Electromagnetic Properties of a Charged Vector Meson*

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A systematic study is made of the electromagnetic properties of charged vector mesons. The various formalisms used to describe charged particles of spin 1 are compared, and a new first-order formulation of the Stückelberg theory is developed. For the most general first-order Proca Lagrangian, subject to the usual symmetry requirements we eliminate the redundant components to obtain a Hamiltonian formulation. The theory is interpreted in the nonrelativistic limit, and the terms corresponding to spin-orbit coupling and electric quadrupole-moment interaction are identified. The analogy to spin-$\frac{1}{2}$ theory has led us to consider spin equations of motion which agree with the quantum-mechanical equations to order m^{-2}.

This general form for the electromagnetic interaction is applied to a recalculation of the $\mu \to e+\gamma$ decay rate through a vector-meson intermediary. We conclude, that the absence of this process is not necessarily an argument against the existence of an intermediary meson in weak interactions.

I. INTRODUCTION

THE charged vector meson that has been proposed as a possible intermediary field (B field) in the weak interactions must, if it exists, have a mass greater than that of the K meson and a very short lifetime.[1] Against such an intermediary field, Feinberg[2] and Gell-Mann[3] have argued that, provided the two neutrinos in μ decay are capable of annihilating each other, such a B field would allow the decay $\mu \to e+\gamma$ in first order in the μ-decay coupling constant G with a rate considerably larger than that experimentally observed.[4] This rate depends very strongly on the nature of the vector-meson electromagnetic coupling which we will investigate in this paper.

The vector-meson field theory differs from the Dirac theory by the appearance of redundant components in the covariant equations of motion, and by the necessity of defining expectation values with an indefinite metric. We begin by demonstrating the equivalence of the various formalisms used for describing charged vector mesons. In particular, we present a new first-order treatment of the Stückelberg theory.[5] Invariance arguments enable us to write down the most general Lagrangian for such particles from which a generalized Sakata-Taketani[6] equation can be derived. The nonrelativistic form (to order m^{-2}) of the theory is readily obtained by a Foldy-Wouthuysen[7] reduction of these Sakata-Taketani equations. As in the Dirac case, the electromagnetic moments are identified with various terms in the nonrelativistic Hamiltonian for the vector meson interacting with an external electromagnetic field. In a uniform electromagnetic field, the equation of motion of a vector meson of magnetic moment $ge\hbar/2mc$ agrees to order m^{-2} with that obtained on invariance grounds for a classical spinning particle.

By way of application, the rate for the unobserved process $\mu \to e+\gamma$ is recalculated for a vector meson of arbitrary (constant) magnetic dipole and electric quadrupole moments. With a suitable choice of these two parameters, the rate for this process, and for the also unobserved $\mu - e$ conversion in a nuclear field, can be made equal to zero.

II. ELECTROMAGNETIC INTERACTIONS OF A CHARGED VECTOR MESON

A. Comparison of the Formulations of the Theory of Spin 1 Fields

1. First-Order Proca Equations

A first-order form of the Proca theory[8] is given by the Lagrangian

$$\mathcal{L}=\tfrac{1}{2}U_{\mu\nu}^\dagger(\partial_\mu U_\nu-\partial_\nu U_\mu)+\tfrac{1}{2}(\partial_\mu U_\nu^\dagger-\partial_\nu U_\mu^\dagger)U_{\mu\nu} -\tfrac{1}{2}U_{\mu\nu}^\dagger U_{\mu\nu}+m^2U_\mu^\dagger U_\mu \quad (2.1)$$

for the case of free fields. In Eq. (2.1), $U_\mu(x)$, $U_{\mu\nu}(x)$ are independent field variables, $U_\mu^\dagger(x)$, $U_{\mu\nu}^\dagger(x)$ are the Hermitian conjugate fields, and m is the mass. The above Lagrangian gives the free-field equations

$$U_{\mu\nu}=\partial_\mu U_\nu-\partial_\nu U_\mu,$$
$$\partial_\mu U_{\mu\nu}=m^2U_\nu.$$

