Bootstraps and the Pion-Nucleon System*

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It has been suggested that all the observable properties of strongly interacting systems can be determined self-consistently if there do not exist any elementary particles in the conventional sense. This conjecture is applied to the nucleon mass and the pion-nucleon coupling constant, which are calculated under the assumption that the nucleon is composite, and that its existence is a consequence of the same forces which produce pion-nucleon resonances. A self-consistent calculation is possible because the nucleon and the $P_{\frac{3}{2}\frac{3}{2}}$ resonance are principally responsible for the forces which produce each other, providing an example of a "bootstrap" or self-supporting mechanism. The N/D method is used for computations. Some ambiguity in the quantitative results due to high-energy processes is unavoidable, but calculations with a large range of "cutoff" values predict the same orders of magnitude as are observed experimentally. It is shown qualitatively that the self-consistent method not only predicts the nucleon and the $P_{\frac{3}{2}\frac{3}{2}}$ resonance, but excludes other combinations which are not observed. This notion is illustrated by consideration of hypothetical interactions between nucleons and scalar mesons and between Λ and Σ hyperons of negative relative parity.

I. INTRODUCTION

THE salient feature of the physics of strong interactions is a rich and rapidly growing collection of particles, both stable and unstable, and a corresponding collection of interaction strengths and coupling constants or resonance widths. The possibility that most or all of these parameters may be determined self-consistently is an attractive conjecture. Attempts to do so have been called *bootstrap* calculations, because they are based on the hypothesis that the system of observed particles is self-supporting, and does not require forces due to elementary particles with arbitrary parameters.

Let us review the principal ingredients of the self-consistent calculation idea. The minimal aim of most theories of strong interactions is to predict scattering amplitudes and bound states from the observed particles and couplings. But not all the observed particles are to be given in the beginning; one generally thinks of starting from a few and finding the rest. Those particles which, in principle, are inserted into a theory at the start, for instance, the elementary fields of field theory or the input poles of S-matrix theory, are conventionally called elementary; their properties cannot be determined within the framework of the theory. All the rest are then "composite." For example, it has long been supposed that if the pion and nucleon be postulated, both the deuteron and the $P_{\frac{3}{2}\frac{3}{2}}$ resonance (also denoted by N^*) at 1238 MeV can be predicted.

Which strongly interacting systems are elementary? They have usually been taken to be the simplest systems with each type of quantum number. These closely correspond to the stable particles: π, K, N, Λ, Σ, etc. There must be, however, some reason for the masses of these particles as well, and it has been hoped that a complete theory of strong interactions will explain their properties also.

An appealing possible answer to the above question, first suggested by Chew and Frautschi,[1] is that there are no elementary particles at all. Or, stated positively, all systems, both stable and unstable, are composite. From this point of view, the same forces which produce resonances also produce bound states, which one may identify with the observed stable particles.

This conjecture is supported by intuitive ideas about centrifugal forces in different angular momentum states of the same system. For example, there appears to be a spin $\frac{5}{2}$, even parity, $I=\frac{1}{2}$, pion-nucleon resonance at about 1680 MeV.[2] One expects that the same forces which produce it should produce an $I=\frac{1}{2}$, even parity, spin-$\frac{1}{2}$ resonance or bound state at a lower mass, because the repulsive centrifugal force is less. No system with these properties is observed except the nucleon, and one is led to the assumption that the nucleon itself is a pion-nucleon bound state. A detailed numerical examination of this proposition is the principal subject of the present work.

If all systems are composite, then the constituent particles, as well as the systems which (by crossing symmetry) give rise to the binding forces, are also composite. One is thus led to the conception of a grand self-consistent scheme, or bootstrap mechanism, from which all the particles and resonances will emerge simultaneously. Their masses and couplings are to be determined simply by requiring consistency when crossing symmetry is applied. Of course, such a calculational program, or even anything approaching it, is at present impossible, and one is not yet able to say whether a unique solution exists or what additional concepts may be needed to complete the picture. Small, well-chosen parts can be attempted, however. Some particles may be taken from experiment; then others may be postulated, and one can see whether a self-

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[1] G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **7**, 395 (1961).

[2] Most of the experimental data used in this paper are taken from W. H. Barkas and A. H. Rosenfeld, Lawrence Radiation Laboratory Report UCRL-8030, 1961 (unpublished).

consistent scheme can be worked out, and semiquantitative predictions made, of either the presence or the absence of systems with particular quantum numbers.

An instructive example is the recent calculation of the ρ meson.[3] In its simplest form, low-energy pion-pion scattering was calculated from the hypothesis that the dominating force is the exchange of a ρ meson. This force was found to be attractive in the $I=1$, $J=1$ state, and an approximate calculation predicted a low-energy resonance, which was then identified with the ρ meson itself. A self-consistent solution is one in which the mass and width of this "output" resonance are the same as those of the exchanged meson. It was found that a unique solution existed which explained qualitatively the properties of the ρ meson, and the quantitative inaccuracy was understandable in terms of the approximations employed.

Suppose that the forces which dominate low-energy pion-nucleon scattering have been identified, that the effect of short-range forces can be imitated by a simple cutoff procedure, and that the dynamics of the system can be solved to yield bound states and resonances. Suppose further that the dynamical problem is solved for the $P_{\frac{3}{2}\frac{3}{2}}$ partial wave and the main features of the N^* are derived in reasonable accord with experiment. One concludes, if the approximations are sound, that the N^* is, indeed, a dynamical effect and that the relevant forces have been correctly recognized.

Suppose, finally, that the same methods, when applied to the $P_{\frac{1}{2}\frac{1}{2}}$ wave, which carries the quantum numbers of the nucleon, predict a bound state with mass equal to the nucleon mass. One has then quantitative evidence that the nucleon itself is a dynamical effect, a consequence of sufficiently strong attractive forces in the same sense that the N^*, or for that matter, the hydrogen atom is a dynamical effect. The forces are, in fact, so strong that the binding energy of the "composite" nucleon is comparable to the masses of the constituents themselves. This is the essential feature that permits a particle to appear as a constituent in a composite state and as the composite itself, thus obscuring, though not invalidating, the analogy to loosely bound states such as the hydrogen atom or an atomic nucleus.

In the present paper, we carry out parallel calculations of the sort indicated above for the nucleon and N^*. The masses and coupling constants enter into the definitions of the forces and then appear as derived quantities associated with singularities of the scattering amplitude, which represent bound states or resonances. The condition that input and output values of the masses and coupling constants be equal determines their numerical values; this is the bootstrap element. These values are found to correspond fairly well with the physical values, though some ambiguity connected with high-energy cutoffs cannot be avoided in this approach.

[3] F. Zachariasen, Phys. Rev. Letters **7**, 112 (1961); F. Zachariasen and C. Zemach, Phys. Rev. **128**, 849 (1962).

The next section discusses some aspects of bootstrap philosophy that can be appreciated with little or no calculation. The calculations for the pion-nucleon system are described in the following sections. Analogous questions related to the $\pi\Lambda\Sigma$ system are taken up in Sec. VI.

II. QUALITATIVE REMARKS

What is the simplest universe that may exist, in the self-supporting bootstrap sense, which includes the nucleon and the pion?[4] In a first try, one might suppose that N, the nucleon, is a bound state of πN with N exchange as the dominant force and that π is a bound state of $N\bar{N}$ with π exchange as the dominant force. This scheme has the virtue of containing a small number of particles, but has few other virtues. We find that N exchange in the $P_{\frac{1}{2}\frac{1}{2}}$ state of πN *does* supply an attractive force, which is encouraging, and it is plausible that couplings of πN to higher mass channels and the exchanges of higher mass systems exert a smaller influence on the N bound state than this mechanism. On the other hand, a dynamical calculation of the type described in the later sections demonstrates that the N exchange force by itself is too weak to bind πN to form N.

Turning to the $N\bar{N}$ system, we note that the π is a very deeply bound state, so that the high-energy (short-range) part of the $N\bar{N}$ force plays an important role. This aspect of the interaction cannot be simply derived from a diagram by Feynman's rules, but must be inferred from something like a Regge description of the particles. Lacking a workable means for incorporating a Regge description into a bootstrap calculation, we cannot even predict the sign of the π exchange force for ranges shorter than a pion Compton wavelength. Until this difficulty is overcome, we cannot evaluate the role of $N\bar{N}$ in forming π mesons, η mesons, etc. A realistic attempt to explain the π as a bound state must, in any case, begin with the lower energy channels $\pi\rho$, $\bar{K}K^*$, etc.

Hence, we restrict our objectives in this paper and attempt only to explain N as a bound state of $N+\pi$. Consider the "crossing table," Table I, which represents an attempt to express as much information about low-energy πN interactions as possible in compact form. The table entries give the signs (positive for attraction, negative for repulsion) and estimate the strengths of the forces, in the various partial wave channels, caused by exchange of different particles, namely, the N, N^*, and ρ meson in πN scattering. (These entries are actually threshold values of Born scattering ampli-

[4] The isotopic spin formalism is taken for granted in this paper. But this notion also arises out of the bootstrap principle. The point is that the bootstrap requirements are inherently symmetrical with respect to particles of like species, i.e., of like character with respect to space-time transformation properties. That such requirements lead to equalities of masses and couplings among particles of like species is not a freak accident, but may be the preferred possibility. We shall discuss this more fully elsewhere.

