Application of Generalized Field Theory to Baryons*

H. C. CORBEN
Space Technology Laboratories, Inc., Redondo Beach, California
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The simplest relativistic wave equations for a particle which in the classical limit possesses moments of inertia about more than one axis are Dirac and Kemmer-Duffin equations containing extra terms which cause these equations to describe a variety of spin states. The classical field theory of such wave equations is developed and the generalized Dirac equation for particles of spin $\frac{1}{2}$ and $\frac{3}{2}$ is examined in detail. It is found that with the choice of two parameters, one of which merely determines the scale, this equation not only correctly describes the spin and charge states of the particles and resonances Ξ^-, Ξ^0, n, p, N^{**}, N^{**0}; it also yields their masses correct to better than 2%. The $\Xi^- - \Xi^0$ and $n-p$ mass differences have the correct sign but are several times their observed values. Choice of one other parameter to give the correct $n-p$ mass difference would lead to even better agreement with experiment for the other states, but would also lead to proton and neutron isobars lying 20 MeV above the ground state.

I. INTRODUCTION

EACH level of theoretical effort, from Newtonian physics to relativity to relativistic quantum field theory, is solidly based on the discipline which preceded it. The correspondence principle shows us how each quantum theory has its classical point-particle limit, and how each second-quantized theory has its classical field-theoretic limit. It is reasonable to expect, then, that a dynamical theory of elementary particles and nuclear forces will also have its roots in classical mechanics and quantum mechanics.

For some years there has existed, within the approximation of classical relativistic particle mechanics, a theory of the dynamics of a spinning particle which possesses moments of inertia about more than one axis.[1] As a consequence of the equations of motion for such a particle, it was found that the mass of the particle is not required to be a constant of the motion and that the intrinsic spin angular momentum is the sum of two vectors, along and perpendicular to the angular velocity.[2] For the case of a pure gyroscope, for which the moments of inertia about all axes normal to the spin axis are zero, the theory reduced to the classical limit of the Dirac and Kemmer-Duffin theories, in so far as it is possible to distinguish spin and quantum effects in going to this limit.

More recently, the quantum theory corresponding to the more general case was formulated.[3] The essential features of the more general classical case were shown to be retained in the quantum theory. The variable mass of the classical particle theory became an operator in the corresponding quantum theory, and, as in the classical case, the spin became the sum of two operators, one of which is the usual spin operator. Thus, the generalized Dirac equation, for example, now includes an extra term and may describe a particle of spin other than $\frac{1}{2}$.

In Ref. 3, the laws of conservation of momentum and angular momentum were shown to lead to an expression for the mass operator, so that we obtained the relativistic wave equation

$$(i\epsilon_\mu P_\mu + Mc)\psi = 0\,, \tag{1.1}$$

where

$$\begin{aligned} P_\mu &= p_\mu - (e/c)A_\mu\,,\\ M &= m + m_0\epsilon_{\mu\nu}\lambda_{\mu\nu} + m'\epsilon_\mu\lambda_\mu\,, \end{aligned} \tag{1.2}$$

and m, m_0, and m' are arbitrary parameters.

The spin of the particle is now

$$S_{\mu\nu} = -i\hbar(\epsilon_{\mu\nu} + \lambda_{\mu\nu})\,, \tag{1.3}$$

where

$$\begin{aligned} [\epsilon_{\mu\nu},\epsilon_\sigma] &= \epsilon_\mu\delta_{\nu\sigma} - \epsilon_\nu\delta_{\mu\sigma}\,,\\ [\lambda_{\mu\nu},\lambda_\sigma] &= \lambda_\mu\delta_{\nu\sigma} - \lambda_\nu\delta_{\mu\sigma}\,, \end{aligned} \tag{1.4}$$

$$\begin{aligned} [\epsilon_{\mu\nu},\epsilon_{\sigma\tau}] &= -(\epsilon_{\mu\sigma}\delta_{\nu\tau} + \epsilon_{\nu\tau}\delta_{\mu\sigma} - \epsilon_{\mu\tau}\delta_{\nu\sigma} - \epsilon_{\nu\sigma}\delta_{\mu\tau})\,,\\ [\lambda_{\mu\nu},\lambda_{\sigma\tau}] &= -(\lambda_{\mu\sigma}\delta_{\nu\tau} + \lambda_{\nu\tau}\delta_{\mu\sigma} - \lambda_{\mu\tau}\delta_{\nu\sigma} - \lambda_{\nu\sigma}\delta_{\mu\tau})\,,\\ [\epsilon_\mu,\lambda_\nu] &= 0\,,\quad [\epsilon_{\mu\nu},\lambda_{\sigma\tau}] = 0\,; \end{aligned} \tag{1.5}$$

hence, $[S_{\mu\nu},M]=0$.

For $m'=0$, the theory exhibits a detailed correspondence with the classical theory. The classical point-particle equations relating the spin $S_{\mu\nu}$, angular velocity $\omega_{\mu\nu}$, and mass M are[1,2]

$$\begin{aligned} \dot S_{\mu\nu} &= Ic(\dot\omega_{\mu\nu} + \dot\Omega_{\mu\nu}) = -(v_\mu P_\nu - v_\nu P_\mu)\,,\\ \dot x_\mu &= v_\mu\,,\quad (v_\mu v_\mu = -1)\,,\\ \dot\Omega_{\mu\nu} &= -(\kappa/Ic)(\omega_{\mu\sigma}\ddot\omega_{\sigma\nu} - \omega_{\nu\sigma}\ddot\omega_{\sigma\mu})\,,\\ \dot M &= (\kappa/2Ic^2)\ddot\omega_{\mu\nu}\dot S_{\mu\nu}\,, \end{aligned} \tag{1.6}$$

$$M = -(\kappa/4c)\omega_{\mu\nu}\ddot\omega_{\mu\nu} + m\,. \tag{1.7}$$

These may be compared with the similar equations derived from (1.1) for $m'=0$:

$$\begin{aligned} \dot S_{\mu\nu} &= -i\hbar(\dot\epsilon_{\mu\nu} + \dot\lambda_{\mu\nu}) = -i(\epsilon_\mu P_\nu - \epsilon_\nu P_\mu)\,,\\ \dot x_\mu &= v_\mu = i\epsilon_\mu\,,\\ \dot\lambda_{\mu\nu} &= -(2im_0c/\hbar)(\epsilon_{\mu\sigma}\lambda_{\sigma\nu} - \epsilon_{\nu\sigma}\lambda_{\sigma\mu})\,,\\ \dot M &= (im_0c/\hbar)\lambda_{\mu\nu}\dot S_{\mu\nu}\,, \end{aligned} \tag{1.8}$$

$$M = m_0\epsilon_{\mu\nu}\lambda_{\mu\nu} + m\,. \tag{1.9}$$

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[1] H. J. Bhabha and H. C. Corben, Proc. Roy. Soc. (London) **A178**, 273 (1941); S. Shanmugadhasan, Proc. Cambridge Phil. Soc. **43**, 106 (1947).

[2] H. C. Corben, Proc. Natl. Acad. Sci. U. S. **48**, 387 (1962). [S, σ of this reference are here replaced by ω, S, respectively.]

[3] H. C. Corben, Proc. Natl. Acad. Sci. U. S. **48**, 1559 (1962).

If we write

$$I c \omega_{\mu\nu} = -i\hbar \epsilon_{\mu\nu},$$
$$I c \Omega_{\mu\nu} = -i\hbar \lambda_{\mu\nu},$$
$$\hbar \kappa \dot{\omega}_{\mu\nu} = 2 i m_0 c^2 I \lambda_{\mu\nu},$$

we note that each of Eqs. (1.6) becomes formally identical with the corresponding equation of (1.8) and that (1.7) assumes the form of (1.9), apart from a factor 2.

Equations (1.5) are satisfied if $\epsilon_{\mu\nu}$ and $\lambda_{\mu\nu}$ assume the form of either of the operators

$$\tfrac{1}{4}\gamma_{\mu\nu} \equiv \tfrac{1}{4}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) \qquad (1.10)$$

or

$$\beta_{\mu\nu} \equiv \beta_\mu\beta_\nu - \beta_\nu\beta_\mu, \qquad (1.11)$$

where γ_μ and β_μ, respectively, satisfy the Dirac and Kemmer-Duffin commutation relations

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu},$$
$$\beta_\mu\beta_\nu\beta_\sigma + \beta_\sigma\beta_\nu\beta_\mu = \beta_\mu\delta_{\nu\sigma} + \beta_\sigma\delta_{\mu\nu}.$$

Equations (1.4) are then satisfied if ϵ_μ, λ_μ are proportional to either γ_μ or β_μ.

If $\epsilon_{\mu\nu}$ is given by (1.10) and ϵ_μ is a constant times γ_μ, Eq. (1.1) becomes a generalization of the Dirac equation to describe a particle of spin given by (1.3),

$$S_{\mu\nu} = -\tfrac{1}{4} i\hbar (\gamma_{\mu\nu} + 4\lambda_{\mu\nu}). \qquad (1.12)$$

On the other hand, if $\epsilon_{\mu\nu}$ is given by (1.11) and ϵ_μ is a constant times β_μ, Eq. (1.1) becomes a generalization of the Kemmer equation to describe a particle of spin

$$S_{\mu\nu} = -i\hbar (\beta_{\mu\nu} + \lambda_{\mu\nu}). \qquad (1.13)$$

If the $\lambda_{\mu\nu}$ are also of the form (1.10) or (1.11) (in a different space, since they commute with $\epsilon_{\mu\nu}$) the generalized Dirac equation will then describe particles of spin 0, $\frac{1}{2}$, 1, $\frac{3}{2}$, and the generalized Kemmer equation will yield all values of the spin up to 2. More general forms for $\lambda_{\mu\nu}$ lead to particles of higher spin.

In this paper, we first develop the classical field theory of the generalized Dirac and Kemmer equations derived from Eq. (1.1) when M is any Hermitean operator which commutes with P_μ but not with ϵ_μ. We then consider in detail the particle states of spin $\frac{1}{2}$ and $\frac{3}{2}$ obtained from the special case of the generalized Dirac equation when we set $\lambda_\mu = \beta_\mu$, $\lambda_{\mu\nu} = \beta_{\mu\nu}$ in Eq. (1.1):

$$(i\epsilon_\mu P_\mu + mc + m_0 c \epsilon_{\mu\nu}\beta_{\mu\nu} + m' c \epsilon_\mu \beta_\mu)\psi = 0, \qquad (1.14)$$

with

$$\epsilon_\mu = \tfrac{1}{2}\gamma_\mu, \quad \epsilon_{\mu\nu} = \tfrac{1}{4}\gamma_{\mu\nu}, \quad \epsilon_\mu\epsilon_\mu = 1 \quad [\text{cf. } (1.6), (1.8)].$$

In general, this equation leads to eight distinct eigenvalues for the rest-energy of particles of spin $\frac{1}{2}$ and four such eigenvalues for particles of spin $\frac{3}{2}$, but for the special case $m'=0$, which exhibits a closer correspondence with the classical point-particle theory, there are four mass eigenvalues for spin $\frac{1}{2}$ and two for spin $\frac{3}{2}$.

For $m'=0$, $\eta_5 = \eta_1\eta_2\eta_3\eta_4$ commutes with the Hamiltonian ($\eta_\mu = 2\beta_\mu^2 - 1$) and it is found that neutral particles are characterized by $\eta_5 = -1$, charged particles by $\eta_5 = +1$, the charge density being $-\frac{1}{2}e_0\psi^*\eta_4(1+\eta_5)\psi$.