In the presence of an electromagnetic field, we perform the usual gauge-invariant replacement[5] $\partial_\mu \to \pi_\mu \equiv \partial_\mu - ieA_\mu$, where $A_\mu(x)$ is the electromagnetic four-

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potential, which yields the field equations

$$U_{\mu\nu}=\pi_\mu U_\nu-\pi_\nu U_\mu\,, \tag{2.2}$$

$$\pi_\mu U_{\mu\nu}=m^2 U_\nu\,. \tag{2.3}$$

The second-order wave equation

$$(\pi^2-m^2)U_\nu-\pi_\mu\pi_\nu U_\mu=0 \tag{2.4}$$

is obtained by substituting Eq. (2.2) into Eq. (2.3). Since a four-vector field must actually possess only three independent components, a subsidiary condition eliminating the unwanted fourth component is needed. This is most easily obtained from Eq. (2.3),

$$\pi_\nu\pi_\mu U_{\mu\nu}=-\tfrac{1}{2}(\pi_\mu\pi_\nu-\pi_\nu\pi_\mu)U_{\mu\nu}=(\tfrac{1}{2}ie)F_{\mu\nu}U_{\mu\nu}=m^2\pi_\mu U_\mu$$

or

$$\pi_\nu U_\nu=(ie/2m^2)F_{\mu\nu}U_{\mu\nu}\,, \tag{2.5}$$

where

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu\,.$$

The second-order wave equation (2.4) then becomes

$$(\pi^2-m^2)U_\nu-(ie/2m^2)\pi_\nu(F_{\mu\lambda}U_{\mu\lambda})+ieF_{\mu\nu}U_\mu=0\,. \tag{2.6}$$

2. *Duffin-Kemmer Formalism*

The first-order Proca equations (2.2) and (2.3) may be written in the matrix form $(\beta_\mu\pi_\mu+m)\psi=0$ by setting

$$\psi=\begin{pmatrix} -(1/m)U_{14}\\ -(1/m)U_{24}\\ -(1/m)U_{34}\\ -(1/m)U_{23}\\ -(1/m)U_{31}\\ -(1/m)U_{12}\\ U_1\\ U_2\\ U_3\\ U_4 \end{pmatrix},$$

$$\beta_1=\begin{pmatrix}
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot
\end{pmatrix},\quad
\beta_2=\begin{pmatrix}
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot
\end{pmatrix},$$

$$\beta_3=\begin{pmatrix}
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot
\end{pmatrix},\quad
\beta_4=\begin{pmatrix}
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-i&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-i&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-i&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
-i&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&-i&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&-i&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot
\end{pmatrix}.$$

These β's satisfy the algebra-defining equation

$$\beta_\mu\beta_\nu\beta_\lambda+\beta_\lambda\beta_\nu\beta_\mu=\beta_\mu\delta_{\nu\lambda}+\beta_\lambda\delta_{\mu\nu}\,.$$

The first-order Proca equations are thus a realization of the Duffin-Kemmer formalism.[5]

3. *Discussion of Second-Order Field Equations*

In a first-order formalism, the subsidiary condition eliminating the timelike vector mesons either is one of the equations of motion or can be derived from them. When the equations of motion are of second order, however, the subsidiary condition must be separately assumed. The second-order equations obtained by the substitution $\partial_\mu\to\pi_\mu$ are then generally not mutually consistent without the addition of suitable $F_{\mu\nu}$ terms. For example, equations

$$(\Box^2-m^2)U_\mu=0 \quad\text{and}\quad \partial_\mu U_\mu=0$$

on $\partial_\mu\to\pi_\mu$ become

$$(\pi^2-m^2)U_\mu=0\,, \tag{2.7}$$

$$\pi_\mu U_\mu=0\,. \tag{2.8}$$

Since $[\pi_\nu,\pi^2]\neq 0$, Eq. (2.7) is inconsistent with Eq. (2.8). A similar difficulty arises with the conventional Stuckelberg formalism[5] in the case of electromagnetic interaction. For these reasons we have preferred to use a Lagrangian giving first-order equations of motion which after $\partial_\mu\to\pi_\mu$ can be iterated so as to yield the consistent second-order equations (2.5) and (2.6).