TABLE I. Threshold values of f/q^{2l} in pion-nucleon amplitudes.

Partial wave Isotopic spin	Angular momentum	N exchange	N^* exchange	ρ exchange	Total	Resonance (if any) (MeV)
	$S_{1/2}$	9.8×10^{-1}	-1.1	1.2×10^{-1}	0.0	
$\frac{1}{2}$	$P_{3/2}$	-5.2×10^{-2}	1.6×10^{-2}	2.8×10^{-3}	-3.3×10^{-2}	
	$D_{5/2}$	3.4×10^{-3}	-2.8×10^{-4}	7.6×10^{-5}	3.2×10^{-3}	
	$F_{7/2}$	-2.3×10^{-4}	5.2×10^{-6}	2.4×10^{-6}	-2.3×10^{-4}	
	$P_{1/2}$	2.6×10^{-2}	8.8×10^{-2}	1.2×10^{-2}	1.3×10^{-1}	940 (N)
$\frac{1}{2}$	$D_{3/2}$	-8.2×10^{-4}	-1.5×10^{-3}	3.0×10^{-4}	-2.3×10^{-3}	1510
	$F_{5/2}$	3.7×10^{-5}	2.6×10^{-5}	8.4×10^{-6}	7.1×10^{-5}	1690
	$S_{1/2}$	-2.0	-2.7×10^{-1}	-6.0×10^{-1}	-2.3	
$\frac{3}{2}$	$P_{3/2}$	1.0×10^{-1}	3.9×10^{-3}	-1.4×10^{-3}	1.0×10^{-1}	1238 (N^*)
	$D_{5/2}$	-6.8×10^{-3}	-6.9×10^{-5}	-3.8×10^{-5}	-6.9×10^{-3}	
	$F_{7/2}$	4.6×10^{-4}	1.3×10^{-6}	-1.2×10^{-6}	4.6×10^{-4}	1920
	$P_{1/2}$	-5.2×10^{-2}	2.2×10^{-2}	-6.2×10^{-3}	-3.6×10^{-2}	
$\frac{3}{2}$	$D_{3/2}$	1.6×10^{-3}	-3.7×10^{-4}	-1.5×10^{-4}	1.1×10^{-3}	
	$F_{5/2}$	-7.4×10^{-5}	6.5×10^{-6}	-4.2×10^{-6}	-7.1×10^{-5}	

tudes, after the factor q^{2l} has been divided out, calculated with the physical values of the masses and coupling constants. They are described in more detail in the next sections.) The relative magnitude of the entries on the same row shows the relative importance of different forces.

The $P_{\frac{1}{2}\frac{1}{2}}$ and $P_{\frac{3}{2}\frac{3}{2}}$ partial waves are the channels in which we expect to find, respectively, the N bound state and the N^* resonance. We observe that the forces due to N and N^* exchange are both attractive in both these channels. Acting cooperatively, the N and N^* jointly supply the justification for their own existence. It is the burden of the following sections to show that these attractive forces are in fact strong enough to produce the results expected of them.

Furthermore, the N and N^* exchanges conspire to give a resultant repulsion in all S and P waves other than their own. The centrifugal barrier effect presumably prevents these forces from causing significant low-energy resonances in higher partial waves. Then the self-consistent scheme we have used is complete in that it does not predict the existence of one-particle exchange forces which have been omitted.

Resonances do exist in some higher partial waves, but at higher energies, so that other channels, e.g., πN^*, $\bar{K}\Lambda$, etc., are likely to affect their interpretation. Thus, the fact that the 1510-MeV $D_{\frac{3}{2}}$ resonance has $I=\frac{1}{2}$ rather than $I=\frac{3}{2}$, whereas the entries of Table I favor the opposite, does not contradict the view that this resonance is a dynamical effect. Without examining the πN^* channel too closely, one may observe from the isotopic crossing matrices that both N and N^* exchanges yield forces with opposite signs in the $I=\frac{3}{2}$ and $I=\frac{1}{2}$ states. Thus, the πN^* channel may effectively reverse the apparent predictions of the πN channel in the $D_{\frac{3}{2}}$ state at higher energies.

The question may be raised as to whether other schemes might justify the existence of the nucleon. How about nucleon interactions with scalar pions? The nucleon is then expected to be a pole in the $I=\frac{1}{2}$, $S_{\frac{1}{2}}$ amplitude. Unlike the observed parity case, the nucleon exchange force turns out to be repulsive in this state, which suggests immediately that a self-supporting mechanism will be difficult to construct. We know that a strong force in the spin-$\frac{1}{2}$ amplitude may come from the exchange of an $I=J=\frac{3}{2}$ particle. The N-exchange force, however, is repulsive in the $P_{\frac{3}{2}\frac{3}{2}}$ state, so the existence of such a force would be difficult to explain; whereas $D_{\frac{3}{2}\frac{3}{2}}$ exchange is repulsive in the $I=\frac{1}{2}$, $S_{1/2}$ amplitude, at least with the experimental N^* mass. There do exist, of course, attractive forces in this state due to particles in whose state N exchange is attractive, such as a hypothetical additional odd-parity "nucleon"; but such a force must produce a bound state with binding energy constrained to be an entire pion mass, as well as overcome the repulsion due to N exchange. Therefore, although the possibility of a scalar coupling cannot be logically excluded without solving the dynamical equations numerically, it is difficult to see how a reciprocal bootstrap mechanism of the type proposed here for odd relative parity can be constructed in this case.

The points mentioned here bring one somewhat closer to a complete bootstrap description of low-energy, nonstrange baryons. By extending and completing the above remarks, one would have been able to predict the existence and properties of the nucleon even if it had never been observed experimentally.

We also take note here of some of the qualitative conclusions reached in Sec. V concerning Σ-Λ-π interactions. In analogy to the pion-nucleon problem, one may hope to interpret the Σ hyperon and the Y_1^* resonance (1380 MeV) as the result of strong forces in the $\pi\Lambda$ system produced, in fact, by Σ and Y_1^* exchange. The assumption of even $\Lambda\Sigma$ relative parity leads to a self-consistent array of forces and particles in the crossing table, Table III, whereas the odd parity alternative does not. Here, then, is theoretical verification of the,

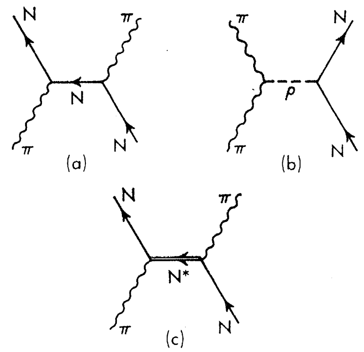

FIG. 1. Exchange diagrams for πN scattering.

by now, well-established experimental fact that the $\Lambda\Sigma$ parity is even.[5]

The bootstrap philosophy, as an integral part of the S-matrix theory, will play a role in any evaluation of the relative merits of field theory and S-matrix theory. Before the properties of the pion were decided by experiment, the principles of conventional field theory permitted one to consider with impartiality the behavior of nucleons interacting with mesons of either parity and with arbitrary mass. Nothing in the framework of field theory suggested that the masses of these two fields could not be chosen independently, or that one of the parity choices was mathematically untenable. One might speculate that a suitable interpretation of mass renormalization could lead to these notions, but in fact this was not done.

III. INPUT FORCES AND POLES

A. The Pion-Nucleon Phase Shift Analysis

We turn now to the details of the calculation.

We have assembled in the Appendix definitions of most of the symbols used in this and following sections, as well as our notation and normalization for the Dirac matrices and spinors.

First, we compute the "input information" for the dynamical calculations in the next section: namely the projections $f_B(W)$ of the Born approximation diagrams into the appropriate angular momentum states. In a strict S-matrix theory approach, the singularities corresponding to the force diagrams, Fig. 1, as well as the pole diagrams, Fig. 2, are obtainable by invariance considerations alone, but it is more convenient to use the conventional rules for Feynman diagrams.[6] In this way one finds the $B(W)$ themselves rather than just their singularities. All five diagrams are not independent, as 1(c) and 2(b) as well as 1(a) and 2(a) are related by crossing symmetry.

[5] R. D. Tripp, M. B. Watson, and M. Ferro-Luzzi, Phys. Rev. Letters **8**, 175 (1962).

[6] Among other advantages, the Feynman graph method eliminates the necessity of complicated arguments relating signs of the discontinuities across the original cuts to the corresponding signs of the partial-wave amplitudes in the crossed channel.

The projections of the pole diagrams, Fig. 2, will be used to compare residues at "output," or calculated, poles to the definitions of lifetimes and coupling constants.

In momentum space we write the pion-nucleon S matrix

$$S=1+\delta^{(4)}(P)\frac{iM}{2\pi E_1E_2}T. \qquad (3.1)$$

The normalization is defined by

$$\langle f|1|i\rangle=\delta_{fi}\delta_3(\mathbf{p}_1-\mathbf{p}_3)\delta_3(\mathbf{p}_2-\mathbf{p}_4), \qquad (3.2)$$

where δ_{fi} is a product of discrete delta functions over all initial and final spin and charge indices.

