\lambda{}^{\text{sing}}=\sum_{n=0}^{\infty}T_\lambda{}^{(n)\text{sing}}=\frac{\lambda}{2}\left[1-\left(1-\frac{g}{\lambda^2}\right)^{1/2}\right]. \tag{A21}$$

We note the branch points at $\lambda=\pm\sqrt{g}$. This result, of course, can be obtained trivially in this case of the r^{-2} potential. The integrals which arose in this case, however, are identical to the ones which arise in the two field theoretic models.

APPENDIX B: THE BETHE-SALPETER CASE FOR VECTOR MESON EXCHANGE

In each case we deal with the scattering of two spin-zero bosons of equal mass, m. The integral equation will be of the same general form as the one in potential scattering, with an extra variable, the relative energy. The integral equation is of the form (see Ref. 1)

$$T_l(p,\omega;p',\omega';s)=V_l(p,\omega;p',\omega')-i\int_{-\infty}^{\infty}d\omega''\int_0^{\infty}dp''V_l(p,\omega;p'',\omega'')[p''^2+m^2-i\epsilon-(\omega''-\tfrac{1}{2}\sqrt{s})^2]^{-1}\times[p''^2+m^2-i\epsilon-(\omega''+\tfrac{1}{2}\sqrt{s})^2]^{-1}T_l(p'',\omega'';p',\omega';s). \tag{B1}$$

Here the scattering amplitude on the mass shell is defined by

$$\exp(i\delta_l)\sin\delta_l=\pi^2[s(s-4m^2)]^{-1/2}T_l((\tfrac{1}{4}s-m^2)^{1/2},0;(\tfrac{1}{4}s-m^2)^{1/2},0;s). \tag{B2}$$

As a propagator for the vector mesons, we take

$$-\frac{i}{(2\pi)^4}g_{\mu\nu}(k^2-\mu^2)^{-1}.$$

Opposite coupling sign to the two scalar bosons is chosen, in order to make the force attractive. The Born term is calculated to be

$$V_l(p,\omega;\,p',\omega')=-\frac{g^2}{2(2\pi)^3}\int_{-1}^{1}dxP_l(x)\frac{s-(\omega+\omega')^2+p^2+p'^2+2pp'x}{p^2+p'^2-2pp'x+\mu^2-(\omega-\omega')^2-i\epsilon},\tag{B3}$$

where μ is the mass of the exchanged particle and g is the coupling constant.

For $l=1,\,2,\,3\cdots$ the expression (B3) may be replaced by $V_l{}^A$:

$$V_l{}^A=\frac{g^2}{(2\pi)^3}(s+2p^2+2p'^2-2\omega^2-2\omega'^2+\mu^2)Q_l\left(\frac{p^2+p'^2+\mu^2-i\epsilon-(\omega-\omega')^2}{2pp'}\right).\tag{B4}$$

Here a term has been dropped which contributes only to the S wave. Equation (B4) may now immediately be generalized to complex l. Our complete form for T_l will be

$$T(l,s)=A(s)\delta_{l,0}+T^A(l,s),\tag{B5}$$

where the analytic part, $T^A(l,s)$, is what is of interest at the moment.

As in the case of the r^{-2} potential the most singular terms will be independent of s; the singularities coming from the high p and ω parts of integrals (when the final answer is taken on the mass shell some s dependence will result from the substitution $p=(\frac{1}{4}s-m^2)^{1/2}$; it is the s dependence in the propagators in (B1) which is ignorable). Taking $s=0$ and rotating the ω integration contours in (B1) counterclockwise to the imaginary axes we obtain

$$\bar{T}_l(p,\omega;\,p',\omega')=\bar{V}_l(p,\omega;\,p',\omega')+\int_{-\infty}^{\infty}d\omega''\int_0^{\infty}dp''\bar{V}_l(p,\omega;\,p'',\omega'')[p''^2+m^2+\omega''^2]^{-2}\bar{T}_l(p'',\omega'';\,p',\omega'),\tag{B6}$$

where

$$\bar{T}_l(p,\omega;\,p',\omega')\equiv T_l{}^A(p,i\omega;\,p',i\omega';\,s=0),$$

$$\bar{V}_l=\frac{g^2}{(2\pi)^3}(2p^2+2p'^2+2\omega^2+2\omega'^2+\mu^2)Q_l\left(\frac{p^2+p'^2+\mu^2+(\omega-\omega')^2}{2pp'}\right).\tag{B7}$$

It is easily seen that the integrals in the Born series solution to (B6) converge for $l>0$. There will be singularities coming from the divergence of these integrals at $l=0$.

