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Stable Longitudinal Oscillations in Anisotropic Plasma*

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The dispersion equation for longitudinal oscillations in an infinite collisionless anisotropic plasma in a uniform magnetic field is analyzed. For a plasma in which the electrons are isotropic and the ions are anisotropic with a velocity distribution given by a two-temperature Maxwellian, it is found that a purely growing mode (as in the case of Rayleigh-Taylor instability) with frequency of the order of the ion cyclotron frequency cannot exist so long as the ion temperature perpendicular to the field is larger than its temperature along the field. This is demonstrated by showing that the dispersion equation has no solutions under these conditions.

It has been shown by Harris[1] that anisotropy in the velocity distribution of a uniform plasma causes longitudinal plasma oscillations with frequencies equal approximately to integral multiples of the ion cyclotron frequency to be unstable. More specifically, if $T_{\perp i}$ and $T_{\perp e}$ are the ion and electron temperatures perpendicular to the uniform magnetic field, and $T_{\parallel i}$ and $T_{\parallel e}$ are their respective temperatures parallel to the field, Harris found that for $T_{\perp i}/T_{\parallel i}=T_{\perp e}/T_{\parallel e}=\infty$, waves propagating parallel ($k_\perp=0$) to or perpendicular ($k_\parallel=0$) to the field are stable; and that they are unstable when neither component of the wave vector is zero. A sufficient condition for the instability was found to be $\omega_{pe}>\omega_{ci}$, where ω_{pe} is the electron plasma frequency and ω_{ci} is the ion cyclotron frequency. Recently, Ozawa *et al.*[2] have shown that in an anisotropic electron plasma instabilities will occur when $T_{\perp e}/T_{\parallel e}>2$. In this article we wish to show that a purely growing longitudinal oscillation (as in the case of Rayleigh-Taylor instability) with frequency of the order of the ion cyclotron frequency cannot exist in a uniform plasma in which the electrons are isotropic ($T_{\perp e}=T_{\parallel e}$) and the ions are anisotropic with $T_{\perp i}>T_{\parallel i}$.

We consider an infinite collisionless plasma in a uniform magnetic field B_0, taken conveniently in the z direction. We assume that the velocity distribution function of both the ions and the electrons in the equilibrium configuration is given by the two-temperature Maxwellian distribution

$$f_0=\frac{1}{\pi^{3/2}\alpha_\perp^2\alpha_z}\exp\left[-\frac{v_\perp^2}{\alpha_\perp^2}-\frac{v_z^2}{\alpha_z^2}\right], \tag{1}$$

where $\alpha_\perp=(2KT_\perp/M)^{1/2}$ and $\alpha_z=(2KT_z/M)^{1/2}$ are the thermal velocities perpendicular and parallel to the field, respectively, K is the Boltzmann constant, and M is the particle mass. We further assume that the system under consideration departs only slightly from an equilibrium configuration in which there is no electric field. In this case a dispersion relation can be readily derived from the linearized Vlasov equations by assuming that all perturbed quantities are of the form $\exp[i\mathbf{k}.\mathbf{r}+i\omega t]$, where ω is the frequency and $\mathbf{k}$ is the wave vector. It can be shown[1,3] that if the particles' thermal velocities $(\alpha_\perp,\alpha_z)$ as well as the wave phase (ω/k) are much smaller than the velocity of light c, the coupling between transverse waves and longitudinal waves can be ignored, and the dispersion equation for the latter can be written as

$$1=\sum_j\frac{\omega_{p_j}{}^2}{k^2}\sum_{n=-\infty}^{+\infty}2e^{-\lambda_j}I_n(\lambda_j)\times\left[-\frac{n\omega_{c_j}}{k_z\alpha_{z_j}\alpha_{\perp_j}{}^2}Y(-\xi_j)-\frac{1}{\alpha_{z_j}{}^2}+\frac{\xi_j}{\alpha_{z_j}{}^2}Y(-\xi_j)\right]. \tag{2}$$

In this equation the wave vector $\mathbf{k}$ is assumed to be in the $x-z$ plane and the first summation is over the plasma species. The remaining quantities are defined as follows: $\omega_p=4\pi Ne^2/M=$plasma frequency, $\omega_c=eB_0/Mc=$cyclotron frequency, $\lambda=\frac{1}{2}\gamma^2n_\perp^2$, $\gamma=k\rho$, $n_\perp=k_\perp/k$, $n_z=k_z/k$, $\rho=$particle radius of gyration, $I_n(\lambda)=I_{-n}(\lambda)$ = the Bessel function of the first kind of imaginary argument, and $\xi=(\omega+n\omega_c)/k_z\alpha_z$. The function $Y(-\xi)$ is the well-known plasma dispersion function[4] usually given by

$$Y(-\xi)=\begin{cases}\displaystyle\int_{-\infty}^{+\infty}\frac{e^{-y^2}dy}{y+\xi}\\ i(\pi)^{1/2}e^{-\xi^2}\operatorname{erfc}(i\xi)\end{cases}, \tag{3}$$

where

$$\operatorname{erfc}(i\xi)=\frac{2}{(\pi)^{1/2}}\int_{i\xi}^{\infty}e^{-y^2}dy. \tag{4}$$

We may note at this point that in the limit $\alpha_{z_j}\to 0$ Eq. (2) reduces to Eq. (49) of Ref. 1. If we now restrict our analysis to a plasma in which the electrons are isotropic, i.e., $\alpha_{z_e}=\alpha_{\perp_e}$, and consider longitudinal