An arbitrary invariant amplitude for the general πN process is written

$$\bar{w}(p_4)[A(s,u)+B(s,u)\gamma\cdot Q]w(p_2), \qquad (3.3)$$

where A and B have two components, one for each total isotopic spin $I=\begin{pmatrix}\frac{3}{2}\\ \frac{1}{2}\end{pmatrix}$. The amplitude in the crossed channel may be obtained by interchanging the two pions and using the substitution rule:

$$A(s,u)=XA(u,s), \qquad (3.4)$$

$$B(s,u)=-XB(u,s), \qquad (3.5)$$

where the isotopic crossing matrix is

$$X=\begin{pmatrix}\frac{1}{3} & \frac{2}{3}\\ \frac{4}{3} & -\frac{1}{3}\end{pmatrix}. \qquad (3.6)$$

If one defines, as usual,

$$f_1=(E+M)[A+(W-M)B]/2W,$$
$$f_2=(E-M)[-A+(W+M)B]/2W, \qquad (3.7)$$

then the partial wave amplitudes are

$$f_{l\pm}=\frac{t_{l\pm}}{q}=\frac{1}{2}\int_{-1}^{1}[f_1(\cos\theta)P_l(\cos\theta)+f_2(\cos\theta)P_{l\pm1}(\cos\theta)]d(\cos\theta), \qquad (3.8)$$

where $f_{l\pm}$ has orbital angular momentum l and total angular momentum $j=l\pm\frac{1}{2}$. $f_{l\pm}$ is related to the phase shift $\delta_{l\pm}$ by $f_{l\pm}=(e^{i\delta_{l\pm}}\sin\delta_{l\pm})/q$.

Finally,

$$f_{l\pm}(-W)=-f_{(l+1)-}(W). \qquad (3.9)$$

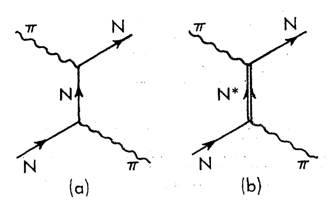

FIG. 2. Pole and resonance diagrams for πN scattering.

B. Nucleon Exchange

First, consider the usual parity assignments, in which the pion-nucleon vertex is the Yukawa interaction:

$$ig\bar{\psi}\gamma_5\tau_i\psi\varphi^i. \quad (3.10)$$

The value of $g^2/4\pi$ is experimentally about 14. A calculation of Fig. 2(a) then shows

$$A^{\text{pole}}=0;\quad B^{\text{pole}}=-\begin{pmatrix}0\\3\end{pmatrix}(g^2/4\pi)/(s-M^2). \quad (3.11)$$

Only the $I=\frac{1}{2}$, $P_{1/2}$ partial wave amplitude has a pole:

$$f_{1-}^{\text{pole}}=-\begin{pmatrix}0\\3\end{pmatrix}\frac{g^2}{4\pi}\frac{1}{2W}\frac{E-M}{W-M}. \quad (3.12)$$

The nucleon exchange force, Fig. 1(a), follows from (3.11) and crossing symmetry:

$$A^N=0;\quad B^N=\begin{pmatrix}2\\-1\end{pmatrix}\frac{g^2}{4\pi}\frac{1}{u-M^2}, \quad (3.13)$$

whence

$$f_{1,2}^N=\begin{pmatrix}2\\-1\end{pmatrix}\frac{g^2}{4\pi}\frac{(E\pm M)(W\mp M)}{2W(u-M^2)}. \quad (3.14)$$

In the center-of-mass frame,

$$u-M^2=2q^2(x_s^N-\cos\theta), \quad (3.15)$$

where

$$x_s^N=1-\frac{s+M^2-\sum}{2q^2}. \quad (3.16)$$

The partial-wave projection is then

$$f_{l\pm}^N=\begin{pmatrix}2\\-1\end{pmatrix}\frac{g^2}{4\pi}\frac{1}{4Wq^2}[(E+M)(W-M)Q_l(x_s^N)+(E-M)(W+M)Q_{l\pm1}(x_s^N)]. \quad (3.17)$$

By taking the limit $W\to W_t$, inserting the physical values for M and $g^2/4\pi$, and dividing by the momentum factor q^{2l}, one obtains the first column in Table I. We shall see later why this provides a good measure of the signs and strengths of the forces.

Next consider the case where the πN coupling is scalar. Now the vertex is just

$$\bar{\psi}\tau_i\psi\varphi^i, \quad (3.18)$$

whereas the propagator for the nucleon remains the same. Again (with primes to denote scalar coupling case),

$$B'^N=\begin{pmatrix}2\\-1\end{pmatrix}\frac{g^2}{4\pi}\frac{1}{u-M^2}, \quad (3.19)$$

but now $A'^N=-2MB'^N$, with the result that the partial waves are

$$f_{l\pm}'^N(W)=\begin{pmatrix}2\\-1\end{pmatrix}\frac{g^2}{4\pi}\frac{1}{4q^2W}[(E+M)(W-3M)Q_l(x_s^N)+(E-M)(W+3M)Q_{l\pm1}(x_s^N)], \quad (3.20)$$

from which, in the limit $W\to W_t$, one infers the remarks made about this interaction in the previous section.

C. N^* Exchange

The description of the diagrams containing an N^* line is more complicated. The N^* can be described by a spinor-vector wave function ψ_μ^* satisfying the Rarita-Schwinger equations in momentum space[7]:

$$(\gamma\cdot p-M^*)\psi_\mu^*(p)=0;\quad \gamma^\mu\psi_\mu^*=0. \quad (3.21)$$

These imply the Lorentz condition

$$p^\mu\psi_\mu^*(p)=0. \quad (3.22)$$

In the N^* pole diagram, Fig. 2(b), the mass of the resonance must be given a small imaginary part $-\frac{1}{2}\Gamma$, where Γ is the full width at half-maximum, i.e.,

$$f_{\frac{3}{2}\frac{3}{2}}=\frac{-R}{W-M^*+\frac{1}{2}i\Gamma}. \quad (3.23)$$

Unitarity at $W=M^*$ relates the residue to the width

$$\Gamma=2q^*R. \quad (3.24)$$

The πNN^* vertex is essentially unique. We shall write it as a constant times

$$(p_\pi+p_N)^\mu\bar{\psi}_N\psi_\mu^*, \quad (3.25)$$

multiplied by an isotopic factor which need not be written explicitly. The N^*, like the nucleon, has even parity; there is no γ_5 in (3.25), because the vector index itself transforms with odd parity.

The general form for the N^* propagator is $P_{\mu\nu}(K)$, where each component of the tensor is a matrix in Dirac space and may be formally written as a Green's function:

$$P_{\mu\nu}(K)=\sum_\alpha \psi_\mu^{(\alpha)}(K)\bar{\psi}_\nu^{(\alpha)}(K)/(K^2-M^{*2}+i\epsilon). \quad (3.26)$$

The sum is over the four wave functions of momentum K. For $K^2\neq M^{*2}$, this definition is ambiguous. Only the value on the mass shell, corresponding to the residue of the pole in the $P_{\frac{3}{2}\frac{3}{2}}$ amplitude, has direct physical meaning. We choose rather arbitrarily a form written down by Mandelstam *et al.*,[8]

$$P_{\mu\nu}(K)=\bar{P}_{\mu\nu}(K)(\gamma\cdot K+M^*)/(K^2-M^{*2}+i\epsilon), \quad (3.27)$$

[7] W. Rarita and J. Schwinger, Phys. Rev. **60**, 61 (1941).

[8] S. Mandelstam, J. E. Paton, R. F. Peierls, and A. Q. Sarker, Ann. Phys. (N. Y.) **18**, 198 (1962).

where

$$\bar{P}_{\mu\nu}(K)=3g_{\mu\nu}-\gamma_\mu\gamma_\nu -\frac{4K_\mu K_\nu}{M^{*2}}+\frac{\gamma_\mu(\gamma\cdot K)K_\nu+K_\mu(\gamma\cdot K)\gamma_\mu}{M^{*2}}. \quad (3.28)$$

The simplest way to verify that (3.28) can be correct is to note that for $K^2=M^{*2}$, and $\mathcal{P}_{\mu\nu}=\bar{P}_{\mu\nu}(\gamma\cdot K+M^*)$,

$$(\gamma\cdot K-M^*)\mathcal{P}_{\mu\nu}=\mathcal{P}_{\mu\nu}(\gamma\cdot K-M^*) =\gamma^\mu\mathcal{P}_{\mu\nu}=\mathcal{P}_{\mu\nu}\gamma_\nu=0. \quad (3.29)$$

Next we introduce a coupling constant C by writing for the pole diagram 2(b) (inserting an isotopic factor to give an isospin-$\frac{3}{2}$ amplitude only)

$$A^*+B^*\gamma\cdot Q=\binom{3}{0}\frac{C}{4q^{*2}(E^*+M)}(p_3-p_4)^\mu \times P_{\mu\nu}(p_1-p_2)^\nu. \quad (3.30)$$