For $m'=0$, $m_0 = -\frac{1}{8}m$, Eq. (1.14) becomes

$$[i\epsilon_\mu P_\mu + mc(1 - \tfrac{1}{4}\textstyle\sum_{\mu<\nu}\epsilon_{\mu\nu}\beta_{\mu\nu})]\psi = 0. \qquad (1.15)$$

It is shown in this paper that for the choice $m = 1297 m_e = m_p/\sqrt{2}$ of the parameter m, Eq. (1.15) with $\epsilon_\mu = \frac{1}{2}\gamma_\mu$ correctly describes the charges and spins of the hyperons and resonances Ξ^-, Ξ^0, n, p, N^{**+}, N^{**0} (assuming spin $\frac{1}{2}$ for Ξ). In addition, the equation leads to mass values of these states which are accurate to better than 2%. The neutron described by Eq. (1.15) is found to be heavier than the proton, and the Ξ^- heavier than the Ξ^0, although the magnitudes of these mass differences are several times the observed values. Hyperons of strangeness ± 1 are not described by this special case of Eq. (1.1) and a study of the other fermions and bosons given by Eq. (1.1) is in progress.

While one should not expect greater accuracy from a classical field theory, the case in which m' is a small imaginary quantity has also been investigated. The choice $m' = 24 i m_e$, coupled with the values $m = 1297 m_e$, $m_0 = -\frac{1}{8}m$ as before, not only gives the correct values for both m_p and m_n and their difference, it also materially improves the agreement with experiment for the masses of other particles described by this equation. However, such a nonzero value for m' would split the proton-antiproton state, giving an excited level of the proton (and its corresponding antiproton) lying approximately 20 MeV above the ground state. An excited neutron state lying at approximately the same height above the ground state would also be predicted by the case $m' = 24 i m_e$, together with some fine structure for the Ξ^0 and N^{**} states.

We emphasize, however, that the main purpose of this paper is to describe the simple generalization of classical field theory which is required if the mass is an operator which commutes with the momentum but not with the γ or β matrices, and to give an example of the hierarchy of states to which it leads. Unless more details emerge correctly from the corresponding quantized field theory, it would be premature to identify these states with those observed in nature.

II. FIELD THEORY OF GENERALIZED DIRAC AND KEMMER EQUATIONS

We first consider the equation

$$(\gamma_\mu \partial_\mu + \kappa)\psi = 0, \qquad (2.1)$$

where γ_μ are the Dirac operators and κ is an operator which commutes with ∂_μ, but not with γ_μ. We define

$$\psi^\dagger = i\psi^*\gamma_4\chi,$$

where χ is an operator possessing the properties

$$[\chi, \partial_\mu] = 0; \quad [\chi, \gamma_\mu] = 0; \quad [\gamma_4\chi, \kappa] = 0. \qquad (2.2)$$

It then follows that, if κ is Hermitean,[4]

$$\partial_\mu \psi^\dagger \gamma_\mu - \psi^\dagger \kappa = 0, \qquad (2.3)$$

[4] The case in which κ has a small anti-Hermitean part must be treated separately.

so that we may define a conserved density

$$s_\mu=\psi^\dagger\gamma_\mu\psi. \tag{2.4}$$

The energy-momentum tensor

$$T_{\mu\nu}=-\tfrac{1}{2}i\hbar c(\psi^\dagger\gamma_\nu\partial_\mu\psi-\partial_\mu\psi^\dagger\gamma_\nu\psi) \tag{2.5}$$

satisfies

$$\partial_\nu T_{\mu\nu}=0,$$

as in the constant mass case, but the symmetrized tensor

$$\tilde{T}_{\mu\nu}=\tfrac{1}{2}(T_{\mu\nu}+T_{\nu\mu}) \tag{2.6}$$

is now no longer conserved. If, however, we introduce the tensor

$$\Sigma_{\mu\nu}=-\Sigma_{\nu\mu}=\tfrac{1}{4}i\hbar c[\psi^\dagger(\gamma_\nu\gamma_\mu\kappa+\kappa\gamma_\mu\gamma_\nu)\psi-2\delta_{\mu\nu}\psi^\dagger\kappa\psi], \tag{2.7}$$

it is found that $\partial_\nu\theta_{\mu\nu}=0$, where

$$\begin{aligned}\theta_{\mu\nu}&=\tilde{T}_{\mu\nu}+\Sigma_{\mu\nu}\\&=T_{\mu\nu}+\tfrac{1}{8}i\hbar c\partial_\rho[\psi^\dagger(\gamma_\rho\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu\gamma_\rho)\psi].\end{aligned} \tag{2.8}$$

The tensor $\Sigma_{\mu\nu}$ vanishes for $\kappa=$const, but here it is nonzero and antisymmetric.

The usual angular momentum of the Dirac theory,

$$\begin{aligned}P_{ik}&=-(i/c)\int(x_i\theta_{k4}-x_k\theta_{i4})dV\\&=-\hbar\int(x_i\psi^\dagger\gamma_4\partial_k\psi-x_k\psi^\dagger\gamma_4\partial_i\psi)dV\\&\quad-\tfrac{1}{4}\hbar\int\psi^\dagger\gamma_4\gamma_{ik}\psi dV,\quad(\gamma_{ik}=\gamma_i\gamma_k-\gamma_k\gamma_i),\end{aligned} \tag{2.9}$$

is no longer conserved, since now the tensor $\theta_{\mu\nu}$ is not symmetrical:

$$\begin{aligned}dP_{ik}/dt&=\int(\theta_{ki}-\theta_{ik})dV\\&=\tfrac{1}{4}i\hbar c\int\psi^\dagger[\gamma_{ik},\kappa]\psi dV.\end{aligned}$$

However,

$$J_{ik}=P_{ik}-\hbar\Lambda_{ik} \tag{2.10}$$

is then conserved if

$$d\Lambda_{ik}/dt=\tfrac{1}{4}ic\int\psi^\dagger[\gamma_{ik},\kappa]\psi dV.$$

Writing

$$\Lambda_{ik}=\int\psi^\dagger\gamma_4\lambda_{ik}\psi dV,$$

where λ_{ik} commutes with γ_j, we then have

$$d\Lambda_{ik}/dt=-ic\int\psi^\dagger[\lambda_{ik},\kappa]\psi dV,$$

so that we require that

$$\int\psi^\dagger[\tfrac{1}{4}\gamma_{ik}+\lambda_{ik},\kappa]\psi dV=0. \tag{2.11}$$

