Feynman Rules for Regge Particles*

T. W. B. KIBBLE
Department of Physics, Imperial College of Science and Technology, London, England
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The analogy between Regge poles and poles due to single-particle exchange is extended to the case of many-particle amplitudes, by considering diagrams with two or more poles. A set of diagrams is obtained in which the internal lines represent Regge particles. The problem of coupling three particles of arbitrary but physical spin is treated first, and coupling constants depending on the helicities are defined. The vertex functions which couple three Regge particles, and which have similar symmetry properties, are defined in terms of the residues of Regge poles. The propagator for a Regge particle with trajectory $\alpha(t)$ is essentially a rotation matrix for spin α, corresponding to a rotation from the initial to the final direction of the center-of-mass momentum, divided by $\sin\pi(\alpha-\sigma)$, where σ is a constant which replaces the signature. The possibility of using this formalism to predict the high-energy behavior of production amplitudes is discussed, in particular, for single-particle production. As for elastic scattering, one can give a unified description of the low-energy and high-energy regions, and the Regge poles in appropriate crossed channels should dominate in the high-energy region.

1. INTRODUCTION

THE Regge pole approximation to scattering amplitudes[1] may be regarded as a modification of the ordinary pole approximation. In place of the exchange of a single light particle, one considers the exchange of a "Regge particle," which represents a whole family of particles or resonances associated with a single Regge trajectory, and which also has a quite different asymptotic behavior. There is, in fact, a close resemblance, which has been noted by various authors,[2] between the contributions of a particular Regge pole and a lowest order Feynman diagram, and it is natural to ask whether the Regge-pole formalism can be extended to production amplitudes by considering diagrams with two or more poles, like that of Fig. 1. The essential difficulty in doing this (apart from the problem of proving the required analyticity) is to know how to couple together particles of variable, and even complex, angular momentum. What is required is a set of "Feynman rules" for Regge particles which will allow the contribution of any such diagram, in which each internal line corresponds to a particular Regge trajectory, to be written down.

It must be emphasized that the analogy between Regge-pole diagrams and ordinary Feynman diagrams must not be carried too far. In conventional perturbation theory, the only free parameters are the masses and coupling constants, and the complete perturbation series is essentially determined by the first Born approximation and the (generalized) unitarity equations.[3] On the other hand, since Regge particles have variable mass, a Regge-pole diagram necessarily contains vertex functions rather than simply coupling constants, and both these functions and the Regge trajectories themselves are *a priori* undetermined. The effect of the unitarity equations is, therefore, likely to be rather different. There is no reason to believe that a better approximation can be obtained by including, in some sense, diagrams with closed loops of Regge particles. Instead, the unitarity equations should serve to determine, at least partially, the arbitrary functions appearing in the Regge formulas, though it is also conceivable that they may require the introduction of additional terms corresponding to cuts rather than poles in the angular momentum plane. An argument which suggests that this may be the case has been presented by Amati, Fubini, and Stanghellini.[4] However, it is quite possible that the cuts are cancelled by other contributions.[5] In any case, even if there are cuts, it is probably still true that there is a region of the invariants in which the pole terms are dominant. For simplicity, we shall assume in this paper that scattering amplitudes, suitably defined, are meromorphic in the right-half angular mometum plane, $\mathrm{Re}\,j>-\frac{1}{2}$.

A further motivation for the present work arises from the suggestion of Chew and Frautschi[6] that all particles are bound states in the Regge sense, that is, that they are all members of Regge families. If this conjecture is correct, then in a Regge-pole diagram the external lines as well as the internal lines should be regarded as representing Regge particles, and the three particles which are coupled together at any vertex should be treated in an essentially symmetric manner. We shall adopt this point of view in the present paper. However, it would clearly be easy to accommodate a number of non-Regge particles of fixed angular momentum.

A very convenient formalism for discussing the angu-

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[2] See, for instance, V. N. Gribov and I. Ya. Pomeranchuk, Phys. Rev. Letters **8**, 412 (1962).

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[5] J. C. Polkinghorne, Phys. Rev. **128**, 2459 (1962).

[6] G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **7**, 394 (1961); **8**, 41 (1962).

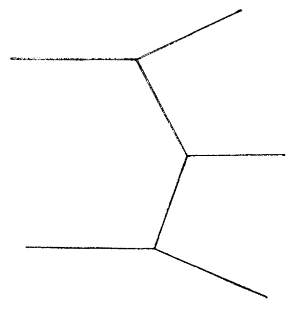

FIG. 1. A typical pole diagram for a production process.

lar momentum decomposition of many-particle amplitudes has been given by Wick[7] in terms of helicity states.[8] We shall use this formalism throughout. However, we find it very convenient for our particular purposes to make some changes of notation and normalization. The relevant definitions and formulas are summarized in Appendix A. In particular, it should be noted that the symbol $|\mathbf{0},sj\mu,\lambda_1,\lambda_2\rangle$ denotes a two-particle state in which the helicities are λ_1 and $-\lambda_2$. This convention may appear rather arbitrary, but, in fact, it is not unnatural to start with a state in which the center-of-mass momentum is in the z direction and the z components of spin are λ_1 and λ_2, and a considerable simplification results from so doing. We also note that a factor $[(2j+1)/4\pi]^{1/2}$ has been absorbed in the normalization of the state and, therefore, does not appear in the partial-wave expansion of the scattering amplitude.

We consider first, in Sec. 2, the problem of coupling together three particles of arbitrary, but physical, spins. This problem is of considerable intrinsic interest, quite apart from its possible generalization to complex spins. We define a set of "coupling constants" $g_{123}(\lambda_1,\lambda_2,\lambda_3)$ depending on the spin components λ_i, and vanishing unless $\lambda_1+\lambda_2+\lambda_3=0$. Apart from normalization factors, g_{123} is essentially the amplitude for the (virtual) decay at rest of particle 3 with z component of spin λ_3 into particles $\bar{1}$ and $\bar{2}$ with momenta in the positive and negative z directions and z components of spin $-\lambda_1$ and $-\lambda_2$, respectively. By rotational invariance, this amplitude is clearly sufficient to determine the general amplitude. It has certain symmetries under permutations of 1, 2, 3, and under change of sign of the three λ_i, which are easy to obtain.

We then consider the coupling of three Regge particles corresponding to the three trajectories $j_i=\alpha_i(t_i)$. This coupling may be specified by a vertex function $\Gamma_{123}(\lambda_1,\lambda_2,\lambda_3;\,t_1,t_2,t_3)$, where $t_i=m_i^2$, which has many of the properties of the coupling constant g_{123}. This function is defined in terms of the residues of Regge poles. In Sec. 3, we examine the contribution of a Regge pole to the scattering amplitude for a process $a+b\rightarrow\bar{c}+\bar{d}$. This discussion serves to define the vertex functions for the case when two of the spins are held fixed and physical, and also to identify the propagator for a Regge particle. The denominator is simply $\sin\pi(\alpha-\sigma)$, where σ is a constant angular momentum which plays the role of the signature, and the numerator consists essentially of a rotation matrix for angular momentum α corresponding to the rotation which takes the initial center-of-mass momentum into the final center-of-mass momentum.