The computation of A^* and B^* is a four- or five-page algebraic exercise, the details of which we leave to the interested reader. The principal point to remember is that all expressions occur between the spinors $\bar{w}(p_4)$ and $w(p_2)$ so that $\gamma\cdot p_2$ on the extreme right and $\gamma\cdot p_4$ on the extreme left may always be replaced by M. A convenient procedure is to evaluate first the coefficients of $\gamma\cdot K$ and of M^*, which occur in the expression for $P_{\mu\nu}$, and then write between the spinors

$$\gamma\cdot K=\gamma\cdot Q+M. \quad (3.31)$$

To find the residue at the pole, it is sufficient to evaluate A^* and B^* at $s=M^{*2}$ only. One obtains, in the center-of-mass frame,

$$A^*=\binom{3}{0}\frac{C}{4q^{*2}(E^*+M)}\frac{1}{s-M^{*2}} \times[a(s)-12q^2(M+M^*)\cos\theta], \quad (3.32)$$

$$B^*=\binom{3}{0}\frac{C}{4q^{*2}(E^*+M)}\frac{1}{s-M^{*2}}[b(s)-12q^2\cos\theta], \quad (3.33)$$

where

$$a(M^{*2})=4(M-M^*)(M+E^*),$$
$$b(M^{*2})=4(M+E^*)^2.$$

The residues vanish except in the $P_{\frac{3}{2}\frac{3}{2}}$ state, for which[9]

$$f_{1+}^{\mathrm{Res}}=-\binom{3}{0}\frac{Cq^2}{2Wq^{*2}}\frac{E+M}{E^*+M}\frac{1}{W-M^*}, \quad (3.34)$$

which explains the choice of factors in (3.30). From (3.24) it follows that C is related to the width by

$$C=M^*\Gamma/3q^*. \quad (3.35)$$

Now it is a straightforward matter to apply crossing symmetry to obtain the amplitude for the exchange force, Fig. 1(c). In analogy with (3.16) we define

$$x_s^*=1-(s+M^{*2}-\textstyle\sum)/(2q^2), \quad (3.36)$$

and introduce the cosine of the angle in the u channel evaluated when $u=M^{*2}$:

$$x_u=1-(s+M^{*2}-\textstyle\sum)/(2q^{*2}). \quad (3.37)$$

Then the partial-wave amplitudes are

$$f_{l\pm}^*=\binom{1}{4}\frac{C}{4Wq^2}\Bigg\{(E+M)Q_l(x_s^*)\Bigg[\frac{3x_u(W-2M-M^*)}{E^*+M}+\frac{(2M-M^*-W)}{E^*-M}\Bigg]+(E-M)Q_{l\pm1}(x_s^*)\times\Bigg[\frac{3x_u(2M+W+M^*)}{E^*+M}+\frac{(M^*-2M-W)}{E^*-M}\Bigg]\Bigg\}$$
$$+\binom{1}{4}\frac{C}{q^{*2}(E^*+M)}\Bigg\{\frac{(E+M)(M^*-W-E^*+M)}{2W}\delta_{l,0}+\frac{(E-M)(E^*-M-W-M^*)}{2W}\delta_{l\pm1,0}\Bigg\}. \quad (3.38)$$

The first term in (3.38) contains the singularities, and of course is identical with that obtained by Frautschi and Walecka.[10] The second term is a consequence of the particular choice of propagator (3.27). It does not have the dynamical singularities, and, as expected, is smaller than the first term in the low-energy region.[11]

D. ρ-Meson Exchange

Only the ρ-exchange diagram, Fig. 1(b), remains to be calculated. The ρ is described by a wave function $\rho_\mu{}^i$, which is a Lorentz 4-vector and an isotopic 3-vector, and is constrained to satisfy the Lorentz condition.

Then the $\pi\pi\rho$ vertex must have the form

$$g_1(p_1+p_3)^\mu T_i\rho_\mu{}^i, \quad (3.39)$$

where the components of T are the 3×3 pion isospin operators. The constant g_1 is simply related to the ρ width, and experimentally $g_1^2/4\pi\approx2$. We shall take the

[9] Note that the sign of the residue in (3.34) is determined by unitarity and is, therefore, a simple check that the sign is correct in both (3.34) and (3.38).

[10] S. Frautschi and D. Walecka, Phys. Rev. **120**, 1486 (1961).

[11] The possibility of a term like the second term in (3.38) illustrates the ambiguities of forces due to exchange of high-spin systems. It would not have appeared in a strict S-matrix derivation of the amplitude. Its magnitude is about 20% of the amplitude at threshold, which indicates the inaccuracy of substituting a pole expression for N^* exchange for an exact description of $(\pi+N)$ exchange.

ρNN vertex to be

$$g_2\bar{\psi}\gamma^\mu\tau_i\psi\rho_\mu{}^i, \tag{3.40}$$

although, as for the photon-nucleon coupling, there could also be a magnetic-moment-type term. There is no good experimental evidence for the magnitude or sign of g_2. If, as has been suggested, the ρ meson is universally coupled to the isospin current, then[12]

$$g_2=\tfrac{1}{2}g_1. \tag{3.41}$$

Equation (3.41) should be modified by form factors, which are also unknown. We shall assume that the order of magnitude and sign of g_2 are given correctly by (3.41) and will verify that the solution for the $P_{\frac{3}{2}\frac{3}{2}}$ amplitude does not change much in the range $g_1g_2/4\pi \lesssim 1$. The propagator for the vector meson is

$$-i(g_{\mu\nu}-K_\mu K_\nu/m_\rho{}^2)/(K^2-m_\rho{}^2+i\epsilon), \tag{3.42}$$

so that the amplitudes corresponding to Fig. 1(b) are

$$A^\rho=0;$$

$$B^\rho=\begin{pmatrix}1\\-2\end{pmatrix}\frac{g_1g_2}{4\pi}\frac{1}{2q^2[1+m_\rho{}^2/(2q^2)-\cos\theta]}. \tag{3.43}$$

Because in the center-of-mass frame,

$$t-m_\rho{}^2=-2q^2[1+m_\rho{}^2/(2q^2)-\cos\theta], \tag{3.44}$$

the partial waves are

$$f_{l\pm}=-\begin{pmatrix}1\\-2\end{pmatrix}\frac{g_1g_2}{4\pi}\frac{1}{2Wq^2}\bigg[(E+M)(W-M)Q_l\bigg(1+\frac{m_\rho{}^2}{2q^2}\bigg)$$
$$+(E-M)(W+M)Q_{l\pm1}\bigg(1+\frac{m_\rho{}^2}{2q^2}\bigg)\bigg]. \tag{3.45}$$

The limits of (3.38) and (3.45) for $W\to W_t$ provide the second and third columns of Table I.

IV. METHODS OF CALCULATION

The validity of the picture of the nucleon as a self-supporting mechanism depends on whether the qualitative description we have presented can be confirmed by an explicit numerical computation of the πN scattering amplitude. Therefore, the difficulty of making quantitative calculations in strong interactions becomes the major obstacle. The best one can do is to choose a reasonable approximate method and try to understand the nature of its limitations.

The one we shall use is the familiar "N/D" method, which is based on the analyticity and unitarity properties of the partial-wave amplitudes alone.[13] The details for the πN amplitude have been worked out by Frazer and Fulco,[14] Frautschi and Walecka,[10] and others,[15,16] who also have shown that W, the total energy, rather than $s=W^2$, must be chosen as the independent variable. The partial-wave amplitude is written $f=\rho h$, where ρ is some kinematical factor, and h as an analytic function has only the "dynamical" singularities and is real-symmetric.[17] h is then represented by the usual ansatz

$$h(W)=N(W)/D(W). \tag{4.1}$$

$N(W)$ contains the singularities due to forces, i.e., exchange of systems in the t and u channels, and is to be determined by them. Both N and D have the real-symmetric property of h. In most previous calculations singularities due to stable "elementary" particles in the s channel, such as the nucleon in the pion-nucleon amplitude, have been put into the numerator function.[10,16] However, in the present treatment one notes that these singularities cannot be thought of as forces, for one expects them to correspond to bound states and appear as zeros in the denominator function.

The singularities in N are given in terms of bound-state poles and scattering amplitudes in the crossed channels; i.e., one is to set $\text{Im}f=\Delta$, where Δ is known in terms of the masses and coupling constants of the particles involved. We shall call the cuts in N the force cuts, the cuts in D the unitary cuts.

It then follows that $\text{Im}h=\Delta/\rho$, or

$$\text{Im}N=D\Delta/\rho, \tag{4.2}$$

because D is analytic in the neighborhood of the cuts in N. Note that if $D=0$, then $\text{Im}N=0$, so that a bound-state pole has a real residue, even if it lies on a cut.

The most useful form of these equations for the present purposes is obtained by defining a function $B(W)$, containing the correct force-cut discontinuities, but analytic in the neighborhood of the unitary cuts. Thus, near the force cuts

$$\text{Im}B=\Delta/\rho=(\text{Im}N)/D=\text{Im}(N/D). \tag{4.3}$$

The function $N-BD$ has only the unitary cuts U, so that N satisfies a dispersion relation of the form (provided no subtractions are required)

$$N(W)=D(W)B(W)-\frac{1}{\pi}\int_U\frac{B(W')\text{Im}D(W')dW'}{W'-W}. \tag{4.4}$$

The D function has two cuts along the real axis, for $W<-W_t$ and $W>W_t$, where $W_t=M+\mu$. As a consequence of the reflection rule (3.9), unitarity for both cuts can be combined in

$$\text{Im}D(W+i\epsilon)$$
$$=-q(|W|)\rho(W)N(W)\theta(|W|-W_t)R(W), \tag{4.5}$$

[12] M. Gell-Mann and F. Zachariasen, Phys. Rev. **123**, 1065 (1961).