This condition is satisfied if $(\frac{1}{4}\gamma_{ik}+\lambda_{ik})$ commutes with κ. The conserved angular momentum J_{ik} is obtained from (2.9) by replacing γ_{ik} in the last term by $\gamma_{ik}+4\lambda_{ik}$. The equation then describes a particle of spin [cf. (1.12)]

$$S_{ik}=\tfrac{1}{4}\hbar\int\psi^\dagger\gamma_4(\gamma_{ik}+4\lambda_{ik})\psi dV. \tag{2.12}$$

If $\lambda_{\mu\nu}$ commutes with γ_σ, the antisymmetrical part of $\theta_{\mu\nu}$ may now be expressed as a divergence:

$$\begin{aligned}\Sigma_{\mu\nu}&=-\tfrac{1}{8}i\hbar c\psi^\dagger[\gamma_{\mu\nu},\kappa]\psi\\&=-\tfrac{1}{2}i\hbar c\partial_\rho(\psi^\dagger\gamma_\rho\lambda_{\mu\nu}\psi).\end{aligned}$$

Further,

$$\theta_{44}=\tilde{T}_{44}=T_{44}=-W(\psi^*\chi\psi), \tag{2.13}$$

for an eigenstate ψ of $i\hbar(\partial/\partial t)$ belonging to the eigenvalue, W.

The conditions

$$[\lambda_{\mu\nu},\gamma_\sigma]=0,\quad[\tfrac{1}{4}\gamma_{\mu\nu}+\lambda_{\mu\nu},\kappa]=0 \tag{2.14}$$

are those used in Ref. 3.

The generalized Kemmer equation[5]

$$(\beta_\mu\partial_\mu+\kappa)\psi=0 \tag{2.15}$$

may be developed in a similar manner. Here, κ is an operator which commutes with ∂_μ but not with β_μ. We now define

$$\psi^\dagger=i\psi^*\eta_4\chi,$$

where $\eta_4=2\beta_4{}^2-1$ and χ possesses properties similar to those of Eq. (2.2): $[\chi,\beta_\mu]=0$, $[\eta_4\chi,\kappa]=0$. It then follows that

$$\partial_\mu\psi^\dagger\beta_\mu-\psi^\dagger\kappa=0 \tag{2.16}$$

and that

$$s_\mu=\psi^\dagger\beta_\mu\psi$$

is conserved. We then have

$$\begin{aligned}\beta_\nu\beta_\mu\kappa\partial_\nu\psi&=\kappa\partial_\mu\psi,\\\partial_\nu\psi^\dagger\kappa\beta_\mu\beta_\nu&=\partial_\mu\psi^\dagger\kappa,\end{aligned}$$

so that

$$\partial_\mu(\psi^\dagger\kappa\psi)=\psi^\dagger\beta_\nu\beta_\mu\kappa\partial_\nu\psi+\partial_\nu\psi^\dagger\kappa\beta_\mu\beta_\nu\psi.$$

In this case we define

$$\begin{aligned}T_{\mu\nu}&=-\tfrac{1}{2}i\hbar c[\psi^\dagger\beta_\nu\partial_\mu\psi-\partial_\mu\psi^\dagger\beta_\nu\psi),\\\theta_{\mu\nu}&=i\hbar c[\psi^\dagger(\beta_\nu\beta_\mu\kappa+\kappa\beta_\mu\beta_\nu)\psi-\delta_{\mu\nu}\psi^\dagger\kappa\psi],\end{aligned} \tag{2.17}$$

so that

$$\partial_\nu T_{\mu\nu}=\partial_\nu\theta_{\mu\nu}=0,$$

and

$$\theta_{\mu\nu}=T_{\mu\nu}+\tfrac{1}{2}i\hbar c\partial_\rho[\psi^\dagger(\beta_\rho\beta_\mu\beta_\nu-\beta_\nu\beta_\mu\beta_\rho)\psi]. \tag{2.18}$$

$\theta_{\mu\nu}$ is symmetrical only in the case in which κ is a c-number, so that we define

$$J_{ik}=-\frac{i}{c}\int(x_i\theta_{k4}-x_k\theta_{i4})dV-\hbar\Lambda_{ik},$$

[5] N. Kemmer, Proc. Roy. Soc. (London) **A173**, 91 (1939).

where

$$\Lambda_{ik}=\int\psi^\dagger\beta_4\lambda_{ik}\psi dV$$

and $(\lambda_{\mu\nu},\beta_\rho)=0$. Thus, J_{ik} is conserved if

$$\int\psi^\dagger(\beta_{ik}+\lambda_{ik},\kappa)\psi dV=0, \tag{2.19}$$

the spin of the particle being [cf. (1.13)]

$$S_{ik}=\hbar\int\psi^\dagger\beta_4(\beta_{ik}+\lambda_{ik})\psi dV. \tag{2.20}$$

We may now define a symmetrical energy-momentum tensor which differs from $\theta_{\mu\nu}$ only by a divergence,

$$\tau_{\mu\nu}=\tfrac{1}{2}i\hbar c\{\psi^\dagger[(\beta_\mu\beta_\nu+\beta_\nu\beta_\mu)\kappa+\kappa(\beta_\mu\beta_\nu+\beta_\nu\beta_\mu)]\psi-2\delta_{\mu\nu}\psi^\dagger\kappa\psi\} \tag{2.21}$$

$$\begin{aligned}&=\theta_{\mu\nu}+\tfrac{1}{2}i\hbar c\psi^\dagger[\beta_{\mu\nu},\kappa]\psi\\&=\theta_{\mu\nu}-\tfrac{1}{2}i\hbar c\psi^\dagger[\lambda_{\mu\nu},\kappa]\psi\\&=\theta_{\mu\nu}+\tfrac{1}{2}i\hbar c\partial_\rho(\psi^\dagger\beta_\rho\lambda_{\mu\nu}\psi).\end{aligned} \tag{2.22}$$

Further,

$$\begin{aligned}\tau_{44}=\theta_{44}&=\tfrac{1}{2}i\hbar c\psi^\dagger(\eta_4\kappa+\kappa\eta_4)\psi\\&=-\tfrac{1}{2}\hbar c\psi^*(\chi\kappa+\kappa\chi)\psi \qquad (2.23)\\&\to -mc^2\psi^*\psi \quad \text{for} \quad \chi=1, \quad \kappa=mc/\hbar=\text{const.}\end{aligned}$$