In Sec. 4 we extend the discussion to the case of a five-particle amplitude, for the process $a+b\rightarrow\bar{c}+\bar{d}+\bar{e}$. We make an angular momentum decomposition in terms of the total angular momentum j and the angular momentum j' of particles $\bar{c}$ and $\bar{d}$ in their center-of-mass. Then, since this amplitude is coupled to the elastic a-b scattering amplitude by unitarity, and since the total angular momentum j is common to both, we must assume that the five-particle amplitude also is a meromorphic function of j in the right-half j plane, $\mathrm{Re}\,j>-\frac{1}{2}$. However, it is also related by crossing to the amplitude for the process $a+b+e\rightarrow\bar{c}+\bar{d}$, for which j' is the total angular momentum, and by a similar argument we may expect that this amplitude is meromorphic in j'. It is, therefore, natural to assume that the amplitude is simultaneously an analytic function of both variables, j and j', meromorphic in the product of the right-half planes. We can then make a double Sommerfeld-Watson transform, and pick up the contributions of the various Regge poles. We shall concentrate on the term arising from a particular pair of Regge poles, and show that it must have precisely the form suggested by the Feynman rules already obtained in the discussion of four-particle amplitudes. This also serves to define the vertex functions when only one of the three particles has fixed physical spin. To define them in complete generality we should have to go to the six-particle amplitudes. It is clear that the discussion could easily be extended to this case, with no essential changes, but we shall not do so explicitly.

In Sec. 5 we discuss the possibility of using the formalism developed here to make predictions about the high-energy behavior of many-particle amplitudes. We consider in particular a five-particle process, such as $N+\pi\rightarrow N+\pi+\pi$. Because there are now five independent invariants, one must be rather careful to say what is meant by the high-energy region. We distinguish three regions of the invariants: the low-energy region, in which all the invariants are small; an intermediate region in which the total energy is large, but the effective mass of at least one pair of final-state particles is not; and the high-energy region, in which all the energy-type invariants are large. Now in the case of a four-particle process, one can give a unified description in terms of Regge poles of the low-energy and high-energy regions. If s is the total-energy invariant, then Regge poles in the s channel are important at low energies (since they describe the resonances), while the poles in the t or u channels dominate at high energies. We give a completely analogous description of the five-particle process.

[7] G. C. Wick, Anns Phys. (N. Y.) **18**, 65 (1962).
[8] M. Jacob and G. C. Wick, Anns. Phys. (N. Y.) **7**, 404 (1959).

In particular, we discuss the high-energy region in which we may expect particular pairs of Regge poles, in channels corresponding to momentum-transfer invariants, to be dominant. At this stage, it seems unlikely that any immediately useful experimental predictions could be made, because of the large number of unknown functions involved. However, if the stage is reached where some of the Regge trajectories are known with reasonable accuracy from elastic scattering data, it should be possible to make rather definite predictions about the asymptotic behavior in this region.

2. THE VERTEX FUNCTION

We shall begin by discussing the coupling of three ordinary particles of spins j_1, j_2, j_3, and masses m_1, m_2, m_3. We consider the virtual decay process $3 \to \bar{1}+\bar{2}$, whose amplitude may be defined in the usual way in terms of the residues of scattering amplitudes at the pole $s=m_3^2$. This amplitude is completely specified [see Eq. (A7)] by the amplitude for the decay at rest of particle 3, with z component of spin λ_3, into particles $\bar{1}$ and $\bar{2}$, with momenta in the positive and negative z directions, and z components of spin $-\lambda_1$ and $-\lambda_2$, respectively, namely,

$$\begin{aligned}\langle 00, -\lambda_1, -\lambda_2 | T(m_3^2) | \lambda_3 \rangle &= \langle -\lambda_1, -\lambda_2 | T^{j_3}(m_3^2) | \rangle \delta(\lambda_1+\lambda_2+\lambda_3, 0) \\ &= N(j_1,\lambda_1) N(j_2,\lambda_2) N(j_3,\lambda_3) \\ &\quad \times f_{12,3}(m_1^2,m_2^2,m_3^2) g_{123}(\lambda_1\lambda_2\lambda_3), \quad (1)\end{aligned}$$

say, thus, defining a set of "coupling constants" g_{123}. Here $f_{12,3}$ is a kinematical factor of the dimensions of a mass, which is introduced to make g_{123} real when all the particles are stable, and to give it maximum symmetry. This factor will be discussed further below. The spin-dependent normalization factors $N(j,\lambda)$ are introduced for reasons which will become clear in the following section, and are defined by

$$N(j,\lambda) = [(j+\lambda)!(j-\lambda)!]^{1/2}. \quad (2)$$

The coupling constants defined in (1) have certain symmetry properties which are easy to derive from parity conservation and invariance under rotations through π. These are

$$\begin{aligned} g_{123}(\lambda_1\lambda_2\lambda_3) &= \eta g_{213}(\lambda_2\lambda_1\lambda_3), \\ &= \zeta g_{213}(-\lambda_2-\lambda_1-\lambda_3), \\ &= \eta\zeta g_{123}(-\lambda_1-\lambda_2-\lambda_3), \quad (3)\end{aligned}$$

where the sign factors are

$$\begin{aligned}\eta &= \epsilon \eta_1 \eta_2 \eta_3, \\ \zeta &= \epsilon (-1)^{j_1+j_2+j_3}. \quad (4)\end{aligned}$$

Here η_i are the intrinsic parities, and

$$\begin{aligned}\epsilon &= +1 \text{ for 3 bosons}, \\ &= -1 \text{ for 2 fermions and 1 boson}.\end{aligned}$$

Note that η_1, η_2 are the parities of particles 1 and 2, not $\bar{1}$ and $\bar{2}$. For fermions these parities are opposite, so that we can also write

$$\begin{aligned}\eta &= \epsilon_{12} \eta_{\bar{1}} \eta_{\bar{2}} \eta_3, \\ \zeta &= \epsilon_{12} (-1)^{j_3-j_1-j_2},\end{aligned}$$

where

$$\begin{aligned}\epsilon_{12} &= -1 \text{ if both 1 and 2 are fermions}, \\ &= +1 \text{ otherwise}.\end{aligned}$$

Thus, η is the relative parity of the vertex in the ordinary sense (e.g., $\eta=-1$ for the $\bar{N}N\pi$ vertex). The sign factor ζ might be called the relative j parity. It is important to note that it is unchanged by adding 2 to any of the spins. Thus, both η and ζ are characteristic of the Regge families to which the particles belong, rather than of the individual members of these families.

Now, we may also consider the virtual crossed process $2 \to \bar{1}+\bar{3}$ and, thus, obtain from crossing symmetry a symmetry of g_{123} under interchange of 2 and 3. If the kinematical factors are chosen so that $f_{12,3}$ and $f_{13,2}$ differ only by an appropriate phase factor, then the complete symmetries of g_{123} are

$$g_{123} = g_{231} = g_{312} = \eta g_{132} = \eta g_{213} = \eta g_{321} \quad (5)$$

and

$$g_{123}(\lambda_1\lambda_2\lambda_3) = \eta\zeta g_{123}(-\lambda_1-\lambda_2-\lambda_3). \quad (6)$$

Let us now return to the kinematical factor $f_{12,3}$, which is required to make g_{123} real when all three particles are stable. The phase of the amplitude (1) is, of course, to some extent a matter of convention. It may always be chosen so that (1) is almost real if m_3 is just above the threshold $m_3=m_1+m_2$. Then, to make g_{123} real below threshold, we must factor out the threshold behavior $k_3^{l_3}$. Here the center-of-mass momentum k_3 is given by

$$2m_3 k_3 = \Delta^{1/2}(m_1^2,m_2^2,m_3^2), \quad (7)$$

where

$$\Delta(s,t,u) = s^2+t^2+u^2-2st-2su-2tu. \quad (8)$$

The orbital angular momentum l_3 is the smallest angular momentum satisfying

$$(-1)^{l_3} = \eta\epsilon_{12},$$

which can be formed out of the three spins. Consider first the case of three bosons ($\epsilon=+1$). Then $l_1=l_2=l_3=l$, say, and a suitably symmetric (but not unique) form for the kinematic factor is simply[9]

$$f_{12,3} = M^{1-2l} \Delta^{\frac{1}{2}l}(m_1^2,m_2^2,m_3^2), \quad (9)$$

where M is some convenient fixed mass introduced for dimensional reasons. The situation is rather more complicated when the particles are two fermions, 1, 2, and one boson, $3(\epsilon=\epsilon_{12}=-1)$. Then we have $l_1=l_2=l$, with $(-1)^l=\eta$ as before, but now $l_3=l\pm1$. However, we

[9] These functions are arbitrary to the extent that they may be multiplied by any symmetric function of the m_i^2 which is real and free of singularities in the region $m_i^2>0$.

can now take[9]

$$f_{12,3}=M^{1-l-l_3}[m_3^2-(m_1+m_2)^2]^{\frac{1}{2}l_3}\times[m_3^2-(m_1-m_2)^2]^{\frac{1}{2}l}. \quad (10)$$

The factors $f_{13,2}$ and $f_{23,1}$ would differ from (10) by a change of sign of each of the expressions inside square brackets, so that each factor is real in the corresponding physical region.