[13] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

[14] W. Frazer and J. Fulco, Phys. Rev. **119**, 1420 (1960).

[15] S. W. MacDowell, Phys. Rev. **116**, 774 (1960).

[16] J. Uretsky, Phys. Rev. **123**, 1459 (1961).

[17] That is, for all complex W, $h(W^*)=[h(W)]^*$.

where R is the ratio of the total to the elastic cross section in the appropriate partial wave and θ is the step function. As a complex function, N is analytic (and real) along this cut, so that, in integral form, and arbitrarily normalizing at $W=W_0$, (4.5) becomes

$$D(W)=1-\frac{W-W_0}{\pi}\int_U \frac{q'N(W')\rho(W')R(W')dW'}{(W'-W)(W'-W_0)}, \quad (4.6)$$

provided the integral converges. The contour U is the entire real axis except for $|W|<W_t$. Inserting (4.6) into (4.4) one obtains

$$N(W)=B(W)+\frac{1}{\pi}\int_U \frac{q'\rho(W')N(W')R(W')}{W'-W} \times\left[B(W')-\frac{W-W_0}{W'-W_0}B(W)\right]dW'. \quad (4.7)$$

Equations (4.6) and (4.7) form a set of equations for determining the full amplitude $h=N/D$.

The phase factor $\rho(W)$ must serve both to eliminate "kinematical" singularities (which have not been put into h), and to guarantee the correct threshold behavior. The partial wave threshold behavior $f\sim q^{2l}$ is not a consequence of partial wave unitarity and analyticity, although it does follow from the double dispersion relations. One is not guaranteed in advance that a suitable factor ρ always exists with both these properties.

Basic to any approximation scheme is the idea that it is most important to know the singularities of the functions N and D near the region of interest, namely within a few pion masses of the physical threshold. The first approximation must be to the unitarity condition (4.5). We shall set $R(W)=1$ for all energies W. This is exactly correct in the elastic region, which extends almost up to the $P_{\frac{3}{2}\frac{3}{2}}$ resonance. One may hope it is reasonable provided one does not attempt to understand quantitatively any of the higher resonances.

Secondly, one must decide which exchange diagrams to consider. We wish to include exchange of all systems whose mass is of the order of the energy range of interest. The nearest singularities are always due to exchange of single-particle states; the only one in the present problem is the nucleon at 940 MeV in the u channel.

How can one represent simply the exchange of two-particle systems, corresponding to the analytic continuation of the physical amplitudes in the crossed channels? We shall adopt the point of view that these forces may be very well represented by the exchange of the relevant resonances (which we have implicitly assumed when we discussed a bootstrap mechanism including the N^*), and then make the approximation that in the exchange diagrams the resonances may be treated like stable particles with real mass. This approximation is valid to the extent that the width of the resonance is small compared to its position. The principal known low-mass pion-pion resonance is the ρ meson at about 750 MeV. In the u channel we shall take into account only the $P_{\frac{3}{2}\frac{3}{2}}$ isobar. Diagrams for these three forces are shown in Fig. 1. The sum of the three Born amplitudes for these forces (multiplied by an appropriate kinematical factor) makes up our approximation to $B(W)$.

In a given partial wave, how can one estimate the relative importance of the several contributions? Because of the denominators in the integrals for N and D, one expects the solution to be most sensitive to the values of $B(W)$ near the region of interest, at low energies. Because the kinematical factor is chosen so that the function $B(W)$ approaches a nonzero constant at threshold, the value of the various terms at $q=0$ is a simple, yet fairly accurate, guide to the signs and relative strengths of the forces, provided the function B is reasonably smooth. In particular, if B is negative in a large enough region including W_t, N will be negative, and D can have no zeros, i.e., the "force" is repulsive. These are the considerations which led us to examine first the threshold values of the Born partial-wave amplitudes in order to obtain qualitative information about the various forces.

For our problem we must write equations of the type (4.6) and (4.7) for the $I=J=\frac{1}{2}$ and $I=J=\frac{3}{2}$ P-wave amplitudes, conventionally denoted f_{11} and f_{33}. First we must find suitable ρ functions in the two cases.

The threshold behavior at the beginning of the left-hand unitary cut is determined by the reflection rule (3.9). Therefore, to write (4.2) for the $P_{\frac{1}{2}\frac{1}{2}}$ partial wave, one may choose the kinematical factor,

$$\rho_{11}(W)=(E-M)/W. \quad (4.8)$$

This factor removes the kinematical double pole at the origin and also guarantees the correct threshold behavior at both the $S_{1/2}$ and $P_{1/2}$ thresholds. That is, the function

$$h_{11}(W)=f_{11}(W)/\rho_{11}(W) \quad (4.9)$$

is expected to have no kinematical zeros and only dynamical singularities throughout the W plane.

Next we must examine the convergence of the integral equations which determine h. The method is applicable without introducing subtraction parameters only if the integral in the D equation (4.6) converges as written. Because $\rho(W)\to\frac{1}{2}$ for large W, the integrand is proportional to $N(W')/W'$, and therefore N must approach zero for large W.

Next consider the N equation, which now has the form

$$N(W)=B(W)+\frac{1}{\pi}\int_U \frac{q'\rho(W')N(W')}{W'-W} \times\left[B(W')-\frac{W-W_0}{W'-W_0}B(W)\right]dW'. \quad (4.10)$$

The convergence of (4.10) depends on the high-energy behavior of the Born function $B(W)$, which is a sum of terms representing the exchanges of various systems.

Consider the Born function f_B, which is related to B just as f is to h. We have calculated f_B from Feynman diagrams as if they were elementary particle diagrams. This gives each term as a function of s and t or s and u of the general form, for exchange of systems of spin j and mass m in the u channel,

$$W^{-3}(u-m^2)^{-1}P_{j+\frac{1}{2}}[1-(s+m^2-\sum)/2q^2], \quad (4.11)$$

plus terms which diverge more slowly with W. The high-energy asymptotic form of (4.11) is proportional to $W^{2j-2}/(u-m^2)$. In particular, the nucleon exchange term goes like $1/W$. S-matrix theory, however, tells one the form of the function only near the pole in u, and the analytic continuation from the singularity is not unambiguous. The Regge pole representation, for instance, suggests that the asymptotic form is proportional to $W^{\alpha(s)}/(u-m^2)$, where $\alpha(m^2)=2j-2$, but in the physical region $u<0$, $\alpha(u)$ is always less than $\frac{1}{2}$. Even in a theory with elementary particles, poles of type (4.11) exist at most only for the elementary particles themselves, and others produced dynamically have damped asymptotic behavior.

A complete dynamical theory of the low-energy properties of a system must come to terms with the high-energy problem. It is probably fair to say that any calculation of, for example, the nucleon problem is not simply a test of the bound-state conjecture, but is simultaneously a test of the adequacy of the treatment of the high-energy contributions.[18] The alternative adopted here is to multiply the input functions $B(W)$ by a damping factor

$$[1+(W^2-W_t^2)/Z^2]^{-1}, \quad (4.12)$$

which is effective for energies larger than some mass Z, hereafter called the cutoff mass. Thus, we remain as noncommittal as possible regarding the actual mechanisms for damping, hoping that through a single parameter the qualitative consequences of the mechanisms can be simulated, and that the quantitative results do not depend sensitively on the cutoff mass.

Both theoretical and experimental indications about Regge trajectories suggest that the damping mechanism becomes operative at energies above a small multiple of the nucleon mass. The different exchange processes are damped at different rates, but we did not wish to manipulate more than one parameter. The most reasonable value of the cutoff is presumably lower for partial waves dominated by N^* exchange than for those dominated by N exchange.

Next let us consider what modifications are necessary to calculate the $P_{\frac{3}{2}\frac{3}{2}}$ amplitude. The basic equations are still (4.6) and (4.7). Because the left-hand unitary cut now describes D waves, the proper kinematical factor, in analogy with (4.8), should be

$$\rho_{33}=q^2(E+M)/W. \quad (4.13)$$

[18] Among methods whose cutoff techniques are different from ours, we may cite those of L. A. P. Balázs, Phys. Rev. **126**, 1220 (1962); and D. Wong, *ibid.* **126**, 1220 (1962).

In order for (4.6) to converge now, $W^2N(W)$ must approach zero. The cutoff B functions will indeed have this property, because they are projections from a form which satisfies a double dispersion relation, but an integral like that in (4.10) cannot decrease faster than $1/W$, except under very special conditions which one cannot hope will be met. To write these equations correctly one would have to make a subtraction, thereby introducing arbitrary parameters. Because this obstacle exists only in the $P_{\frac{3}{2}\frac{3}{2}}$ amplitude, which is somewhat peripheral to our main subject, we shall, like previous authors, go around it.[10] If (4.13) is replaced by

$$\rho_{33}(W)=q^2/W^2, \quad (4.14)$$

the convergence arguments now become identical to those for the $P_{\frac{1}{2}\frac{1}{2}}$ equations. The $P_{3/2}$ threshold behavior is still correct, but the $D_{3/2}$ amplitude goes like q^2 instead of q^4. If one considers only the P waves, the error is far away from the region of interest, and the approximation has the same validity as the others we have made, provided one can verify that the D-wave amplitude is reasonably small. Of course, our results will be meaningless as a calculation of the $D_{\frac{3}{2}\frac{3}{2}}$ amplitude, whereas the $P_{\frac{1}{2}\frac{1}{2}}$ calculation also gives information about the low-energy S-wave amplitude.