4. *Stückelberg Formalism*

There is one other dynamical form of the vector-meson theory, introduced by Stückelberg,[5] which is well known in the neutral-meson case. There has apparently been, however, no consistent treatment of the electromagnetic interaction of charged mesons in the Stückelberg formalism. The original Stückelberg theory is a second-order formalism involving a four-vector field Z_μ and a scalar field B.[5] In the absence of interaction, these fields are related to the Proca field U_μ by the equation $U_\mu=Z_\mu+m^{-1}\partial_\mu B$. By the subsidiary condition

$$\partial_\mu Z_\mu+mB=0,$$

the scalar field B cancels out the fourth component of the vector-meson field. In the conventional formulation, when the electromagnetic interaction is introduced by the minimal substitution $\partial_\mu\rightarrow\pi_\mu$, this separately imposed subsidiary condition becomes inconsistent with the field equations. We will consider here a new first-order formulation of this theory which is internally consistent automatically and turns out to be identical with Proca theory.

For free mesons consider the Lagrangian

$$\begin{aligned}\mathcal{L}=&\tfrac{1}{2}Z_{\mu\nu}{}^\dagger[\partial_\mu Z_\nu-\partial_\nu Z_\mu+m^{-1}(\partial_\mu\partial_\nu-\partial_\nu\partial_\mu)B]\\&+\tfrac{1}{2}[\partial_\mu Z_\nu{}^\dagger-\partial_\nu Z_\mu{}^\dagger+m^{-1}(\partial_\mu\partial_\nu-\partial_\nu\partial_\mu)B^\dagger]Z_{\mu\nu}\\&-\tfrac{1}{2}Z_{\mu\nu}{}^\dagger Z_{\mu\nu}+m^2Z_\mu{}^\dagger Z_\mu+mZ_\mu{}^\dagger\partial_\mu B+m\partial_\mu B^\dagger Z_\mu\\&+C_\mu{}^\dagger\partial_\mu B+\partial_\mu B^\dagger C_\mu-C_\mu{}^\dagger C_\mu,\end{aligned}\tag{2.9}$$

where $Z_{\mu\nu}$, B, Z_μ, and C_μ are independent field variables. On variation of $\mathcal{L}$ we obtain the equations

$$\partial_\nu Z_{\nu\mu}-m^2Z_\mu-m\partial_\mu B=0,\tag{2.10}$$

$$Z_{\mu\nu}=\partial_\mu Z_\nu-\partial_\nu Z_\mu,\tag{2.11}$$

$$\partial_\nu Z_\nu+m^{-1}\partial_\mu C_\mu=0,\tag{2.12}$$

$$C_\mu=\partial_\mu B.\tag{2.13}$$

By operating on Eq. (2.10) with ∂_μ we obtain Eq. (2.12) on using Eq. (2.13). Substitute Eq. (2.11) into Eq. (2.10) to obtain

$$(\Box^2-m^2)Z_\mu-\partial_\mu(\partial_\nu Z_\nu+mB)=0,$$

and, using Eqs. (2.12) and (2.13), we find

$$(\Box^2-m^2)(Z_\mu+m^{-1}\partial_\mu B)=0.\tag{2.14}$$

Set $U_\mu=Z_\mu+m^{-1}\partial_\mu B$ so that Eq. (2.14) along with the condition $\partial_\mu U_\mu=\partial_\mu Z_\mu+m^{-1}\Box^2B=0$ [which is identical to Eqs. (2.12) and (2.13)] reduces to the Proca equations. Thus, the internally consistent equations,

$$\partial_\nu Z_{\nu\mu}-m^2Z_\mu-m\partial_\mu B=0,\tag{2.15}$$

$$Z_{\mu\nu}=\partial_\mu Z_\nu-\partial_\nu Z_\mu\tag{2.16}$$

together with (2.14), are equivalent to the Proca equations.