We now turn to the problem of coupling three Regge particles, corresponding to the trajectories $j_i=\alpha_i(t_i)$. Since the masses are no longer constants, we must have in place of a coupling constant a vertex function $\Gamma_{123}(\lambda_1,\lambda_2,\lambda_3;\, t_1,t_2,t_3)$, where $t_i=m_i^2$. Clearly, these vertex functions must be defined in terms of the residues of Regge poles, just as the coupling constants are determined in terms of the residues of ordinary poles. We shall do this in the following sections. Nevertheless, it is convenient to anticipate the discussion by considering here the properties we may expect them to possess.

The first of these properties is a relation to the coupling constants already defined. When the Regge trajectory $j_i=\alpha_i(t_i)$ passes through a physical spin, there is (in general) a corresponding physical particle. We shall denote the mass and spin of the lowest member of this family of particles by m_i and σ_i. The spins of the higher members are then σ_i+2, σ_i+4, $\cdots$. It is important to notice that the signature of a Regge trajectory is defined by σ_i and, in fact, we shall find it convenient to use σ_i in place of the signature. (This avoids the necessity of using different conventions for boson and fermion trajectories.) Now when all three trajectories pass through physical values, the vertex functions Γ_{123} must be proportional to the corresponding coupling constants g_{123}, since the Regge formula must have the correct residues at the ordinary poles $t_i=m_i^2$. We may, in fact, normalize Γ_{123} by requiring that, at the positions of the lowest members of each family, it should be equal to the coupling constant:

$$\Gamma_{123}(m_1^2,m_2^2,m_3^2)=g_{123}. \quad (11)$$

We shall find that for higher members of the families an additional normalization constant, related to the slope of the trajectory, is required, that is, that

$$\Gamma_{123}(m_{1^*}^2,m_{2^*}^2,m_{3^*}^2)=n_{1^*}n_{2^*}n_{3^*}g_{1^*2^*3^*}, \quad (12)$$

where the n_{i^*} will be determined explicitly in the following section.

The next property we may expect the vertex functions to have is some symmetry under permutations of (123) or change of sign of the helicities λ_i. In view of the remarks about the sign factors η and ζ above, it would be consistent to suppose that Γ_{123} has precisely the same symmetries as g_{123}, namely, (5) and (6), provided that the relative j parity ζ is regarded as a function only of the signatures (that is, of the σ_i) and not of the variable spins j_i. It is, of course, essential that the definition of the kinematic factors should be extended to unphysical spins in such a way as to make Γ_{123} real in the region below all thresholds; for, otherwise the symmetry relations could involve a variable phase factor equal to unity only at the physical points. We define the function

$$F_{12,3}(t_1,t_2,t_3)=(k_1/M)^{j_1-\sigma_1}(k_2/M)^{j_2-\sigma_2}\times(k_3/M)^{j_3-\sigma_3}f_{12,3}(t_1,t_2,t_3), \quad (13)$$

where $f_{12,3}$ is the factor (9) or (10) appropriate to spins σ_i. It is easy to see that this factor has the correct behavior $k_3^{(j_3-\sigma_3)+l_3}$ at the threshold $t_3=(t_1^{1/2}+t_2^{1/2})^2$.[10] However, it is not possible to factor out completely the behavior at $t_3=0$, and Γ_{123} therefore is not, in general, real in the region where t_3 is negative. In order to obtain functions which are real in that region it is generally necessary to make linear combinations of the Γ_{123}, and the appropriate factors necessarily depend on the helicities. The expression (13) does, however, include all the factors which depend on the variable spins, and the only remaining factors required are of the form $t_i^{1/2}$. (This is the reason for this particular choice.)

One problem which arises on going from physical spins to continuous spins concerns the range of values of the λ_i. Clearly, this range cannot change discontinuously as we go along the trajectory, and we must, therefore, allow each λ_i to range (by integral steps) from $-\infty$ to $+\infty$. For physical values of j_i, only those values of λ_i satisfying $|\lambda_i|\leq j_i$ are physical. In this connection, it may happen that all vertex functions vanish at a particular j_i for physical λ_i; then there is no particle associated with the trajectory at this point.[11]

3. FOUR-PARTICLE AMPLITUDES

We consider here a scattering process $a+b\rightarrow\bar{c}+\bar{d}$. The scattering amplitude is given by Eq. (A8) of Appendix A, which may be written in the form

$$\begin{aligned}\langle\theta_f\phi_f,\,-\lambda_c,\,-\lambda_d|\,T(t)\,|\theta_i\phi_i,\lambda_a,\lambda_b\rangle &=\textstyle\sum_j\langle-\lambda_c,\,\lambda_d|\,T^j(t)\,|\lambda_a,\lambda_b\rangle\\ &=\textstyle\sum_j\langle-\lambda_c,\,-\lambda_d|\,T^j(t)\,|\lambda_a,\lambda_b\rangle\, e^{-i\lambda\phi}d_{\lambda,-\mu}{}^j(-\theta)e^{-i\mu\psi}, \quad (14)\end{aligned}$$

where $\lambda=-\lambda_c-\lambda_d$, $\mu=-\lambda_a-\lambda_b$, θ is the scattering angle, and the angles ϕ and ψ are defined in Appendix A, Fig. 8.

Now, if we set

$$d_{\lambda,\mu}(j,\cos\vartheta)=N(j,\lambda)N(j,\mu)d_{\lambda,\mu}{}^j(-\vartheta), \quad (15)$$

[10] It is assumed here that the orbital angular momentum increases linearly with j_3, and thus increases by 2 on going from one member of the Regge family to the next. This is generally true, with the reservation that for some j_3 some of the "possible" values of l_3 become negative and therefore unphysical. (Compare Ref. 11.) Note that the possible values of l_3 for fixed j_3 differ by 2. It is unimportant which of these values we choose since changing l_3 by 2 introduces only the real rational function k_3^2.