To what extent can one use these methods now to decide whether the nucleon and the N^* resonance alone can support themselves dynamically, and if so, what masses and couplings a self-consistent calculation predicts? One might imagine proceeding in the following fashion: Choose a nucleon mass M and a pion-nucleon coupling constant g and solve the $P_{\frac{3}{2}\frac{3}{2}}$ equations self-consistently for the mass M^* and the width Γ of the N^* resonance. Then use this M^* and Γ to solve the $P_{\frac{1}{2}\frac{1}{2}}$ equations self-consistently to obtain a new M and g. Finally, repeat this sequence for various M and g until a completely self-consistent solution for all four quantities is reached.

Although this could be, in principle, a method for predicting the nucleon mass from the pion mass alone, the necessity of introducing a cutoff in the present calculation precludes anything so sensational. Here we shall compute the properties of the nucleon and the N^* separately, assuming the other has been done and taking its parameters from experiment. Thus, we cannot really predict the nucleon, but will be able to offer a quantitative explanation of its dynamical origin.[19]

V. RESULTS OF THE CALCULATION

A. The $P_{\frac{3}{2}\frac{3}{2}}$ Partial Wave

We are now prepared to solve (4.6) and (4.10) to find the $P_{\frac{3}{2}\frac{3}{2}}$ amplitude, which has been defined as

$$h_{33}=(W^2/q^2)f_{33}=f_{33}/\rho_{33}=N_{33}/D_{33}. \quad (5.1)$$

[19] The plausibility of such a "reciprocal bootstrap" mechanism has been suggested by Chew, using an effective-range-type argument. G. F. Chew, Phys. Rev. Letters **9**, 233 (1962).

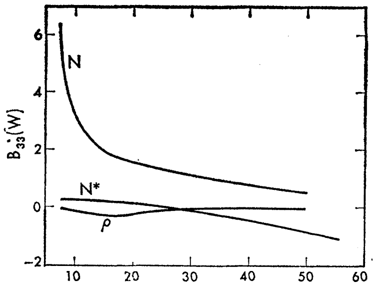

FIG. 3. The three contributions to the Born function $B(W)$ for the $P_{\frac{3}{2}\frac{3}{2}}$ partial wave, in pion mass units, and with experimental masses and couplings.

For given input, a resonance occurs at M^* when $\mathrm{Re}D(M^*)=0$, near which

$$h_{33}(W) =\frac{N(M^*)/\mathrm{Re}D'(M^*)}{(W-M^*)-iq^*\rho(M^*)N(M^*)/\mathrm{Re}D'(M^*)}. \quad (5.2)$$

By comparing this with (3.34) and (3.35), one obtains the width:

$$\Gamma=-2q^{*3}N(M^*)/M^{*2}\,\mathrm{Re}D'(M^*). \quad (5.3)$$

The three contributions to the function $B_{33}(W)$ are shown in Fig. 3. Nucleon exchange is indeed the most important, and it is not surprising that the gross features of the $P_{\frac{3}{2}\frac{3}{2}}$ resonance have been explained in terms of this force alone. Figure 4 illustrates the method of searching for a self-consistent resonance at a particular cutoff value. Here we have used $Z=10M$, or about 9.4 BeV. The nucleon and ρ constants were chosen at their physical value [and to satisfy (3.41)]. B_{33} was calculated for a variety of masses M^* and widths Γ. For each input Γ, we found an "output" position and width, which is plotted as a function of the "input" mass in Fig. 4 for a sequence of widths.

The solution can be self-consistent only when a curve crosses the dashed diagonal line, along which input mass equals output mass. For each width one thus finds a self-consistent mass. The correct self-consistent solution occurs where the output width at this mass equals the input width.

For the problem of Fig. 4, the solution is self-consistent when $M^*=1160$ MeV and $\Gamma\approx 49$ MeV. The value of Γ is not so far off as it might seem, because it is related to the residue by a kinematical factor containing q^{*3}. Define instead a constant proportional to the residue at the N^* pole:

$$G=M^{*2}\Gamma/2q^{*3}. \quad (5.4)$$

Experimentally G is about 8.5. The value $\Gamma=49$ MeV at $M^*=1160$ MeV corresponds to a G of about 11.

The "low-energy" approximations, especially to the unitary condition, should not be so good at $W=M^*$ as they are at the nucleon mass. Furthermore, in the $P_{\frac{3}{2}\frac{3}{2}}$ problem, the divergent behavior of the N^* exchange was not fully offset by the damping factor. The lowest cutoff mass for which a solution could be found was $8M=7.5$ BeV. Of course, the effective cutoff for N^* exchange should be lower than that for N exchange, as noted earlier.

TABLE II. Self-consistent mass and width of N^* resonance.

Cutoff (MeV)	Mass (MeV)	Width (MeV)	G
7500	1300	293	10.0
8450	1186	82	9.2
9400	1160	49	10.0

The self-consistent results for three cutoff values are summarized in Table II.

It has long been known that one could explain the gross features of low-energy πN scattering, namely, the existence of the $P_{\frac{3}{2}\frac{3}{2}}$ resonance and the relation between its width and position, by considering nucleon exchange as the only, or at least the principal, attractive force.[20] Therefore, for comparison of our calculation method to earlier ones we have also solved our equations without a cutoff including the nucleon exchange force alone.

The solution for the phase shift in this case is shown in curve I, Fig. 5. The D function has a zero at about 1064 MeV, i.e., slightly below threshold. That is, the undamped nucleon force is strong enough to predict not a resonance but a bound state. It is instructive to compare this result with the calculation of Frautschi and Walecka.[10] They used an N/D method and wrote down essentially the same equations; but whereas we have solved the integral equation numerically, they approximated B by a few poles and then solved the coupled equations algebraically. Their result is curve II of Fig. 5. They, indeed, found a resonance, but only slightly above threshold and still well below the observed value. Nevertheless, the similarity of curves I and II is encouraging to those who try to get a semi-quantitative idea of the solution to equations like the coupled N/D equations by approximating the integral by a few wisely chosen poles.

If the nucleon is a bound state, it is not surprising that the undamped force predicts a resonance at too low a mass. For a composite particle the short-range or high-energy part must be damped, i.e., there should be a cutoff even through the equations are soluble

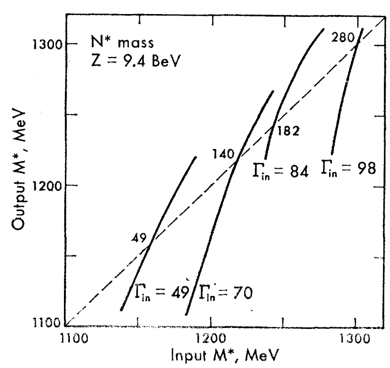

FIG. 4. Output position of the $P_{\frac{3}{2}\frac{3}{2}}$ resonance as a function of input position, for four different input widths, computed with the cutoff parameter Z taken to be 10 nucleon masses. The output widths at the self-consistent masses are indicated along the diagonal line. All widths are in MeV.

[20] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956).

without one. The correct force is, therefore, not so attractive, and the true position of the resonance is higher than predicted in this way.

The other calculation we shall compare with is the calculation of the $P_{\frac{3}{2}\frac{3}{2}}$ amplitude by means of the "determinantal" method. This type of calculation was first done by Baker[21] and by Bali *et al.*[22] One may think of this approximation as the first order of an iterative solution to our N/D equations. If in zero order $N_0=0$ and $D_0=1$, (4.6) gives in first order $N_1=B$, and D_1 follows from (4.7). To insure approximate crossing symmetry over the short cut, which is due to nucleon exchange alone, one must subtract at $W=M$. The discontinuity across the whole force cut is then $\mathrm{Im}(B/D)$, whereas in the coupled equations it is just $\mathrm{Im}B$. Because from (4.7) it follows that $D(W)$ approaches infinity logarithmically for large W, the effect of this first-order approximation is to make the force less attractive; i.e., the same effect as a cutoff, and, therefore, a higher position will be predicted for the resonance. The phase shift calculated in this way is shown in curve III, Fig. 5. Our result is slightly different from Ref. 21 and 22, because we have used a different value for the πN coupling constant in order to compare with the other curves; the calculation predicts a resonance at about 1540 MeV.