III. THE GENERALIZED DIRAC EQUATION

If one of the ϵ_μ, λ_ν is a set of Dirac operators $\frac{1}{2}\gamma_\mu$ and the other is a set of Kemmer-Duffin operators β_ν, the resulting equation describes fermions with spin tensor given by

$$S_{\mu\nu}=-i\hbar[\beta_{\mu\nu}+\tfrac{1}{4}\gamma_{\mu\nu}].$$

Although the mass operator is the same in each case, the operators multiplying the p_μ are different, and we therefore obtain two distinct equations describing particles of spin $\frac{1}{2}$ and $\frac{3}{2}$. In this paper, we consider only one of these [Eq. (1.14)]. We use the notation $\gamma_i=\rho_2\boldsymbol{\sigma}(i=1, 2, 3)$, $\gamma_4=\rho_3$, and

$$\begin{aligned}\boldsymbol{\Sigma}&=-i(\beta_{23},\beta_{31},\beta_{12}),\\\boldsymbol{\lambda}&=-i(\beta_{14},\beta_{24},\beta_{34}),\\\boldsymbol{\beta}&=(\beta_1,\beta_2,\beta_3),\end{aligned} \tag{3.1}$$

so that the spin of the particle is

$$S=\hbar(\tfrac{1}{2}\boldsymbol{\sigma}+\boldsymbol{\Sigma}). \tag{3.2}$$

The spin-$\frac{1}{2}$ states are therefore characterized by

$$\boldsymbol{\sigma}\cdot\boldsymbol{\Sigma}=-2 \quad (\Sigma^2=2) \quad (\downarrow,\uparrow), \tag{3.3}$$

or by

$$\boldsymbol{\sigma}\cdot\boldsymbol{\Sigma}=0 \quad (\Sigma^2=0) \quad (\uparrow,\cdot). \tag{3.4}$$

The spin-$\frac{3}{2}$ states are similarly characterized by

$$\boldsymbol{\sigma}\cdot\boldsymbol{\Sigma}=1 \quad (\Sigma^2=2) \quad (\uparrow,\uparrow). \tag{3.5}$$

If in Eq. (1.14) m, m_0, m' are real parameters, the conserved density (2.4) may be written

$$s_\mu=-ic\psi^*\gamma_4\eta_4\gamma_\mu\psi, \tag{3.6}$$

where $\eta_4=2\beta_4^2-1$, so that η_4 commutes with β_4 and anticommutes with β_1,β_2,β_3. For the special case $m'=0$, however, we note that $\eta_5\equiv\eta_1\eta_2\eta_3\eta_4$ commutes with the Hamiltonian, so that in this case we may define another conserved four-vector j_μ which we identify with the charge-current density

$$j_\mu=-\tfrac{1}{2}e_0c\psi^\dagger(1+\eta_5)\gamma_\mu\psi, \tag{3.7}$$

where $\psi^\dagger=i\psi^*\gamma_4\eta_4$. The charge density is, therefore, for $m'=0$,

$$\rho=-ij_4/c=-\tfrac{1}{2}e_0\psi^*\eta_4(1+\eta_5)\psi. \tag{3.8}$$

In the 5×5 representation of the β_μ, η_5 is diagonal with the value -1 for the first four elements and $+1$ for the fifth, while in the 10×10 representation it is $+1$ for the first six elements and -1 for the others.

We note that for m' imaginary, the four-vector $\psi^\dagger\eta_5\gamma_\mu\psi$ is strictly conserved, and that for small imaginary m' the currents (3.6), (3.7) are separately conserved only approximately.

In the rest system of the particle ($\mathbf{p}=0$), the energy operator according to Eq. (1.14) is given by

$$(W/c^2)\psi=[2\rho_3m-2m_0(\rho_3\boldsymbol{\sigma}\cdot\boldsymbol{\Sigma}+i\rho_2\boldsymbol{\sigma}\cdot\boldsymbol{\lambda})-m'(i\rho_1\boldsymbol{\sigma}\cdot\boldsymbol{\beta}-\beta_4)]\psi. \tag{3.9}$$

The eigenvalues of W for the case $m'=0$ have been computed in Ref. 6 (in an attempt to apply this equation to the electron, muon, and two types of neutrino). More generally, we now write

$$\frac{W-2mc^2}{2m_0c^2}=\alpha,\quad \frac{W+2mc^2}{2m_0c^2}=\beta,\quad \frac{m'}{2m_0}=\epsilon.$$

For the 5×5 representation[5] of the β_μ, the 20-component spinor ψ decomposes into four separate 5-component spinors, e.g.,

$$\begin{pmatrix}\alpha & 1 & -i & 1 & i\epsilon\\-1 & \beta & -i & 1 & i\epsilon\\-i & i & \beta & i & -\epsilon\\-1 & 1 & -i & \beta & i\epsilon\\-i\epsilon & i\epsilon & \epsilon & i\epsilon & \alpha\end{pmatrix}\begin{pmatrix}\psi_1\\\psi_{17}\\\psi_{18}\\\psi_{14}\\\psi_5\end{pmatrix}=0. \tag{3.10}$$

In this subspace,

$$\boldsymbol{\sigma}\cdot\boldsymbol{\Sigma}=\begin{pmatrix}0, & 0, & 0, & 0, & 0\\0, & 0, & i, & -1, & 0\\0, & -i, & 0, & -i, & 0\\0, & -1, & i, & 0, & 0\\0, & 0, & 0, & 0, & 0\end{pmatrix},$$

$$S_z=\tfrac{1}{2}\hbar\begin{pmatrix}1, & 0, & 0, & 0, & 0\\0, & -1, & -2i, & 0, & 0\\0, & 2i, & -1, & 0, & 0\\0, & 0, & 0, & 1, & 0\\0, & 0, & 0, & 0, & 1\end{pmatrix}. \tag{3.11}$$

The characteristic equations for the other spinors are obtained by replacing α by $-\beta$, β by $-\alpha$ (i.e., changing the sign of W), and by reversing the spin direction.