[11] M. Gell-Mann, in *Proceedings of the 1962 Annual International Conference on High-Energy Physics at CERN* (CERN, Geneva, 1962), p. 533.

where $N(j,\lambda)$ is given by Eq. (2), then $d_{\lambda,\mu}(j,x)$ is a meromorphic function of j in the entire j plane, and as a function of x is (apart from a phase factor) the boundary value of a function $d_{\lambda,\mu}(j,z)$, which, if j is not at a pole, is holomorphic in the plane cut from $-\infty$ to $+1$. (See Appendix B for the details.) As in the preceding section, we use, in place of the signature, a fixed angular momentum σ, which may take the values 0 to 1 if the j in (14) are integers, and $\frac{1}{2}$ and $\frac{3}{2}$ if the j are half-odd-integers. We define two amplitudes $T^\sigma(j,t)$ by

$$\langle -\lambda_c, -\lambda_d | T^\sigma(j,t) | \lambda_a,\lambda_b \rangle = \tfrac{1}{2}[1+(-1)^{j-\sigma}][N(j,\lambda)N(j,\mu)]^{-1} \times \langle -\lambda_c, -\lambda_d | T^j(t) | \lambda_a,\lambda_b \rangle. \quad (16)$$

Then our basic assumption is that each of the functions $T^\sigma(j,t)$ is an analytic function of j, meromorphic in the right-half j plane $\mathrm{Re}\,j > -\frac{1}{2}$. This assumption is consistent with the results which have been proved in Schrödinger theory with suitable potentials.[12] We can then transform Eq. (14) by a Sommerfeld-Watson transform, provided that the functions (16) are appropriately bounded for large j. In doing this we have to show that the lower limit on the sum over j, namely, $\max(\lambda,\mu)$, is unimportant; that is, that the integrand of the contour integral has no poles (other than Regge poles) for positive j less than this value. This is proved in Appendix B. In deforming the contour to run parallel to the imaginary axis, we shall pick up the contributions of the Regge poles. Here we wish to consider the contributions of a particular Regge pole at $j=\alpha(t)$, with "signature" σ. If the residue of $T^\sigma(j,t)$ at this pole is $\beta(t)$, then the contribution to the amplitude (14) is (see Appendix B)

$$\frac{\pi}{2\sin\pi(\alpha-\sigma)} \langle -\lambda_c, -\lambda_d | \beta(t) | \lambda_a,\lambda_b \rangle \times e^{-i\lambda\phi}[d_{\lambda,-\mu}(\alpha,\cos\vartheta)+(-1)^{\sigma-\lambda}d_{\lambda,\mu}(\alpha,-\cos\vartheta)]e^{-i\mu\psi}. \quad (17)$$

Now the general arguments which have been used to show that the residue $\beta(t)$ must factorize into two factors, depending, respectively, on the final and on the initial state,[13] are equally applicable to the case of general spins. Thus, $\beta(t)$ has the form of the product of two vertex functions. The vertex functions are effectively defined by the relation

$$\langle -\lambda_c, -\lambda_d | \beta(t) | \lambda_a,\lambda_b \rangle \delta(\lambda_c+\lambda_d+\lambda, 0)\delta(\lambda_a+\lambda_b+\mu, 0) = N\Gamma_{cd1}(\lambda_c,\lambda_d,\lambda; m_c^2,m_d^2,t)F_{cd,1}(m_c^2,m_d^2,t) \times \Gamma_{\bar{1}ab}(\mu,\lambda_a,\lambda_b; t,m_a^2,m_b^2)F_{ab,1}(m_a^2,m_b^2,t), \quad (18)$$

where the subscript 1 labels the trajectory, and N is a normalization constant to be determined. We note that only vertex functions with two of the particles physical can be defined by this equation. To determine the constant N, we examine the residue of the pole at the position $t=m_1^2$ of the lowest particle on this trajectory. If we assume that the external particles are the lowest members of their respective families, then we may use the normalization condition (11). Clearly the residue at this pole should be effectively the product of two decay amplitudes, which may be expressed in terms of the coupling constants by Eq. (1). In this way we find

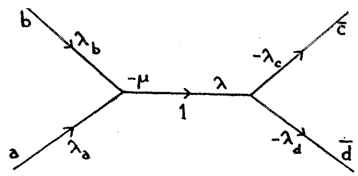

FIG. 2. Regge-pole diagram for a four-particle process.

$$N = N_a N_b N_c N_d \alpha'(m_1^2),$$

where $N_a = N(j_a,\lambda_a)$, etc., and α' is the slope of the Regge trajectory. Similarly, by examining the residue at a higher pole $t=m_{1^*}^2$, we may evaluate the additional normalization factor n_{1^*} which appears in (12). We find

$$n_{1^*} = [\alpha'(m_{1^*}^2)/\alpha'(m_1^2)]^{1/2}.$$

Finally, the contribution of this Regge pole may be obtained by substituting (18) into (17). It may be represented diagrammatically by the diagram of Fig. 2, and is (omitting the constant mass arguments)

$$N_a N_b N_c N_d \sum_{\lambda\mu} \Gamma_{cd1}(\lambda_c,\lambda_d,\lambda; t)\Gamma_{cd,1}(t) \times G_{1;\lambda\mu}(\phi,\vartheta,\psi; t)\Gamma_{\bar{1}ab}(\mu,\lambda_a,\lambda_b; t)F_{ab,1}(t), \quad (19)$$

where the summations over λ and μ are dummy summations eliminated by the δ-function factors in the vertex functions, and the "propagator" G_1 is given by

$$G_{1;\lambda\mu}(\phi,\vartheta,\psi; t) = \alpha'(m_1^2)e^{-i\lambda\phi}G_{\lambda\mu}{}^\sigma[\alpha(t),\cos\vartheta]e^{-i\mu\psi}, \quad (20)$$

where

$$G_{\lambda\mu}{}^\sigma(\alpha,x) = \frac{\pi}{2\sin\pi(\alpha-\sigma)} \times [d_{\lambda,-\mu}(\alpha,x)+(-1)^{\sigma-\lambda}d_{\lambda,\mu}(\alpha,-x)]. \quad (21)$$

The set of "Feynman rules" is essentially obvious from this example. Note that the internal line is labeled by two helicities, representing its spin components in the directions of the initial and final center-of-mass momenta [which are related by the rotation $(\phi, -\vartheta, -\psi)$].

We conclude this section with a discussion of the asymptotic behavior of (19) for large $\cos\vartheta$. Using Eq. (B10) we find, for $\mathrm{Re}\,\alpha > -\frac{1}{2}$,

$$G_{\lambda\mu}{}^\sigma(\alpha,z) \simeq \frac{(-i)^{\lambda+\mu}\pi\Gamma(2\alpha+1)[1+e^{-i\pi(\alpha-\sigma)}]}{2^{\alpha+1}\sin\pi(\alpha-\sigma)} z^\alpha \quad (22)$$

as $z\to\infty$, $\mathrm{Im}\,z>0$. In terms of the Mandelstam invariant s, which is related to z by

$$s \simeq 2kk'z \quad \text{as} \quad z\to\infty,$$

[12] J. M. Charap and E. J. Squires, Ann. Phys. (N. Y.) **20**, 145 (1962); **21**, 8 (1963), and to be published; J. Hartle (unpublished).

[13] M. Gell-Mann, Phys. Rev. Letters **8**, 263 (1962); V. N. Gribov and I. Ya. Pomeranchuk, *ibid.* **8**, 343 and 412 (1962); J. M. Charap and E. J. Squires, Phys. Rev. **127**, 1387 (1962).

where k and k' are the initial and final center-of-mass momenta, we, therefore, have for large s

$$F_{cd,1}G_{\lambda\mu}{}^{\sigma}(\alpha,z)F_{ab,1}\simeq f_{cd,1}f_{ab,1}(M^2/kk')^{\sigma}(-i)^{\lambda+\mu}\times\frac{\pi\Gamma(2\alpha+1)[1+e^{-i\pi(\alpha-\sigma)}]}{2^{2\alpha+1}\sin\pi(\alpha-\sigma)}\left(\frac{s}{M^2}\right)^{\alpha}. \quad (23)$$