The advantage of the above first-order formulation is the possibility of introducing the electromagnetic interaction consistently. Put $\partial_\mu\rightarrow\pi_\mu$ in Eq. (2.9) to obtain

$$\begin{aligned}\mathcal{L}=&\tfrac{1}{2}Z_{\mu\nu}{}^\dagger[\pi_\mu Z_\nu-\pi_\nu Z_\mu-(ie/m)F_{\mu\nu}B]\\&+\tfrac{1}{2}[\pi_\mu{}^\dagger Z_\nu{}^\dagger-\pi_\nu{}^\dagger Z_\mu{}^\dagger+(ie/m)F_{\mu\nu}B^\dagger]Z_{\mu\nu}\\&-\tfrac{1}{2}Z_{\mu\nu}{}^\dagger Z_{\mu\nu}+m^2Z_\mu{}^\dagger Z_\mu+mZ_\mu{}^\dagger\pi_\mu B+m\pi_\mu B^\dagger Z_\mu\\&+C_\mu{}^\dagger\pi_\mu B+\pi_\mu B^\dagger C_\mu-C_\mu{}^\dagger C_\mu.\end{aligned}\tag{2.17}$$

From Eq. (2.17) follow the equations

$$\pi_\nu Z_{\nu\mu}-m^2Z_\mu-m\pi_\mu B=0,\tag{2.18}$$

$$Z_{\mu\nu}=\pi_\mu Z_\nu-\pi_\nu Z_\mu-(ie/m)F_{\mu\nu}B,\tag{2.19}$$

$$\pi_\nu Z_\nu+m^{-1}\pi_\mu C_\mu-(ie/2m^2)F_{\mu\nu}Z_{\mu\nu}=0,\tag{2.20}$$

$$C_\mu=\pi_\mu B.\tag{2.21}$$

As in the free-field case [if we use Eq. (2.21)] operating on Eq. (2.18) with π_μ gives Eq. (2.20). Substitute Eq. (2.19) into Eq. (2.18) to find

$$(\pi^2-m^2)Z_\mu-\pi_\nu\pi_\mu Z_\nu-m\pi_\mu B-(ie/m)\pi_\nu(F_{\nu\mu}B)=0.$$

When Eqs. (2.20) and (2.21) are used, this latter equation becomes

$$\begin{aligned}(\pi^2-m^2)(Z_\mu+m^{-1}\pi_\mu B)+ieF_{\nu\mu}(Z_\nu+m^{-1}\pi_\nu B)\\-(ie/2m^2)\pi_\mu(F_{\lambda\nu}Z_{\lambda\nu})=0\end{aligned}\tag{2.22}$$

on making use of the commutation relations

$$[\pi_\mu,\pi^2]=-ie\pi_\nu F_{\mu\nu}-ieF_{\mu\nu}\pi_\nu.$$

If we set $U_\mu=Z_\mu+m^{-1}\pi_\mu B$, then $Z_{\mu\nu}=U_{\mu\nu}$, and Eq. (2.22) becomes

$$(\pi^2-m^2)U_\mu-(ie/2m^2)\pi_\mu(F_{\lambda\nu}U_{\lambda\nu})+ieF_{\nu\mu}U_\nu=0,$$

which is identical with Eq. (2.6) in the Proca theory. In addition, the subsidiary condition Eq. (2.5) in the Proca theory is readily seen to be identical to Eq. (2.20).

B. Most General Lagrangian for a Charged Vector Meson

1. *Divergence Transformations*

The theories we have just considered possess, as we shall see in Sec. D, a "normal" magnetic moment, i.e., their gyromagnetic ratio g is 1. The Lagrangians we have been using are not unique, however. In the Proca theory the divergence

$$\mathcal{L}'=\gamma\partial_\nu[\partial_\mu U_\nu{}^\dagger U_\mu-\partial_\mu U_\mu{}^\dagger U_\nu],\tag{2.23}$$

where γ is a dimensionless constant, may be added to the free-field Lagrangian (2.1). The divergence $\mathcal{L}'$ will not change the field equations derived from the Lagrangian. However, the Lagrangian $\mathcal{L}+\mathcal{L}'$ will have, as field equations in the presence of electromagnetic interaction,