Three solutions of (3.10) for states of spin $\frac{1}{2}$ ($\boldsymbol{\sigma}\cdot\boldsymbol{\Sigma}=-2$ or 0) are

$$\Psi_{1/2,\uparrow}=\begin{bmatrix}\alpha(\beta-1)\\ \alpha(\alpha+1)\\ i\alpha(\alpha+1)\\ \alpha(\alpha+1)\\ i\epsilon(\beta-3\alpha-4)\end{bmatrix}, \tag{3.12}$$

with

$$\alpha(\alpha\beta+2\alpha+3)=\epsilon^2(\beta-3\alpha-4). \tag{3.13}$$

These are also eigenstates of S_z belonging to the eigenvalue $\frac{1}{2}\hbar$.

For $m'=0$, Eq. (3.13) breaks up into three states

$$\begin{aligned}&\alpha=0,\quad W=2mc^2,\quad \eta_5=1,\quad \eta_4=1,\quad \gamma_4=1,\\ &\qquad \alpha\beta+2\alpha+3=0,\quad \eta_5=-1,\quad \gamma_4\eta_4=1.\end{aligned} \tag{3.14}$$

According to (3.8), the first of these is negatively charged and has a mass $2m=2594m_e$, and we would identify it with the Ξ^- hyperon. The other two particles of Eq. (3.14) are neutral ($\eta_5=-1$), and we would identify them with the antineutron and Ξ^0, of calculated masses $-1876m_e$ and $2524m_e$, respectively.

Equation (3.11) also gives rise to two states of spin $\frac{3}{2}$ ($\boldsymbol{\sigma}\cdot\boldsymbol{\Sigma}=1$):

$$\Psi_{3/2,-3/2}=\begin{bmatrix}0\\1\\-i\\0\\0\end{bmatrix};\quad \Psi_{3/2,1/2}=\begin{bmatrix}0\\1\\i\\-2\\0\end{bmatrix}, \tag{3.15}$$

where the second suffix on Ψ refers to the eigenvalue of S_z. These states have the same mass

$$\beta=1,\quad W=2(m_0-m)c^2\quad (\uparrow,\uparrow), \tag{3.16}$$

with $\eta_4=-1$, $\eta_5=-1$. We identify this state with the neutral component of the anti-N^{**} resonance $\gamma_4=-1$ [calculated mass $-2918m_e$ for the values of m_0, m given in Eq. (1.15)].

In the 10×10 representation of the β_μ, the spinor Ψ has 40 components, decomposing into four spinors of ten components each, e.g.,

$$\begin{bmatrix}
\alpha & -i & 1 & 0 & 1 & i & i\epsilon & 0 & 0 & -i\epsilon\\
i & \alpha & -i & -1 & 0 & 1 & 0 & i\epsilon & 0 & -\epsilon\\
1 & i & \alpha & i & -1 & 0 & 0 & 0 & i\epsilon & i\epsilon\\
0 & 1 & i & \beta & i & -1 & 0 & -i\epsilon & \epsilon & 0\\
-1 & 0 & 1 & -i & \beta & i & i\epsilon & 0 & -i\epsilon & 0\\
i & -1 & 0 & -1 & -i & \beta & \epsilon & i\epsilon & 0 & 0\\
-i\epsilon & 0 & 0 & 0 & i\epsilon & -\epsilon & \alpha & -i & 1 & 1\\
0 & -i\epsilon & 0 & -i\epsilon & 0 & i\epsilon & i & \alpha & -i & -i\\
0 & 0 & -i\epsilon & -\epsilon & -i\epsilon & 0 & 1 & i & \alpha & -1\\
-i\epsilon & \epsilon & i\epsilon & 0 & 0 & 0 & -1 & -i & 1 & \beta
\end{bmatrix}
\begin{bmatrix}\psi_1\\ \psi_2\\ \psi_{13}\\ \psi_{24}\\ \psi_{25}\\ \psi_{36}\\ \psi_7\\ \psi_8\\ \psi_{19}\\ \psi_{40}\end{bmatrix}=0. \tag{3.17}$$

$$\boldsymbol{\sigma}\cdot\boldsymbol{\Sigma}=\begin{bmatrix}x & 0 & 0 & \cdot\\ 0 & x & 0 & \cdot\\ 0 & 0 & x & \cdot\\ \cdot & \cdot & \cdot & \cdot\end{bmatrix},\quad S_z=\tfrac{1}{2}\hbar\begin{bmatrix}y & 0 & 0 & \cdot\\ 0 & y & 0 & \cdot\\ 0 & 0 & y & \cdot\\ \cdot & \cdot & \cdot & -1\end{bmatrix},$$

with

$$x=\begin{bmatrix}0 & -i & 1\\ i & 0 & -i\\ 1 & i & 0\end{bmatrix},\quad y=\begin{bmatrix}1, & -2i, & 0\\ 2i, & 1, & 0\\ 0, & 0, & -1\end{bmatrix},$$

the dots representing a single row and column.

Four solutions of (3.17) for states of spin $\frac{1}{2}$ ($\boldsymbol{\sigma}\cdot\boldsymbol{\Sigma}=-2$ or 0) are given by

$$\begin{bmatrix}\psi\\ -i\psi\\ -\psi\\ \phi\\ -i\phi\\ -\phi\\ \chi\\ -i\chi\\ -\chi\\ \lambda\end{bmatrix},\quad \text{with}\begin{cases}(\alpha-2)\psi-2i\phi+i\epsilon\chi-i\epsilon\lambda=0,\\ -2i\psi+(\beta+2)\phi-2\epsilon\chi=0,\\ -i\epsilon\psi+2\epsilon\phi+(\alpha-2)\chi+\lambda=0,\\ -3i\epsilon\psi-3\chi+\beta\lambda=0,\end{cases}$$

so that

$$\begin{aligned}&(\alpha\beta+2\alpha-2\beta)(\alpha\beta-2\beta+3)\\ &\qquad+\epsilon^2(7\alpha\beta-\beta^2+6\alpha-2\beta-24)+12\epsilon^4=0\\ &\qquad\qquad (\downarrow,\uparrow)\quad (\uparrow,\cdot).\end{aligned} \tag{3.18}$$