Following the same lines as outlined in Appendix A we find that the most singular contribution at $l=0$ comes only from the leading term in the series development of $Q_l(z)$ in powers of z^{-2}. We define a "most singular," $V_l{}^s$, as

$$V_l{}^s=\frac{g^2}{(2\pi)^3}\frac{\pi^{1/2}\Gamma(1+l)}{\Gamma(l+\frac{3}{2})}(2p^2+2p'^2+2\omega^2+2\omega'^2+\mu^2)\left[\frac{pp'}{p^2+p'^2+\mu^2+(\omega-\omega')^2}\right]^{l+1}.\tag{B8}$$

The argument is a little more delicate in this case. Note that the singularities at $l=0$ now come completely from the integrals; the kernel is regular at $l=0$. It is still possible to prove, however, that the higher terms in the series development of $Q_l(z)$ (of order z^{-l-3}) behave sufficiently better as $p,\,\omega\to\infty$ to ensure that the most singular contributions do indeed come from the substitution of $V_l{}^s$, (B8), in place of V_l in the integral equation, (B6).

The factors $\pi^{1/2}\Gamma(1+l)/\Gamma(l+\frac{3}{2})$ can now be evaluated at $l=0$; μ^2 may be set equal to zero.

The nth term in the Born series, inserting the "most singular" kernel, (B8), and changing variables to $\xi_i=p_i^2$, $\eta_i=\omega_i^2$, is

$$\begin{aligned}T_l{}^{(n)s}(\xi,\eta;\,\xi',\eta')=&\frac{g^{2n}}{(2\pi)^{3n}}\int_0^{\infty}d\xi_1\cdots d\xi_{n-1}d\eta_1\cdots d\eta_{n-1}\xi^{(l+1)/2}\frac{(\xi+\xi_1+\eta+\eta_1)[\xi+\xi_1+\eta+\eta_1-2(\eta\eta_1)^{1/2}]^{-l-1}}{(\xi_1+\eta_1+m^2)^2}\\&\times\xi_1{}^{l+\frac{1}{2}}\eta_1{}^{-1/2}\frac{(\xi_1+\xi_2+\eta_1+\eta_2)[\xi_1+\xi_2+\eta_1+\eta_2-2(\eta_1\eta_2)^{1/2}]^{-l-1}}{(\xi_2+\eta_2+m^2)^2}\xi_2{}^{l+\frac{1}{2}}\eta_2{}^{-1/2}\cdots\xi_{n-1}{}^{l+\frac{1}{2}}\eta_{n-1}{}^{-1/2}(\xi_{n-1}+\xi'+\eta_{n-1}+\eta')\\&\times[\xi_{n-1}+\xi'+\eta_{n-1}+\eta'-2(\eta_{n-1}\eta')^{1/2}]^{-l-1}(\xi')^{(l+1)/2}.\end{aligned}\tag{B9}$$

We set $m^2=1$ and change variables to t_i, ξ_i where $\eta_i=t_i(\xi_i+1)$. The variables η and η' may be set equal to zero and ξ, ξ' to unity, with the exception of those factors standing to the extreme right and left in (B9).

We obtain

$$\begin{aligned}T_l{}^{(n)s}(\xi,0;\,\xi',0)=&\frac{g^{2n}}{(2\pi)^{3n}}\int_0^{\infty}\prod_{i=1}^{n-1}\frac{t_i^{-1/2}}{(1+t_i)^2}dt_i\int_0^{\infty}d\xi_1\cdots d\xi_{n-1}(\xi\xi')^{(l+1)/2}(1+\xi_1)^{-1}[\xi_1(1+t_i)+\xi_2(1+t_2)]\\&\times[\xi_1(1+t_1)+\xi_2(1+t_2)-2(t_1t_2\xi_1\xi_2)^{1/2}]^{-l-1}(1+\xi_2)^{-1+l}\cdots,\end{aligned}\tag{B10}$$

where some minor modifications have again been made which do not change the nature of the singular contributions.

It may be seen that in the representation (B10) the singularities at $l=0$ come from the ξ integrals. Furthermore, the most singular term of the ξ integrations are independent of the variables t_i; we may evaluate the inner integrals at $t_i=0$ to obtain the leading singularity. From the ξ integrals alone then we obtain the chain

$$I_n=\int_0^\infty d\xi_1\cdots d\xi_{n-1}(\xi_1+1)^{-1}(\xi_1+\xi_2)^{-l}\times(1+\xi_2)^{-1+l}(\xi_2+\xi_3)^{-l}\cdots(1+\xi_{n-1})^{-1}. \quad \text{(B11)}$$

This is the same chain as encountered in the r^{-2} potential, Eq. (A14).