B. The $P_{\frac{1}{2}\frac{1}{2}}$ Partial Wave

Let us now turn to the calculation of the self-consistent nucleon pole in the $P_{\frac{1}{2}\frac{1}{2}}$ channel. We assume the resonance to have been found exactly and carry out a similar computation for the nucleon. The $P_{\frac{1}{2}\frac{1}{2}}$ amplitude is defined as

$$h_{11}(W)=N_{11}/D_{11}=Wf_{11}/(E-M). \qquad (5.5)$$

If one finds $D(M_0)=0$, with M_0 below the physical threshold, there is a $P_{1/2}$, $I=\frac{1}{2}$, bound state coupled to

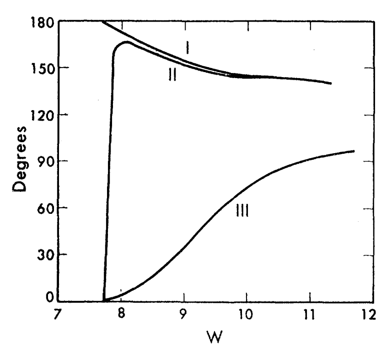

FIG. 5. The $P_{\frac{3}{2}\frac{3}{2}}$ πN phase shift calculated from undamped nucleon exchange alone, as a function of the center-of-mass energy W. Curve I is our numerical solution of the integral equations. Curve II is the pole approximation to the same equations by Frautschi and Walecka. Curve III is a first-order "determinantal" calculation.

[21] M. Baker, Ann. Phys. (N. Y.) 4, 271 (1958).

[22] N. Bali, C. Garibotti, J. J. Giambiagi and A. Pignotti, Nuovo Cimento **20**, 1209 (1961).

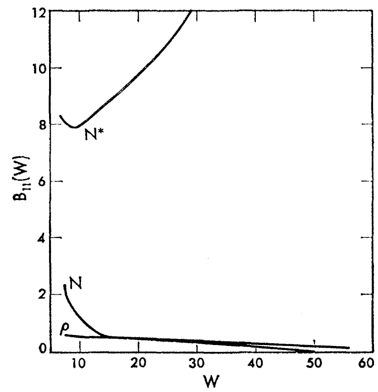

FIG. 6. The three contributions to the Born function $B(W)$ for the $P_{\frac{1}{2}\frac{1}{2}}$ partial wave, in pion mass units, and with experimental masses and couplings.

the pion and nucleon with pseudoscalar coupling constant

$$g^2/4\pi=-\ N(M_0)/D'(M_0). \qquad (5.6)$$

One does not expect to find an additional particle, but rather to identify the bound state with the nucleon itself. Therefore, one must require $M_0=M$.

Figure 6 shows the contributions to $B_{11}(W)$. One may verify that the contribution of the nonsingular polynomial term to the N^* exchange force is indeed, as hoped, a small fraction of the total.

Notice from Fig. 6 that the ρ-exchange force is quite small for reasonable values of the couplings. Its contribution to the self-consistent solution will be almost negligible. [Chew has shown that in an effective-range approach the ρ contribution vanishes near $W=M^{19}$; the same conclusion can be reached by taking the appropriate static limits in (3.44).] This is a fortunate circumstance, for then the solution will depend very little on the unknown ρ-nucleon coupling and the form factors.

The N^* exchange is now the dominating force. Thus, the nucleon cannot be self-supporting, as in a first approximation the ρ meson may be imagined to be, but the nucleon and the N^* can support each other. The nucleon-exchange force is, however, not negligible, and cannot be ignored. The method of computation is analogous to that in the preceding section, except that here the variable mass M appears as the mass of an external particle as well as of an exchanged particle.

We found, as anticipated, that solutions exist for rather lower cutoff values than were necessary for the $P_{\frac{3}{2}\frac{3}{2}}$ problem; this too is fortunate, as the ambiguities connected with the N^* exchange are diminished.

The method of searching for a self-consistent solution is identical to that used in the N^* calculation. Figure 7 shows a typical self-consistency plot, for a cutoff of 2.82 BeV. The mass of the output pole is plotted against the mass of the input pole for a range of coupling constants. Self-consistency is attained for $M=887$ MeV, $g^2/4\pi=19$.

The principal result is contained in Figs. 8 and 9, which show self-consistent mass and coupling constant versus cutoff. The two curves in Fig. 9 correspond to

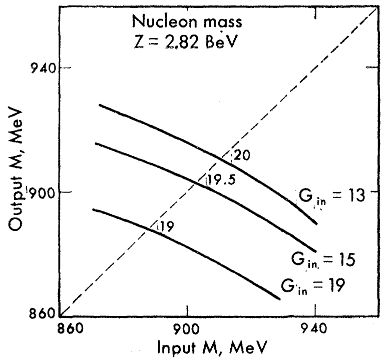

FIG. 7. Output versus input nucleon mass for three input coupling constants $G=g^2/4\pi$, with cutoff parameter 2.82 BeV. Output $g^2/4\pi$ at the self-consistent masses is indicated along the diagonal line.

two values for the ρ-nucleon coupling. The corresponding two curves in Fig. 8 would not be distinguishable in this scale.

There are several ways one might approach these results. At one extreme one may inquire how accurately this calculation has succeeded in predicting M and $g^2/4\pi$ theoretically; or one may be much less ambitious and simply ask to what extent the results of Figs. 8 and 9 confirm the qualitative conjectures concerning the composite nature of the nucleon.

The first question is difficult to answer, for one has no way of knowing at present which value of the cutoff is preferable, beyond a qualitative idea that it is more likely to be 3 or 4 nucleon masses than 30 or 40. In a cutoff range 2 to 5 BeV, the coupling and the mass difference M^*-M vary by about a factor of 2 on either side of the experimental values. Whether one thinks this is a quantitatively good result or not depends on one's expectations.

On the other hand, if the nucleon were truly elementary, we would have had no reason to expect any correlation at all between the observed nucleon parameters and the results of this type of calculation.

The numerical computations were done on the IBM 7090 at the Lawrence Radiation Laboratory. The integral in the N equation was approximated by a finite sum according to Simpson's rule, and the resulting matrix equation solved by the subroutine MATINV. The matrix sizes found to be practical ranged from 60×60 to 80×80. The computing time for all the calculations described in this paper was about 1 h, not including time wasted by programming errors. We are obliged to David Wong for discussions about numerical methods.

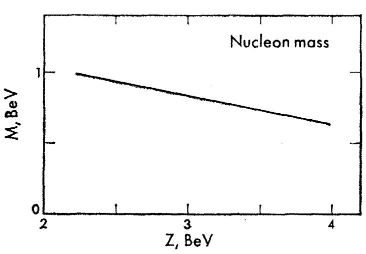

FIG. 8. Self-consistent nucleon mass as a function of cutoff parameter Z.

VI. THE Λ AND Σ HYPERONS

If all particles are composite, one should be able to predict or at least explain them all by calculations related to the one just presented for the nucleon or to the previous vector meson calculations. In this final section, we shall discuss briefly and in less quantitative detail a possible dynamical explanation of the Σ hyperon.

The lowest mass two-particle system in whose scattering amplitude the Σ may appear as a pole is the $\pi\Lambda$ system. The nearest force-type singularities are due to Σ exchange in the u channel, so that one has a bootstrap-type situation.

What can one say about the ΛΣ parity? If it is even, then the system closely resembles the πN system we have just studied. In that case the $\pi\Lambda\Sigma$ vertex is

$$ig_{\pi\Lambda\Sigma}\bar{\psi}_\Lambda\gamma_5\psi_\Sigma\pi+\text{H.c.}, \tag{6.1}$$

and the partial wave Born functions are

$$f_{l\pm}(W)=\frac{g_{\pi\Lambda\Sigma}{}^2}{4\pi}\frac{1}{4q^2W}[(E+M)(W+Y-2M)Q_l(x)$$
$$+(E-M)(W+2M-Y)Q_{l\pm1}(x)], \tag{6.2}$$

where now E and M refer to the Λ, and Y is the Σ mass. x is defined by (3.16), but with M^2 replaced by Y^2. For $Y=M$, (6.2) reduces to (3.17) without the isotopic factor. Under the even-parity assumption, one expects to find the Σ in the $P_{1/2}$ wave of the $\pi\Lambda$ system. But the force from (6.2) is repulsive in the $P_{1/2}$ wave, in contrast to the analogous πN situation which is affected by isotopic spin factors. At first sight, this might suggest the parity cannot be even; but experience with the nucleon problem shows that Σ exchange may produce a large $P_{3/2}$, $I=1$, resonance. The obvious candidate is the Y_1^* resonance at 1385 MeV and width about 50 MeV.

The force due to Y_1^* exchange is given by a formula exactly like (3.38) except that the isotopic factor is omitted, and M and M^* now refer to the Λ and Y^*, etc. The constant C is related to the Y^* width by

$$C=M^*\Gamma/q^*. \tag{6.3}$$

This force may be sufficiently attractive in the $P_{1/2}$ state to overcome the Σ exchange repulsion. Therefore, a bound state can perhaps still be managed here, because less binding energy is necessary than in the nucleon case where it was constrained to be exactly one pion mass.

The numerical basis for these remarks is contained in the first two rows of Table III, which is an abbreviated crossing table for $\pi\Lambda$ interactions. As in Table I the entries are the threshold values of Born amplitudes, except for the momentum factor, giving the sign and strength of the relevant exchange forces. We have set, rather arbitrarily, $g^2/4\pi=14$.