4. MANY-PARTICLE AMPLITUDES

We now consider a process

$$a+b\to\bar{c}+\bar{d}+\bar{e}. \quad (24)$$

With a particular choice of axes, the amplitude for this process may be written [see Eq. (A9)] in the form

$$\langle 00,(t'\vartheta'\phi',-\lambda_c,-\lambda_d),(-\lambda_e)|T(t)|-\vartheta 0,\lambda_a,\lambda_b\rangle$$
$$=\sum_{jj'\mu'}\langle(t'j'-\mu',-\lambda_c,-\lambda_d),(-\lambda_e)|T^j(t)|\lambda_a,\lambda_b\rangle\times d_{\lambda',-\mu'}{}^{j'}(-\vartheta')e^{-i\mu'\phi'}d_{\lambda,-\mu}{}^{j}(-\vartheta), \quad (25)$$

where

$$\lambda'+\lambda_c+\lambda_d=0,$$
$$\mu'+\lambda+\lambda_e=0, \quad (26)$$
$$\mu+\lambda_a+\lambda_b=0.$$

Now according to the general principles of S-matrix theory, this same function should also represent, if analytically continued in an appropriate way, the amplitude for the crossed process

$$a+b+e\to\bar{c}+\bar{d}, \quad (27)$$

The amplitude for this process is

$$\langle\vartheta'\phi',-\lambda_c,-\lambda_d|T(t')|00,(\lambda_e),(t-\vartheta 0,\lambda_a,\lambda_b)\rangle$$
$$=\sum_{jj'\lambda}\langle-\lambda_c,-\lambda_d|T^{j'}(t')|(\lambda_e),(tj\lambda,\lambda_a,\lambda_b)\rangle\times d_{\lambda',-\mu'}{}^{j'}(-\vartheta')e^{-i\mu'\phi'}d_{\lambda,-\mu}{}^{j}(-\vartheta), \quad (28)$$

where λ', μ', μ are given by Eqs. (26). Comparing Eqs. (25) and (28), one sees that $T^j(t)$ and $T^{j'}(t')$ should be analytic continuations (in t and t') of the same function, except possibly for a constant phase factor. We define four amplitudes $T^{(\sigma,\sigma')}$, suppressing the helicity labels, by

$$T^{(\sigma,\sigma')}(j,t;j',t')$$
$$=\tfrac{1}{4}[1+(-1)^{j-\sigma}][1+(-1)^{j'-\sigma'}]\times[N(j,\lambda)N(j,\mu)N(j',\lambda')N(j',\mu')]^{-1}\times\langle(t'j'-\mu',-\lambda_c,-\lambda_d),(-\lambda_e)|T^j(t)|\lambda_a,\lambda_b\rangle.$$

Since each of the angular momenta j and j' is the total angular momentum in one process, we shall assume that these four amplitudes are analytic functions of both variables, meromorphic in the product of the right-half planes. Then we can make a double Sommerfeld-Watson transform, picking up Regge poles in each variable. We shall consider the contribution of a particular pair of Regge poles, $j=\alpha_1(t)$ and $j'=\alpha_2(t')$. This contribution may easily be written down in terms of the residue

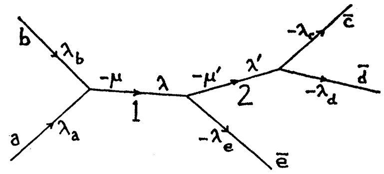

FIG. 3. Regge-pole diagram for a five-particle process.

$\beta_{12}(t,t')$. Now from the general requirements on factorization of residues, β_{12} must be the product of a factor depending on the initial state, that is essentially $\Gamma_{\bar{1}ab}$, and a factor depending on the final state. Applying the same argument to the process (27), we see that β_{12} must be a product of three distinct factors,

$$\beta_{12}(t,t')=N\Gamma_{cd2}F_{cd,2}\Gamma_{\bar{2}e1}F_{2e,1}\Gamma_{\bar{1}ab}F_{ab,1}.$$

As before, the normalization factor N may be found by examining the residues at the poles $t=m_1{}^2$, $t'=m_2{}^2$. One obtains in this way precisely the expression one would expect on the basis of the "Feynman rules" described in Sec. 3, for the contribution of the diagram of Fig. 3, namely,

$$N_aN_bN_cN_dN_e\sum_{\lambda',\mu',\lambda,\mu}\Gamma_{cd2}(\lambda_c\lambda_d\lambda';t')F_{cd,2}(t')\times G_{2,\lambda'\mu'}(0,\vartheta',\phi';t')\Gamma_{\bar{2}e1}(\mu'\lambda_e\lambda;t',t)F_{2e,1}(t',t)\times G_{1,\lambda\mu}(0,\vartheta,0;t)\Gamma_{\bar{1}ab}(\mu\lambda_c\lambda_d;t)F_{ab,1}(t). \quad (29)$$

The only difference between the contributions to the amplitudes for the processes (24) and (27) is a constant phase factor which arises from the fact that for the latter amplitude $F_{2e,1}$ is replaced by $F_{e1,2}$.

One important distinction between this expression and the contribution (19) to a two-particle scattering amplitude is the fact that only three of the summations over helicities are removed by the δ functions in the vertex functions. One genuine summation from $-\infty$ to $+\infty$ remains, though, in practice, it may be possible to neglect all but a small number of terms. Moreover, so long as the sum converges reasonably rapidly, this difference is unimportant for predictions of high-energy behavior, since every term has essentially the same asymptotic behavior. It is also possible that one might be able to convert this infinite sum, like the sums over j, into an integral, by using analyticity properties in λ; however, we shall not consider this question here.

5. DISCUSSION

One of the most promising features of the Regge-pole formalism is that it allows a unified description of the low-energy (resonance scattering) and high-energy (diffraction scattering) regions. In a scattering process in which the total energy is $s^{1/2}$, the contributions of the resonances at low energies are described by the Regge poles in the s channel; whereas at high energies the scattering is dominated by the Regge poles in the t or u channel, depending on which of these invariants is numerically small, that is, for near-forward or near-backward scattering. (There is, of course, a basic difference between the two regions, namely, that in the low-energy region one does not expect the Regge poles to describe the whole of the scattering, since the background integral term is certainly not, in general, negligible.) This kind of description can readily be extended to production processes and, to be specific, we shall consider a five-particle process $a+b\rightarrow\bar{c}+\bar{d}+\bar{e}$. There are five independent Mandelstam invariants for this process. We shall use the letter s to denote the (necessarily positive) energy invariants, and t to denote the momentum-transfer invariants. Thus, for instance,

$$s_{ab}=(p_a+p_b)^2\,,$$
$$t_{ac}=(p_a-p_c)^2\,.$$

Because of the larger number of invariants, one has to be careful to specify precisely what is meant by 'low-energy' and 'high-energy' regions. We shall find it useful to distinguish three different regions, specified by the magnitudes of the energy invariants s_{ab}, s_{cd}, s_{ce}, s_{de} (with one linear relation between them):

I. *Low-energy regions:* s_{ab} small. Then all the other invariants must clearly be small also.

II. *Intermediate region:* s_{ab} large, but at least one of the other energy invariants small.

III. *High-energy region:* all energy invariants large.

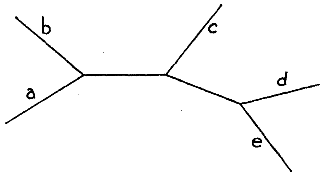

FIG. 4. A typical diagram for the low-energy region.

Here "large" means much larger than the squared masses, and "small" means of the order of the squared masses. Note that the total energy is not necessarily larger in region III than in region II.

Different kinds of Regge diagrams will be important in each of these three regions, though as in the four-particle case it is only in the high-energy region that one may expect the Regge pole terms alone to give a good approximation to the entire amplitude. In region I the important diagrams will be like those of Fig. 4. It makes little difference in this region whether one treats the internal lines in these diagrams as short-lived particles in the ordinary sense or as Regge particles; the important contribution in each case occurs for s_{ab} and s_{de} near the resonance positions.