$$U_{\mu\nu}=\pi_\mu U_\nu-\pi_\nu U_\mu,\tag{2.24}$$

$$\pi_\mu U_{\mu\nu}-m^2U_\nu+ie\gamma F_{\mu\nu}U_\mu=0.\tag{2.25}$$

The term proportional to γ in Eq. (2.25) will correspond to an additional magnetic-moment interaction.[5] We see then that there are infinitely many free-particle Lagrangians leading to the same free-field equations but differing in the distribution of charge density. Thus, the principle of minimal electromagnetic interaction does not define a "normal" magnetic moment unless the free-particle Lagrangian is specified. Since, for any choice of γ, the theory is nonrenormalizable,[9] this criterion too (unlike the spin-$\frac{1}{2}$ case) is not usable to define a preferred electromagnetic interaction.

2. *Electric Quadrupole Moment Interaction*

Group theoretical considerations allow a particle of spin 1 to possess an electric quadrupole moment in addition to a magnetic dipole moment. We now proceed to show how an electric quadrupole-moment interaction can be added to the first-order Proca Lagrangian. We require that such an interaction be bilinear in the meson field variables U_μ and $U_{\mu\nu}$, and linear in the electric charge e and the derivatives of the electromagnetic field $\partial_\lambda F_{\mu\nu}$. Since these derivatives are constrained by the Maxwell equations

$$\partial_\nu F_{\mu\lambda}-\partial_\mu F_{\nu\lambda}=\partial_\lambda F_{\mu\nu}\,,$$

only the form

$$\mathcal{L}''=aeU_{\mu\nu}{}^\dagger U_\lambda\partial_\lambda F_{\mu\nu}+a^*eU_\lambda{}^\dagger U_{\mu\nu}\partial_\lambda F_{\mu\nu} \qquad (2.26)$$

satisfies these requirements along with the requirements of Lorentz and gauge invariance. The multiplication factor a is now determined by demanding invariance of this electromagnetic interaction under time reversal.

We define the time-reversed fields (apart from arbitrary phases, which are the same for all terms in the total Lagrangian) by

$$\begin{aligned} &A_i{}^T=A_i(\mathbf{r},-t)\,, && A_0{}^T(\mathbf{r},t)=-A_0(\mathbf{r},-t)\,,\\ &U_i{}^T=U_i(\mathbf{r},-t)\,, && U_0{}^T(\mathbf{r},t)=-U_0(\mathbf{r},-t)\,,\\ &\partial_i{}^T=\partial_i\,, && \partial_4{}^T=-\partial_4\,,\quad a^T=a^*. \end{aligned}$$

Applying these definitions to Eq. (2.26), we have

$$(\mathcal{L}'')^T=\mathcal{L}''=a^*eU_{\mu\nu}{}^\dagger U_\lambda\partial_\lambda F_{\mu\nu}+aeU_{\mu\nu}U_\lambda{}^\dagger\partial_\lambda F_{\mu\nu}\,,$$

and thus, in complete analogy to the β-decay Hamiltonian, all coupling constants must be relatively real, and a pure imaginary. Choosing $a=iq/4m^2$, where q is an arbitrary dimensionless constant, we obtain the electric quadrupole-moment interaction

$$\mathcal{L}''=(ieq/4m^2)[U_{\mu\nu}{}^\dagger-U_\lambda{}^\dagger U_{\mu\nu}]\partial_\lambda F_{\mu\nu}. \qquad (2.27)$$

We have been unable to introduce a term like (2.27) in a "normal" way by suitable choice of a free-particle Lagrangian without going to derivatives of third or higher order. The quadrupole moment is, nevertheless, subject to the same degree of ambiguity as the magnetic moment, since, as we shall see in Sec. D, the "normal" interaction (2.23) already implies a certain amount of quadrupole moment.