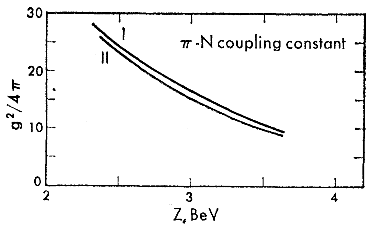

FIG. 9. Self-consistent πN coupling constant $g^2/4\pi$ as a function of cutoff parameter Z, for two values of the ρN coupling constant g_2. Curve I, $g_2=\frac{1}{2}$; curve II, $g_2=1$.

TABLE III. Threshold values f/q^{2l} in pion-lambda amplitudes.

Partial wave	Σ exchange	$Y_1{}^*$ exchange	Assumed $\Lambda\Sigma$ parity
$P_{3/2}$ ($Y_1{}^*$ channel)	2.2×10^{-2}	5.7×10^{-3}	$+1$
$P_{1/2}$ (Σ channel)	-1.1×10^{-2}	2.9×10^{-2}	$+1$
$P_{3/2}$ ($Y_1{}^*$ channel)	-1.6×10^{-2}	5.7×10^{-3}	-1
$S_{1/2}$ (Σ channel)	6.0×10^{-1}	-4.1×10^{-1}	-1

If, on the other hand, the Σ parity is odd relative to the Λ, the $\pi\Lambda\Sigma$ coupling would be scalar, and the factor $i\gamma_5$ should be omitted from the vertex (6.1). The Born functions can be obtained simply by changing the sign of Y in (6.2), and now one looks for the Σ pole in the $S_{1/2}$ amplitude. In this state the force is moderately attractive and, with a sufficiently large coupling constant, a self-consistent solution can be obtained. (We have obtained one by a determinantal computation.) The exchange of a $P_{3/2}$ resonance is repulsive in this state, but not necessarily enough to overcome the Σ-exchange attraction.

On the other hand, the force in the $P_{3/2}$ state is strongly repulsive. Therefore, unless the $P_{3/2}$ spin and parity assignment for the $Y_1{}^*$ turns out to be incorrect, we can predict with some confidence that the $\Lambda\Sigma$ parity is even, and that the dynamics are similar to those of the pion-nucleon system. These remarks are illustrated by the last two rows of Table III, which gives the crossing relations for the negative-parity case in the relevant partial waves. We have taken $g^2/4\pi=1$.[23] It should be noted that the $\frac{3}{2}$ and $\frac{1}{2}$ threshold values are not really comparable.

With even $\Lambda\Sigma$ parity the similarity between the one-channel $\pi\Lambda$ problem and the πN problem is so close computationally that only trivial modifications of the computer programs were necessary to solve these equations as well. We have attempted to find the $P_{1/2}$ bound state only, and put in the properties of the Λ and $Y_1{}^*$ from experiment. The Σ mass and the $\pi\Lambda\Sigma$ coupling constant were looked for self-consistently as a function of cutoff. A minor simplification is that there is no force corresponding to ρ exchange, because the ρ meson cannot couple directly to the Λ.

The computation was done for a range of coupling constants, and graphs like Fig. 7 were drawn. Figure 10 shows the results for cutoff at about five Λ masses, or 5.6 BeV. The self-consistent Σ mass is 1180 MeV, and $g_{\pi\Lambda\Sigma}{}^2/4\pi=18.5$. The experimental Σ mass is about 1190 MeV.

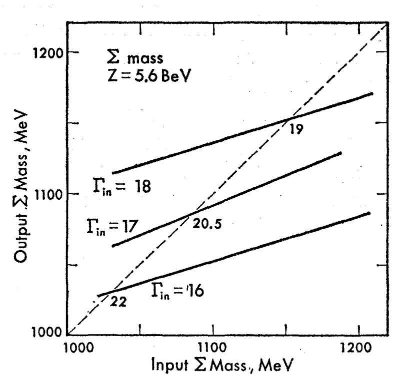

FIG. 10. Output versus input Σ mass for three input $\pi\Lambda\Sigma$ coupling constants with cutoff parameter taken to be 5 lambda masses. Output coupling constants at self-consistent masses are indicated along the diagonal line.

A self-consistent solution could be found only in the immediate neighborhood of this (quite reasonable) cutoff value, so that no graphs such as Figs. 8 and 9 can be presented for this problem. In any case, this calculation can only be considered as a partial illustration of the dynamics of the Σ problem. We have omitted an important effect which has no analog in pion–nucleon scattering; namely, the competing $\pi\Sigma$ state. The elastic unitarity relation, as written, is correct only for a rather small range of the physical region. A quantitatively correct treatment must doubtless include $\pi\Sigma$ scattering and $\pi\Sigma-\pi\Sigma$ transitions as well, and include Σ, Λ, and $Y_1{}^*$ exchange in all channels. Such a problem might be framed in a matrix ND^{-1} formalism.

The Λ pole can appear only in elastic $\pi\Sigma$ scattering, which is, therefore, a one-channel problem; but one cannot solve this problem alone for the lambda as a bound state, because the $\pi\Sigma\Sigma$ coupling is also unknown. To obtain enough conditions to be able to find all the quantities self-consistently, one must simultaneously solve the two-channel $I=1$ problem and the one-channel $I=0$ problem. The Σ and Λ would then both be found as self-supporting bound states.

We are indebted to F. Zachariasen for many discussions on the topics in this paper.

APPENDIX. PION-NUCLEON KINEMATICS AND NOTATION

First, we identify the principal symbols we have used.

μ = pion mass, M = nucleon (or Λ) mass,
$\sum=2M^2+2\mu^2$, $\beta=(M^2-\mu^2)^2$,
$M^*=N^*$ (or $Y_1{}^*$) mass, $\Gamma=N^*$ (or $Y_1{}^*$) width,
$g=\pi N$ coupling constant, $g_1=\pi\rho$ coupling constant,
$g_2=\rho N$ coupling constant, q = c.m. momentum,
W = total c.m. energy, E = nucleon (or Λ) energy,
θ = c.m. scattering angle, W_t = threshold energy,
W_0 = subtraction energy, Z = cutoff energy,
$m_\rho=\rho$ mass.

[23] The choice of the value of the scalar coupling constant is arbitrary. Because it is related differently to the residue at the pole, its value must be rather smaller than a pseudoscalar coupling constant to produce forces of similar magnitude in the crossed channels.

E^*, q^*, etc., are the values of E, q, etc., evaluated for $W=M^*$. We use a Lorentz scalar product in which $x\cdot y=x^0y^0-\mathbf{x}\cdot\mathbf{y}=x^\mu g_{\mu\nu}y^\nu$, and then define the Dirac gamma matrices by

$$\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=2g^{\mu\nu}.$$

The incident and final four-momenta of the pions are p_1, p_3, respectively, and of the nucleons are p_2, p_4.

The scalar invariants are

$$s=(p_1+p_2)^2,\quad t=(p_1-p_3)^2,\quad u=(p_1-p_4)^2,$$

which satisfy

$$s+t+u=\sum.$$

In the center-of-mass system,

$$t=-2q^2(1-\cos\theta),$$
$$u=\sum-s+2q^2(1-\cos\theta),$$
$$4q^2=s-\sum+\beta/s,$$
$$\cos\theta=1-(s+u-\sum)/2q^2,$$
$$E\pm M=\frac{(W\pm M)^2-\mu^2}{2W}.$$

We define the positive-energy Dirac spinors by

$$(\gamma\cdot p-M)w(p)=0,$$

normalized so that $\bar{w}w=1$, and set

$$Q=\tfrac{1}{2}(p_1+p_3),$$
$$\gamma_5=i\gamma_1\gamma_2\gamma_3\gamma_4.$$

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Kinetic Approach to Condensation

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A kinetic formalism including a Boltzmann-like equation is introduced to study classical condensation phenomena in gases. Force laws which include a repulsive core together with an r^{-N} attractive tail are examined for all integral N greater than unity. The theory considers perturbations whose wavelengths are large compared to the diameter of the core. The results fall into two categories depending on whether $N\leq 3$ or $N\geq 4$, respectively. For the first class of long-range forces, there are no stable thermodynamic states. For the second class of short-range forces, phase-equilibrium curves are found which are in accord, qualitatively, with classical results. In the limit as $N\to\infty$, all states are stable. A discussion of the effects of random collisions is included.

I. INTRODUCTION AND SUMMARY OF RESULTS

THE principal formalisms by which gas condensation has, in the past, been investigated separate into three distinct areas of study. The widest of these is the statistical-mechanics approach[1–8] which, in turn, is centered about the construction of a partition function or higher order virial coefficients. A second formalism is that of Becker and Döring[9] which is concerned primarily with the development of droplets in a condensing gas. A third avenue of investigation is a fluid-dynamical one which was first suggested by Jeans[10,11] in studies of gravitational instabilities. In the present analysis another kinetic formalism is initiated, which is centered about a Boltzmann-like equation. This equation stems from the first-order reduced Liouville[12] equation and is derived (cf. Appendix) by expanding the integral over the two-particle interaction in terms of the correlation between the particles. The lowest order equation so obtained contains a collective force term over noncorrelated particles.[13]

This equation is used to uncover the stability of Maxwellian equilibrium states. If these instabilities are interpreted as being the origin of condensation phenomena (gas $\to$ liquid), then the related stability criteria readily yield phase-equilibrium curves. That this is indeed the case has been demonstrated[14] (to within second-virial-coefficient standards) through exhibiting

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