In region II, one should really distinguish three subregions, according to which of the three final-state invariants is small. If s_{ab} is large enough, these regions will not overlap, though for moderate values of s_{ab}, more than one pair of final-state particles may be in the resonance region. In any case, we are well above the resonance region for the two incident particles, and

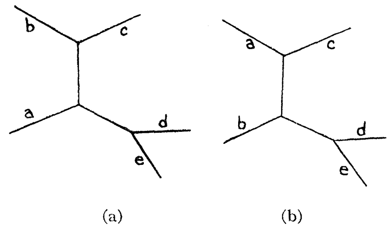

FIG. 5. Typical diagrams for the intermediate region where the total energy is high but particles d and e are resonant.

their scattering will be dominated by an exchanged Regge particle. In the subregion where s_{de} is small, we should expect diagrams like those of Fig. 5 to predominate. Here the Regge trajectories are defined by t_{bc} or t_{ac}, and by s_{de}. Whether it is the t_{bc} or t_{ac} poles which dominate must depend (as in the choice between t and u in the four-particle case) on which of these invariants is numerically small. (If neither is small, we should expect the cross section to be very small indeed.)

Finally, we come to the region III, in which we are above the region of resonances between pairs of particles in the final state. Then only Regge poles in the momentum-transfer variables can give an important contribution, and we have to consider diagrams such as that of Fig. 6, in which the Regge trajectories are function of t_{bc} and t_{ae}. This diagram will, therefore, be important in the subregion where these particular invariants are small. Clearly, the subregions corresponding to different pairs of momentum-transfer invariants are generally well separated from one another, as are the forward- and backward-scattering regions in the four-particle case.

The region III is particularly interesting from the present point of view, since it is only in this region that one can make experimental predictions by considering the pole terms alone. We shall, therefore, examine the asymptotic form of a contribution like that of Fig. 6. Apart from a different labeling of the particles, this is given by the expression (29) with $t=t_{ae}$ and $t'=t_{bc}$. For large values of s_{cd} and s_{de} we can use the asymptotic form (22) for the propagator. Thus, we might expect that the dependence on the invariants s_{de} and s_{cd} is through a factor of the form

$$(s_{de}/M^2)^{\alpha_1(t)}\quad (s_{cd}/M^2)^{\alpha_2(t')}\,, \tag{31}$$

with $t=t_{ae}$, $t'=t_{bc}$. However, it has been pointed out by Halliday and Polkinghorne[14] that this is not necessarily true if the order of the limits $s_{de}\rightarrow\infty$, $s_{cd}\rightarrow\infty$ is im-

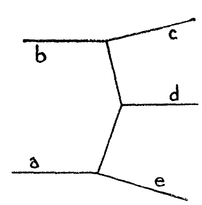

FIG. 6. Typical diagram for the region where all energy invariants are large.

[14] I. G. Halliday and J. C. Polkinghorne (to be published). I am indebted to Dr. Polkinghorne for informing me of their results prior to publication.

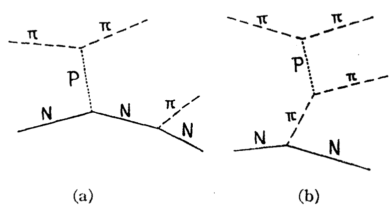

FIG. 7. Regge-pole diagrams for pion production in π-N collisions in two regions. P denotes the Pomeranchuk trajectory, and N, π on internal lines label the nucleon and pion Regge trajectories.

portant. This particular form corresponds to taking the two limits successively (in either order). The physically interesting situation corresponds rather to the case where both s_{cd} and s_{de} are made large simultaneously, that is, to the limit $s_{cd}\rightarrow\infty$, $s_{de}\rightarrow\infty$ keeping the ratio finite. This limit is equal to the sequential limits only if certain uniformity properties are satisfied. Halliday and Polkinghorne have shown that in the case of a certain sum of Feynman diagrams the various possible ways of taking the limits are actually inequivalent. In fact, none of the limits they consider is precisely the physically interesting one, which is

$$s_{cd}\sim s_{de}\sim s_{ab}^{1/2}\rightarrow\infty\,,$$

with t_{ae}, t_{bc} fixed. Their limit (ii), namely,

$$s_{ab}\sim s_{cd}\sim s_{de}\rightarrow\infty\,,$$

in fact, takes one outside the physical region for the process.[15] It will require a more detailed investigation of the limiting procedure to determine whether, in fact, the asymptotic form in this limit is given by the expression (31). It is hoped to examine this point more thoroughly in a subsequent paper. In any case, if the trajectories $\alpha_1(t)$ and $\alpha_2(t')$ are known, the contribution of these Regge poles can be found from the expression (29), in which the vertex functions are treated as unknown functions, and this would yield information about the dependence on the energy variables.

The description given above incorporates a number of apparently inconsistent models into a coherent whole.[16,17] Consider, for example, the process

$$N+\pi\rightarrow N+\pi+\pi\,.$$

It has been suggested that this process should be dominated at high energies by inelastic diffraction scattering in which the nucleon is excited to one of the higher nucleon-pion resonances. This clearly corresponds to the diagram of Fig. 7(a). On the other hand, according to the peripheral model, it is dominated by a single-pion-exchange process in which at high energies the pion-pion scattering may be treated as diffraction scattering. This corresponds to the diagram of Fig. 7(b). If the description suggested in this paper is correct, each of these models has its range of validity. They apply, in fact, in certain subregions of the regions II and III, respectively. Other diagrams will apply in the other subregions.

ACKNOWLEDGMENTS

I am indebted to Professor A. Salam for stimulating my interest in Regge poles, and to J. Hartle for helpful discussions of analyticity in the helicity formalism. Part of this work was performed at the Summer Institute for Theoretical Physics in Seattle, and it is a pleasure to thank Professor R. Geballe, Professor E. M. Henley, and Professor B. A. Jacobsohn for the hospitality of the University of Washington, and Professor K. W. Ford for the hospitality of Brandeis University.

APPENDIX A

Helicity Amplitudes

We summarize here the relevant definitions concerning helicity states,[7,8] and introduce some useful notations.

The states of a single particle of mass m and spin j are labeled by the momentum $\mathbf{p}$, or p, ϑ, ϕ, and the helicity λ. They are defined in terms of the states $|\mathbf{0},\lambda\rangle$ of momentum zero and z component of spin λ by

$$|\mathbf{p},\lambda\rangle=|p\vartheta\phi,\lambda\rangle=U(\phi\vartheta p/m)|\mathbf{0},\lambda\rangle\,, \tag{A1}$$

where $U(\phi\vartheta p/m)$ is the unitary operator corresponding to a Lorentz transformation taking the vector $(m,\mathbf{0})$ into the actual momentum vector, namely,

$$U(\phi\vartheta\sinh\chi)=e^{-iJ_3\phi}e^{-iJ_2\vartheta}e^{-iM_{03}\chi}.$$

We use the covariant normalization

$$\langle\mathbf{p}',\lambda'|\mathbf{p},\lambda\rangle=(2\pi)^3 2(\mathbf{p}^2+m^2)^{1/2}\delta_3(\mathbf{p}'-\mathbf{p})\delta_{\lambda'\lambda}. \tag{A2}$$

For two particles of masses m_1, m_2, and spins j_1, j_2, one first constructs center-of-mass states of total mass $s^{1/2}$ by combining two one-particle states in the form

$$|\mathbf{0},s\vartheta\phi,\lambda_1,\lambda_2\rangle=|p\vartheta\phi,\lambda_1\rangle|-p\vartheta\phi,\lambda_2\rangle. \tag{A3}$$

Here we have used the notation (A1) also for negative p. Note that the helicity of particle 2 is $-\lambda_2$; this apparently unsymmetrical convention avoids some unnecessary complications in later formulas, particularly in connection with crossing symmetry. Note also that we have not included any kinematical factor in (A3); thus, the normalization in terms of s contains a factor $2s^{1/2}/p$. The momentum p is given as usual by

$$p^2=\Delta(m_1^2,m_2^2,s)/4s\,, \tag{A4}$$

where

$$\Delta(s,t,u)=s^2+t^2+u^2-2st-2su-2tu. \tag{A5}$$

[15] I am indebted to Dr. J. C. Taylor for pointing this out. The statement in the preprint of this paper that in region III $s_{cd}s_{de}/s_{ab}$ may be treated as large is, in fact, inconsistent with the requirement that both t_{ae} and t_{bc} are fixed. Thus, the remark made there that in this region the amplitude becomes independent of s_{ab} is incorrect.