The most singular term in the nth order in the present case is, thus,

$$T_l^{(n)s}(\xi,\xi')=\frac{g^{2n}}{(2\pi)^{3n}}\frac{(\xi\xi')^{1/2}}{l^{n-1}}\times\left[\int_0^\infty\frac{t^{-1/2}}{(1+t)^2}dt\right]^{n-1}C_{n-1}{}^0, \quad \text{(B12)}$$

in terms of the coefficients, C, defined in (A20).

The expression which generates the most singular term in each order is

$$2(\xi\xi')^{1/2}\frac{l}{\pi}\left[\frac{1}{2}-\left(\frac{1}{4}-\frac{g^2}{64\pi^2 l}\right)^{1/2}\right]. \quad \text{(B13)}$$

On the mass shell we set $(\xi\xi')^{1/2}=(s-4m^2)/4$. The singular part of (B13) is now

$$T_l{}^s(s)=-\frac{s-4m^2}{4\pi}\left[l\left(l-\frac{g^2}{16\pi^2}\right)\right]^{1/2}. \quad \text{(B14)}$$

APPENDIX C: THE $g\varphi^4$ THEORY

The bubble exchange graphs of Fig. 1 are generated for $l=1, 2, 3\cdots$ by the integral equation (B1) with[1,2]

$$V_l(p,\omega;p',\omega')=\int_{4m_0^2}^\infty dy\rho(y)Q_l\left(\frac{p^2+p'^2+y-it-(\omega-\omega')^2}{2pp'}\right),$$

where

$$\rho(y)=\frac{g^2}{8(2\pi)^5}\left[\frac{y-4m^2}{y}\right]^{1/2}. \quad \text{(C1)}$$

This expression for V_l is the form for the exchange of a scalar particle of mass, $y^{1/2}$, weighted by the appropriate mass spectral function for the bubble. The basic bubble (the first graph of Fig. 1) is of course divergent, but this divergence is only in the S wave. As before we perform our continuation to complex l from the analytic interpolating function for $l=1, 2, 3\cdots$.

As in the vector meson case the integrals in the iteration solution converge for Re$l>0$. Again they diverge for $l=0$; singularities at $l=0$ will result from continuation of the integrals.

Again the leading term in the expansion of $Q_l(z)$ gives rise to the most singular terms. Also, only the asymptotic form of the spectral function will matter.

$$\lim_{y\to\infty}\rho(y)=\frac{g^2}{8(2\pi)^5}. \quad \text{(C2)}$$

It is the large y part of the integral in (C1) which gives rise to poles at $l=0$.

We again may set $s=0$ without chaging the singular terms. The relevant part of V_l, $\bar{V}_l$, is given by

$$\bar{V}_l=\frac{g^2}{8(2\pi)^5}\frac{\pi^{1/2}\Gamma(1+l)}{l\Gamma(l+\frac{3}{2})}(pp')^{l+1}\times[p^2+p'^2+4m^2-(\omega-\omega')^2]^{-l}. \quad \text{(C3)}$$

The difference, $V_l-\bar{V}_l$ is regular at $l=0$ and bounded by the form

$$|V_l-\bar{V}_l|<M(pp')^{l+1}[p^2+p'^2+4m^2-(\omega-\omega')^2]^{-l}. \quad \text{(C4)}$$

Therefore, the most singular terms will come from $\bar{V}_l$.

We note the strong similarity between the expression (C3) for the $g\varphi^4$ case and the expression (B8) for the vector meson exchange case. The behaviors as p, p', ω, ω' approach infinity are essentially the same. However note that in the $g\varphi^4$ case V_l has a pole at $l=0$ while in the vector meson case the singularities come from the integrals alone. Thus, for the $g\varphi^4$ case the order of the pole at $l=0$ will increase by two in each successive order of perturbation theory. The situation in this respect is identical to that in the r^{-2} potential case.

A series of steps exactly analogous to those following (B8) will suffice to demonstrate that the most singular terms in each order are generated from a chain exactly of the form (B10).

In this case the equation analogous to (B11) is

$$T_l^{(n)}(\xi,\xi')=\frac{2}{\pi}\frac{(\xi\xi')^{1/2}g^{2n}}{l^{2n-1}}\frac{C_{n-1}{}^0}{[32(2\pi)^4]^n}. \quad \text{(C5)}$$

On the mass shell, we have

$$\sum_{n=1}^\infty T_l^{(n)}(s)=\frac{s-4m^2}{2\pi}l\left[\frac{1}{2}-\left(\frac{1}{4}-\frac{g^2}{32(2\pi)^4l^2}\right)^{1/2}\right], \quad \text{(C6)}$$

the singular part of which is

$$T_l{}^s(s)=-\frac{s-4m^2}{4\pi}\left(l^2-\frac{g^2}{8(2\pi)^4}\right)^{1/2}. \quad \text{(C7)}$$