For $m'=0$, Eq. (3.18) separates into four states, two of which (Ξ^0 and N) also appear in the 5×5 representation [Eq. (3.14)]:

$$\begin{aligned}&\alpha\beta-2\beta+3=0,\quad \eta_5=-1,\\ &\alpha\beta+2\alpha-2\beta=0,\quad \eta_5=+1.\end{aligned} \tag{3.19}$$

The first of Eqs. (3.19) is obtained from the second of Eqs. (3.14) by replacing α by $-\beta$ and β by $-\alpha$, i.e., by changing the sign of the energy. The second of Eqs. (3.19) now represents charged ($\eta_5=1$) particle and antiparticle states

$$W=\pm 2c^2(m^2+4mm_0)^{1/2},$$

which, with the same value as before for m and m_0, have the mass of the proton, i.e., $\pm\sqrt{2}m=\pm 1835m_e$.

The corresponding spin-$\frac{3}{2}$ states are described by the following six solutions of the same equations (3.17) for the case $\boldsymbol{\sigma}\cdot\boldsymbol{\Sigma}=1$:

$$\Psi_{3/2,3/2}=\begin{pmatrix}\psi\\ i\psi\\ 0\\ \phi\\ i\phi\\ 0\\ \chi\\ i\chi\\ 0\\ 0\end{pmatrix},\quad \Psi_{3/2,-1/2}=\begin{pmatrix}\psi\\ -i\psi\\ 2\psi\\ \phi\\ -i\phi\\ 2\phi\\ \chi\\ -i\chi\\ 2\chi\\ 0\end{pmatrix},$$

with

$$(\alpha+1)\psi+i\phi+i\epsilon\chi=0,$$
$$i\psi+(\beta-1)\phi+\epsilon\chi=0,$$
$$-i\epsilon\psi-\epsilon\phi+(\alpha+1)\chi=0,$$

so that

$$(\beta-2)\phi=-i(\alpha+2)\psi,$$
$$(\alpha+1)(\beta-2)\chi=-i\epsilon(\alpha-\beta+4)\psi,$$

and

$$(\alpha+1)(\alpha\beta+\beta-\alpha)+\epsilon^2(\alpha-\beta+4)=0\quad(\uparrow,\uparrow).\quad(3.20)$$

For $m'=0$, the spin-$\frac{3}{2}$ solution corresponding to the proton state [second of Eqs. (3.19)] is $(\chi=0)$

$$\alpha\beta+\beta-\alpha=0,\quad \eta_5=+1,$$

again representing charged particle and antiparticle states

$$W=\pm 2c^2(m^2-2mm_0)^{1/2}.$$

Again with the same values of m and m_0, this leads to a spin-$\frac{3}{2}$ excited proton state at $2900m_e$. The corresponding neutral state from Eq. (3.20) is given by $\alpha=-1$, for which $\psi=0$, $\phi=0$, so that $\eta_5=-1$, and its mass is $2918m_e$ [cf. Eq. (3.16)]. We are therefore led to identify the solutions of Eq. (3.20) with the N^{**} resonance.

Finally, for the 1×1 representation of the $\beta_\mu(\beta_\mu=0)$, Eq. (3.9) describes a particle of spin $\frac{1}{2}$, mass $2m=-2594m_e$ ($\eta_5=1$, $\gamma_4\eta_4=1$). According to (3.8), its charge is positive so that it would represent the $\bar{\Xi}^\dagger$.

In general, then, apart from the sign of W, there are eight distinct values of the rest-energy given by Eq. (3.9) for states of spin $\frac{1}{2}$. These are solutions of the equations

$$W=2m,\quad(3.21)$$

$$W^3+2(2m_0-m)W^2+2(6m_0^2-8mm_0-2m^2+m'^2)W$$
$$+8m(m-m_0)(m+3m_0)-8m'^2(m-m_0)=0,\quad(3.22)$$

$$W^4-4m_0W^3+2(6m_0^2-12mm_0-4m^2+3m'^2)W^2$$
$$+4(4m_0m^2+16mm_0^2-mm'^2+2m_0m'^2)W$$
$$+16m(m+4m_0)(m^2+2mm_0-3m_0^2)$$
$$-32m'^2(m^2+mm_0+3m_0^2)+12m'^4=0.\quad(3.23)$$

For the case $m'=0$, these equations reduce simply to four distinct eigenvalues:

$$\left.\begin{array}{l}W=2m,\\ W=-2m_0\pm2(m^2+2mm_0-2m_0^2)^{1/2},\end{array}\right\}(3.21),(3.22)$$

$$\left.\begin{array}{l}W=2m_0\pm2(m^2+2mm_0-2m_0^2)^{1/2},\\ W=\pm2[m(m+4m_0)]^{1/2}.\end{array}\right\}(3.23)\quad(3.24)$$

TABLE I. Eigenvalues of $\boldsymbol{\sigma}\cdot\boldsymbol{\Sigma}$ and $(\boldsymbol{\sigma}\cdot\boldsymbol{\lambda})^2$.

Equation number	Components	S_1	S_2	$(\boldsymbol{\sigma}\cdot\boldsymbol{\lambda})^2$
(3.12)	First four	−2	0	3
(3.12)	Fifth	0	0	0
(3.15)	All the nonzero	1	1	0
(3.18)	First six	−2	−2	4
(3.18)	Last four	−2	0	3
(3.20)	First six	1	1	1
(3.20)	Last four	1	1	0

Similarly, for states of spin $\frac{3}{2}$, there are in general four eigenvalues, which are given by

$$W=2(m_0-m),\quad(3.25)$$

$$W^3+2(m_0-m)W^2+4m(2m_0-m)W$$
$$+4(m-2m_0)(2m^2-2mm_0-m'^2)=0.\quad(3.26)$$

For $m'=0$, these reduce to two distinct values

$$W=2(m_0-m),\quad(3.25)$$

$$\left.\begin{array}{l}W=-2(m_0-m),\\ W=\pm2[m(m-2m_0)]^{1/2}.\end{array}\right\}(3.26)$$

All of these eigenvalues for the case $m'=0$ are given by the formula

$$W=m_0(S_2-S_1)$$
$$\pm\{[m_0(S_1+S_2)-2m]^2-4m_0^2(\boldsymbol{\sigma}\cdot\boldsymbol{\lambda})^2\}^{1/2},\quad(3.27)$$

where S_1 and S_2 are the eigenvalues of $\boldsymbol{\sigma}\cdot\boldsymbol{\Sigma}$ given in Table I. This leads to the mass relation

$$2\Xi^-+\Xi^0=n+2N^{**0}$$

which is accurate to better than 0.5%.