[16] M. L. Good and W. B. Walker, Phys. Rev. **120**, 1857 (1960); P. T. Matthews and A. Salam, Nuovo Cimento **21**, 126 (1961).

[17] G. F. Chew and F. E. Low, Phys. Rev. **113**, 1640 (1959); F. Bonsignori and F. Selleri, Nuovo Cimento **15**, 853 (1960). See also, S. Drell and K. Hiida, Phys. Rev. Letters **7**, 199 (1961).

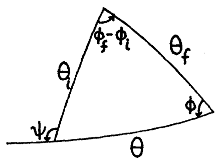

FIG. 8. Relations between the angles in a scattering process; θ_i,ϕ_i and θ_f,ϕ_f are the polar angles of the initial and final center-of-mass momenta, and θ is the scattering angle.

The general two-particle state may be obtained by applying a Lorentz transformation $U(\phi'\vartheta'p'/s^{1/2})$ to (A3).

Three-particle states are formed similarly by first combining particles 1 and 2 to form the state (A3), and then combining this with particle 3 to obtain a state

$$|\mathbf{0},s'\vartheta'\phi',(s\vartheta\phi,\lambda_1,\lambda_2),(\lambda_3)\rangle = |p'\vartheta'\phi',s\vartheta\phi,\lambda_1,\lambda_2\rangle|-p'\vartheta'\phi',\lambda_3\rangle.$$

Here λ_1 and $-\lambda_2$ are the helicities of 1 and 2 in their center of mass and $-\lambda_3$ is the helicity of 3 in the center of mass of all three particles.

Angular momentum states for two particles are defined, with an unconventional normalization, by

$$|\mathbf{0},sj\mu,\lambda_1,\lambda_2\rangle=\frac{2j+1}{4\pi}\int d\Omega|\mathbf{0},s\vartheta\phi,\lambda_1,\lambda_2\rangle \times\langle\mu|D^j(\phi\vartheta 0)|\lambda_1+\lambda_2\rangle^*. \quad \text{(A6)}$$

With this normalization, the inverse relation is simply

$$|\mathbf{0},s\vartheta\phi,\lambda_1,\lambda_2\rangle=\sum_{j,\mu}|\mathbf{0},sj\mu,\lambda_1,\lambda_2\rangle\langle\mu|D^j(\phi\vartheta 0)|\lambda_1+\lambda_2\rangle.$$

A similar decomposition may be made for a three-particle state in terms of two angular momenta.

For the S matrix, we use the notation

$$i\langle p',f|S-1|p,i\rangle=(2\pi)^4\delta_4(p'-p)\langle f|T(s)|i\rangle$$

with $s=p'^2=p^2$, and

$$\langle j'\mu',f|T(s)|j\mu,i\rangle=\delta_{j'j}\delta_{\mu'\mu}\langle f|T^j(s)|i\rangle.$$

The angular dependence of the amplitude for a decay process $c\rightarrow a+b$ is given by

$$\langle\vartheta\phi,\lambda_a,\lambda_b|T(m_c^2)|\lambda_c\rangle =\langle\lambda_a,\lambda_b|T^{j_c}(m_c^2)|\rangle\langle\lambda_a+\lambda_b|D^{j_c}(0,-\vartheta,-\phi)|\lambda_c\rangle. \quad \text{(A7)}$$

For a scattering process $a+b\rightarrow c+d$, we have

$$\langle\vartheta_f\phi_f,\lambda_c,\lambda_d|T(s)|\vartheta_i\phi_i,\lambda_a,\lambda_b\rangle =\sum\nolimits_j\langle\lambda_c,\lambda_d|T^j(s)|\lambda_a,\lambda_b\rangle \times\langle\lambda_c+\lambda_d|D^j(\phi,-\vartheta,-\psi)|\lambda_a+\lambda_b\rangle, \quad \text{(A8)}$$

where the angles ϕ, ϑ, ψ are defined by the spherical triangle of Fig. 8. In particular, ϑ is clearly the scattering angle. Finally, for a production process $a+b\rightarrow c+d+e$, the amplitude is

$$\langle\vartheta_f\phi_f,(s'\vartheta'\phi',\lambda_c,\lambda_d),(\lambda_e)|T(s)|\vartheta_i\phi_i,\lambda_a,\lambda_b\rangle =\sum_{jj'\mu'}\langle(s'j'\mu',\lambda_c,\lambda_d),(\lambda_e)|T^j(s)|\lambda_a,\lambda_b\rangle \times\langle\lambda_c+\lambda_d|D^{j'}(0,-\vartheta',-\phi')|\mu'\rangle \times\langle\mu'+\lambda_e|D^j(\phi,-\vartheta,-\psi)|\lambda_a+\lambda_b\rangle, \quad \text{(A9)}$$

where ϕ, ϑ, ψ are again defined by Fig. 8.

APPENDIX B

Analytic Properties of Rotation Matrices

Since the rotation matrices are expressible in terms of the hypergeometric function, their analyticity properties are easy to obtain. It is convenient however to collect the relevant formulas here.[18] We define functions $p_{\lambda\mu}(j,z)$ related to the Jacobi polynomials[19] by

$$\begin{aligned}p_{\lambda\mu}(j,z)&=[\tfrac{1}{2}(z+1)]^{\frac{1}{2}(\lambda+\mu)}[\tfrac{1}{2}(z-1)]^{\frac{1}{2}(\lambda-\mu)}P_{j-\lambda}{}^{(\lambda-\mu,\lambda+\mu)}(z)\\ &=\frac{\Gamma(j-\mu+1)}{\Gamma(j-\lambda+1)\Gamma(\lambda-\mu+1)}\\ &\quad\times[\tfrac{1}{2}(z+1)]^{\frac{1}{2}(\lambda+\mu)}[\tfrac{1}{2}(z-1)]^{\frac{1}{2}(\lambda-\mu)}\\ &\quad\times F(\lambda-j,\lambda+j+1;\lambda-\mu+1;\tfrac{1}{2}[1-z]). \quad \text{(B1)}\end{aligned}$$

This is an everywhere meromorphic function of j, λ, and μ, and a holomorphic function of z in the plane cut from $-\infty$ to $+1$. However, our discussion is restricted to the case where $\lambda\pm\mu$ are integral. It is then an entire function of j, and has zeros as follows:

$$p_{\lambda\mu}(j,z)=0$$

$$\text{at}\quad\begin{cases}j=\mu,\ \mu+1,\ \cdots,\ \lambda-1 & \text{if}\quad \mu<\lambda\,,\\ j=-\mu,\ -\mu+1,\ \cdots,\ -\lambda-1 & \text{if}\quad \mu>\lambda\,.\end{cases} \quad \text{(B2)}$$

It has the symmetry properties

$$p_{-\lambda,-\mu}(j,z)=p_{\lambda,\mu}(j,z)\,, \qquad N^2(j,\mu)p_{\mu\lambda}(j,z)=N^2(j,\lambda)p_{\lambda\mu}(j,z)\,, \quad \text{(B3)}$$

where

$$N^2(j,\lambda)=\Gamma(j+\lambda+1)\Gamma(j-\lambda+1)$$

and also

$$p_{\lambda\mu}(-j-1,\,z)=(-1)^{\lambda-\mu}p_{\mu\lambda}(j,z)\,. \quad \text{(B4)}$$