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1 E. G. Harris, J. Nucl. Energy **C2**, 138 (1961).
2 Y. Ozawa, I. Kaji, and M. Kito, J. Nucl. Energy **C** (to be published).
3 I. B. Bernstein, Phys. Rev. **109**, 10 (1958).
4 B. D. Fried and S. D. Conte, *The Plasma Dispersion Function* (Academic Press Inc., New York, 1961).

oscillations with frequency of the order of the ion cyclotron frequency, i.e., $\omega/\omega_{c_i}\sim 1$, we can utilize the second of Eqs. (3) to rewrite Eq. (2) in the form

$$1+k^2d_i^2+\frac{T_{\perp i}}{T_e}(1-i\omega' g)$$
$$=\int_0^\infty \left[i\omega'+\left(\frac{t_i^2-1}{t_i^2}\right)\frac{\gamma_i^2t_i^2n_z^2}{2}x\right]$$
$$\times\exp[-i\omega' x-\tfrac{1}{4}\gamma_i^2t_i^2n_z^2x^2-\gamma_i^2n_\perp^2\sin^2(\tfrac{1}{2}x)]dx, \quad (5)$$

where d_i is the Debye length given by $d_i^2=KT_{\perp_i}/4\pi Ne^2$,

$$g=\frac{(\pi)^{1/2}}{\gamma_i n_z}\left(\frac{M_e}{M_i}\right)^{1/2}\left(\frac{T_{\perp i}}{T_e}\right)^{1/2}, \quad \omega'=\frac{\omega}{\omega_{c_i}}, \quad \text{and} \quad t_i=\frac{\alpha_{z_i}}{\alpha_{\perp_i}}.$$

In view of the assumptions made earlier, $g|\omega'|$ is much smaller than unity and represents in effect the ratio of the wave phase velocity to the electron thermal velocity. Letting $i\omega'=S=i\beta+\nu$, it is possible to demonstrate that if $t_i^2<1$ Eq. (5) has no roots for which $\beta=0$ and $\nu>0$ and, hence, no waves which only grow in time. To do this it is convenient to rewrite Eq. (5) in the form

$$1+k^2d^2+\frac{T_\perp}{T_e}(1-gS)$$
$$=Se^{-\mu}\int_0^\infty \exp[-Sx-\tfrac{1}{4}\gamma^2t^2n_z^2x^2+\mu\cos x]dx$$
$$+\left(\frac{t^2-1}{t^2}\right)\frac{\gamma^2t^2n_z^2}{2}\int_0^\infty x\exp[-Sx-\tfrac{1}{4}\gamma^2t^2n_z^2x^2$$
$$-\mu\sin^2(\tfrac{1}{2}x)]dx, \quad (6)$$

where we have let $\mu=\gamma^2n_\perp^2=2\lambda$, and have dropped the subscript i since there is no longer need for it. Noting that

$$e^{\mu\cos x}=\sum_{n=-\infty}^{+\infty} I_n(\mu)e^{inx}, \quad (7)$$

we can write the first integral in Eq. (6) as

$$Se^{-\mu}\int_0^\infty \exp[-Sx-\tfrac{1}{4}\gamma^2t^2n_z^2x^2+\mu\cos x]dx.$$
$$=S\sum_{-\infty}^{+\infty} e^{-\mu}I_n(\mu)\int_0^\infty \exp[-(S+in)x-\tfrac{1}{4}\gamma^2t^2n_z^2x^2]dx.$$

Following Bernstein,[3] we now introduce

$$\phi(u)=\frac{\rho n_z t}{(\pi)^{1/2}}\exp\left(-\frac{u^2}{\rho^2t^2n_z^2}\right) \quad (8)$$

and note that

$$\int_{-\infty}^{+\infty} du\phi(u)=1.$$

For $\nu>0$ we observe that

$$\int_{-\infty}^{+\infty}\frac{\phi(u)du}{S+i(n+\mathbf{k}u)}$$
$$=\int_{-\infty}^{+\infty}\phi(u)du\int_0^\infty dx\exp[-S-i(n+ku)]x$$
$$=\int_0^\infty dx\exp[-(S+in)x-\tfrac{1}{4}\gamma^2t^2n_z^2x^2]. \quad (9)$$

Substituting Eq. (9) into Eq. (6) and letting $\beta=0$, it becomes

$$k^2d^2+\frac{T_\perp}{T_e}(1-\nu g)+\sum_{-\infty}^{+\infty} e^{-\mu}I_n(\mu)\int_{-\infty}^{+\infty} du\phi(u)\frac{(n+ku)^2}{\nu^2+(n+ku)^2}$$
$$=\left(\frac{t^2-1}{t^2}\right)\frac{\gamma^2t^2n_z^2}{2}\int_0^\infty x\exp[-\nu x-\tfrac{1}{4}\gamma^2t^2n_z^2x^2$$
$$-\mu\sin^2(\tfrac{1}{2}x)]dx, \quad (10)$$

where we have utilized Eq. (7) with $x=0$. Recalling that $I_n(\mu)=I_{-n}(\mu)$, the integral term on the left-hand side of the above equation is positive definite; the integral over x is also positive. Since $\nu g\ll 1$, it is clear that Eq. (10) has no solution for $t^2<1$ since the left-hand side is always greater than the right. A solution is possible, however, for $t^2>1$. For $t^2=T_\perp/T_e=1$, and $\beta\neq 0$, a similar argument can be employed to show that as long as $g|\omega'|\ll 1$ Eq. (6) cannot have roots with $\nu>0$.

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