While the calculated masses agree with experimental observation to better than 2%, the mass differences $n-p$ and $\Xi^--\Xi^0$ are not accurately described by the case $m'=0$. We may therefore suppose that m' is a small imaginary quantity, the magnitude of which we adjust to give the correct neutron-proton mass difference. Writing $m_0=-am$ $(a=\frac{1}{8})$ as before, $m'=-i\delta m$ $(\delta\ll1)$, and neglecting terms of higher order than δ^2, we find that Eq. (3.23) becomes, with $W=2mc^2x$,

$$\left[x^2-\frac{\delta^2}{2a^2}(2a+1)x+4a-1-\frac{\delta^2}{2a^2}(1-4a^2)\right]$$
$$\times\left[x^2+\left(2a+\frac{\delta^2}{2a^2}(2a+1)\right)x+(3a^2+2a-1)-\frac{\delta^2}{2a^2}(3a^2-2a-1)\right]=0.$$

The masses of the n, Ξ^0, p states are, therefore,

$$M_n = (1.446 - 79.51\delta^2)m,$$
$$M_{\Xi^0}' = (1.946 + 0.49\delta^2)m, \quad (3.28)$$
$$M_p,\, M_p' = [1.414 + \delta^2(42.43 \pm 40.0)]m.$$

We would, therefore, obtain *two* proton states, and the mass difference between the neutron and the lower of these is

$$\delta m = (0.03137 - 81.94\delta^2)m.$$

This has the experimental value of $2.53m_e$ if $\delta^2 = 3.59\times 10^{-4}$, and the lower proton state then has the correct experimental value of $1836.1m_e$ if, as before, $m = 1297m_e$. Such a term would then lead to an excited proton state lying $37.2m_e = 19$ MeV above the ground state.

The mass m_{Ξ^0}' of the Ξ^0 particle given by (3.28) is $2524m_e$, but the Ξ^0 particle of Eq. (3.22) would then have a slightly different mass. From this equation we obtain

$$m_{\Xi^-} = \left(2 - \frac{80\delta^2}{3}\right)m \qquad = 2581m_e,$$
$$m_{\Xi^0} = (1.946 + 27.68\delta^2)m = 2537m_e,$$
$$m_n' = (1.446 + 1.01\delta^2)m \ = 1875m_e,$$

giving an excited neutron state also lying 19 MeV above the ground state.

The neutral and charged spin-$\frac{3}{2}$ resonances given by Eqs. (3.25) and (3.26) would now split as follows:

$$N^{**0}, \quad 2.250m = 2918m_e; \quad (3.25)$$
$$\left.\begin{array}{ll} N^{**0}, & (2.222m = 2882m_e) \\ N^{**+-}, & \begin{bmatrix} 2.265m = 2941m_e \\ 2.236m = 2888m_e \end{bmatrix} \end{array}\right\}. \quad (3.26)$$

These values lie below the observed spin-$\frac{3}{2}$ resonances by one or two percent.

These effects of a small term proportional to m' in Eq. (1.1) are presented only as an illustration of the type of splitting such a term produces. It would be expected that the eigenvalues of the bare Hamiltonian computed here would be changed by amounts larger than those considered above when radiative corrections are taken into account.

The particles discussed in this paper represent only a small fraction of those observed. The other equations [Ref. 6, Eqs. (14), (16), and (17)] that may be obtained from Eq. (1.1) by choosing ϵ_μ or λ_μ to be Dirac or Kemmer-Duffin operators lead between them to 5 distinct spin 0 rest-energies, 2 of spin $\frac{1}{2}$, 7 of spin 1, 1 of spin $\frac{3}{2}$, and 2 of spin 2. For a number of these, as in the case investigated in this paper, a neutral state is accompanied by a charged state whose rest-energy differs from it only by terms of order $(m_0/m)^2$. Other neutral states do not possess such a corresponding charged state. It is in this way that isobaric spin enters the theory.[7]

Whether this approach will ultimately lead to a meaningful description of elementary particles cannot be known until the full consequences of the theory, including interactions, have been investigated. The class of classical field theories developed here is solidly based on relativistic classical mechanics and is consistent with relativity theory and quantum theory. By relaxing the condition that M in Eq. (1.1) is a c number, we have been led to mass-eigenstates which are not required to be simple multiples of each other. The main purpose of this paper has been to point out this hitherto unsuspected richness which field theory can offer.

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[6] H. C. Corben, Proc. Natl. Acad. Sci. U. S. **48**, 1746 (1962).

[7] Thus, for example, ignoring all of the states in the 10×10 representation [Eq. (3.17)] other than those to which we have referred as proton and neutron states, and considering only the large components (for which $\beta_4^2 = 1$), it follows that $\beta_4\psi_p = -i\psi_n$, $\beta_4\psi_n = i\psi_p$. Since in general $\eta_5\psi_p = \psi_p$, $\eta_5\psi_n = -\psi_n$, and η_5 anticommutes with β_4, we may then write the usual components of the isobaric spin, thus, $\tau_1 = i\beta_4\eta_5$, $\tau_2 = -\beta_4$, and $\tau_3 = \eta_5$. Although operators of the *same* β algebra contribute a term to the ordinary spin $\mathbf{S}$ [Eq. (3.2)], it is found that $\mathbf{S}$ and $\boldsymbol{\tau}$ commute.