A limit function $\bar{p}_{\lambda\mu}(j,x)$ on the real interval $-1\leq x\leq 1$ may be defined by

$$\bar{p}_{\lambda\mu}(j,x)=(\mp i)^{\lambda-\mu}p_{\lambda\mu}(j,\,x\pm i0)\,,$$

and for physical j the rotation matrices may be expressed in terms of this function by[20]

$$\begin{aligned}d_{\lambda\mu}{}^j(-\vartheta)&=\frac{N(j,\lambda)}{N(j,\mu)}\bar{p}_{\lambda\mu}(j,\cos\vartheta)\,,\\ &=\frac{N(j,\mu)}{N(j,\lambda)}\bar{p}_{-\mu,-\lambda}(j,\cos\vartheta),\\ &=\frac{1}{N(j,\lambda)N(j,\mu)}d_{\lambda\mu}(j,\cos\vartheta)\,,\quad\text{say}. \quad \text{(B5)}\end{aligned}$$

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[18] Some of these analyticity properties have also been discussed by J. M. Charap and E. J. Squires, Ann. Phys. (N. Y.) **20**, 145 (1962).

[19] See *Higher Transcendental Functions*, Bateman Manuscript Project (McGraw-Hill Book Company, Inc., New York, 1953), Vol. II, pp. 168–174. Most of the results of this section are immediately applications of the properties of the hypergeometric function. See, *ibid.*, Vol. I, Chap. 2.

[20] A. R. Edmunds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, 1957), p. 58.

It is also useful to define a second function

$$q_{\lambda\mu}(j,z)=[\tfrac{1}{2}(z+1)]^{\frac{1}{2}(\lambda+\mu)}[\tfrac{1}{2}(z-1)]^{\frac{1}{2}(\lambda-\mu)}Q_{j-\lambda}{}^{(\lambda-\mu,\lambda+\mu)}(z)$$
$$=\frac{N^2(j,\mu)}{2\Gamma(2j+2)}\left[\frac{z+1}{z-1}\right]^{\frac{1}{2}(\lambda+\mu)}\left[\frac{z-1}{2}\right]^{-j-1}$$
$$\times F(\lambda+j+1,\,\mu+j+1;\,2j+2;\,2/[1-z])\,, \tag{B6}$$

which is meromorphic in the entire j plane, and as a function of z has the same domain of holomorphy as $p_{\lambda\mu}(j,z)$. It also satisfies the symmetry relations (B3), though not (B4). The functions are related by

$$p_{\lambda\mu}(j,z)=\pi^{-1}\tan\pi(j-\lambda)$$
$$\times[(-1)^{\lambda-\mu}q_{\lambda\mu}(j,z)-q_{\mu\lambda}(-j-1,\,z)]\,, \tag{B7}$$

and, in addition, the discontinuity of $q_{\lambda\mu}$ across the cut from -1 to $+1$ is expressible in terms of $p_{\lambda\mu}$ according to

$$i^{\mu-\lambda}q_{\lambda\mu}(j,\,x+i0)-i^{\lambda-\mu}q_{\lambda\mu}(j,\,x-i0)=-i\pi\bar{p}_{\lambda\mu}(j,x)\,. \tag{B8}$$

The asymptotic behavior of these functions is most easily expressed in terms of $q_{\lambda\mu}$. For large values of z, the asymptotic behavior follows at once from (B6), and is

$$q_{\lambda\mu}(j,z)\simeq\frac{2^jN^2(j,\mu)}{\Gamma(2j+2)}z^{-j-1}\,, \tag{B9}$$

whence it follows from (B5) and (B7) that for $\mathrm{Re}j>-\frac{1}{2}$

$$d_{\lambda\mu}(j,z)\simeq(\mp i)^{\lambda-\mu}\Gamma(2j+1)(\tfrac{1}{2}z)^j \tag{B10}$$

according as $\mathrm{Im}z>0$ or <0. For large j, such that $|\arg j|<\pi-\delta<\pi$, and with z fixed and not on the cut, we have[21]

$$q_{\lambda\mu}(j,\,\cosh\xi)\simeq[\pi/2j\,\sinh\xi]^{1/2}e^{-(j+\frac{1}{2})\xi}\,. \tag{B11}$$

Now consider a function $T_{\lambda\mu}(x)$ defined on the real interval $-1\leq x\leq 1$, and let

$$T_{\lambda\mu}{}^j=(j+\tfrac{1}{2})\int_{-1}^{1}T_{\lambda\mu}(\cos\vartheta)d_{\lambda\mu}{}^j(-\vartheta)d\cos\vartheta\,.$$

The inverse relation is

$$T_{\lambda\mu}(\cos\vartheta)=\sum_{j=j_0}^{\infty}T_{\lambda\mu}{}^jd_{\lambda\mu}{}^j(-\vartheta)\,,$$

[21] This follows from Ref. 19, Vol. I, p. 77, Eq. (16).

where $j_0=\max(|\lambda|,|\mu|)$. If we define

$$A_{\lambda\mu}(j)=\frac{N(j,\mu)}{N(j,\lambda)}T_{\lambda\mu}{}^j$$
$$=(j+\tfrac{1}{2})\int_{-1}^{1}T_{\lambda\mu}(x)\bar{p}_{\lambda\mu}(j,x)\,dx\,, \tag{B12}$$

then we can write

$$T_{\lambda\mu}(x)=\sum_{j=j_0}^{\infty}A_{\lambda\mu}(j)\bar{p}_{-\mu,-\lambda}(j,x)\,. \tag{B13}$$

Using the symmetries for physical j, we then have

$$T_{\lambda\mu}(x)=\sum_{j=j_0}^{\infty}A_{\lambda\mu}(j)(-1)^{j-\lambda}\bar{p}_{\mu,-\lambda}(j,\,-x)\,. \tag{B14}$$

In this form [though not in the form (B13)] the lower limit of the summation may be set equal to 0 or $\frac{1}{2}$ since either $\bar{p}_{\lambda\mu}$ or $\bar{p}_{\mu,-\lambda}$ vanishes at all the additional points.

Now suppose that $T_{\lambda\mu}(x)$ is analytic in x (except for square-root branch points at $x=\pm1$ which are present if $\lambda\mp\mu$ is odd) and polynomially bounded in the x plane with a cut only on the positive real axis. Then by replacing $\bar{p}_{\lambda\mu}$ in (B12) by the discontinuity of $q_{\lambda\mu}$ as given by (B8), and integrating from $-\infty$ to $+1$, we obtain an analytic continuation $A_{\lambda\mu}(j)$ which is bounded by an appropriate exponential in the right-half j plane. Thus, the conditions for a Sommerfeld-Watson transformation are satisfied and we can first write

$$T_{\lambda\mu}(x)=\tfrac{1}{2}i\int_C\frac{dj}{\sin\pi(j-\lambda)}A_{\lambda\mu}(j)\bar{p}_{\mu,-\lambda}(j,\,-x)\,.$$

where C is a contour encircling the positive real axis in a clockwise sense, and excluding any poles of $A_{\lambda\mu}(j)$, and then deform the contour to run parallel to the imaginary axis, picking up Regge poles in the usual way. A typical Regge pole term at $j=\alpha$ has the form

$$\frac{-\pi\beta}{\sin\pi(\alpha-\lambda)}d_{\lambda,-\mu}(\alpha,\,-x)\,,$$

where β is the residue of $N^{-2}(j,\mu)A_{\lambda\mu}(j)$ at $j=\alpha$.