Adding Eqs. (2.1), (2.23) (with $\partial_\mu\to\pi_\mu$), and Eq. (2.27), we now have as the total Lagrangian

$$\begin{aligned}\mathcal{L}=\tfrac{1}{2}U_{\mu\nu}{}^\dagger(\pi_\mu U_\nu-\pi_\nu U_\mu)+\tfrac{1}{2}(\pi_\mu{}^\dagger U_\nu{}^\dagger-\pi_\nu{}^\dagger U_\mu{}^\dagger)U_{\mu\nu}\\ -\tfrac{1}{2}U_{\mu\nu}{}^\dagger U_{\mu\nu}+m^2U_\mu{}^\dagger U_\mu+(ie\gamma/2)(U_\mu{}^\dagger U_\nu-U_\nu{}^\dagger U_\mu)F_{\mu\nu}\\ +(ieq/4m^2)[U_{\mu\nu}{}^\dagger U_\lambda-U_\lambda{}^\dagger U_{\mu\nu}]\partial_\lambda F_{\mu\nu}.\end{aligned} \qquad (2.28)$$

Except for the possibility of letting γ and q have form-factor space-time dependence, this Lagrangian is the most general charged vector-meson Lagrangian consistent with the ordinary invariance requirements. The vector-meson theory tacitly used in the original $\mu\to e+\gamma$ argument[2,3] corresponded to the choice $\gamma=q=0$. As discussed in Sec. IIB1, we know of no physical criterion justifying a particular choice of γ.

In the next two sections we investigate more fully the physical content of this theory.

C. Generalized Sakata-Taketani Equation

The Lagrangian (2.28) furnishes the field equations

$$\pi_\mu U_{\mu\nu}-m^2U_\nu+ie\gamma U_\mu F_{\mu\nu}+(ieq/4m^2)U_{\mu\lambda}\partial_\nu F_{\mu\lambda}=0\,, \qquad (2.29)$$

$$U_{\mu\nu}=\pi_\mu U_\nu-\pi_\nu U_\mu+(ieq/2m^2)U_\lambda\partial_\lambda F_{\mu\nu}. \qquad (2.30)$$

A meson field satisfying first-order wave equations is expected to have six dynamically independent components, corresponding to the three independent field variables and their time derivatives. Equations (2.29) and (2.30) must, therefore, contain four redundant components which we wish to eliminate. Since in Eqs. (2.29) and (2.30) $U_{ij}(i,j=1,2,3)$ and U_4 do not contribute to the time development of the meson field, these are the four components to be eliminated. After this elimination we will possess a Hamiltonian form of the theory. For simplicity, we consider the electromagnetic fields time-independent, and the magnetic field spatially constant, in the terms proportional to q only. The terms not proportional to q can be considered completely general.

From Eq. (2.29) we have

$$U_4=(1/m^2)(\pi_iU_{i4}+ie\gamma U_iF_{i4})\,.$$

Let $m\phi_i\equiv U_{i4}$, so that we have

$$U_4=(1/m)\boldsymbol{\pi}\cdot\boldsymbol{\phi}+(e\gamma/m^2)\mathbf{U}\cdot\mathbf{E}\,,$$

where $\mathbf{E}$ is the electric field strength. Also from Eq. (2.29),

$$\begin{aligned}\pi_jU_{ji}-m^2U_i+\pi_4U_{4i}=-ie\gamma F_{ji}U_j-ie\gamma U_4F_{4i}\\ -(ieq/2m^2)U_{4j}\partial_iF_{4j}-(ieq/4m^2)U_{lm}\partial_iF_{lm}\,,\end{aligned}$$

which becomes

$$\begin{aligned}i\frac{\partial\phi_i}{\partial t}=e\varphi\phi_i+mU_i+m^{-1}[\boldsymbol{\pi}\times(\boldsymbol{\pi}\times\mathbf{U})]_i+ie\gamma m^{-1}(\mathbf{U}\times\mathbf{H})_i\\ +e\gamma m^{-2}\mathbf{E}_i(\boldsymbol{\pi}\cdot\boldsymbol{\phi})+e^2\gamma^2m^{-3}\mathbf{E}_i(\mathbf{U}\cdot\mathbf{E})\\ -e(\tfrac{1}{2}q)m^{-2}\phi_j\partial_iE_j\,,\end{aligned} \qquad (2.31)$$

[9] H. Umezawa and S. Kamefuchi, Nucl. Phys. 23, 399 (1960).

where φ is the scalar potential, and $\mathbf{H}$ is the magnetic field strength. We wish to write this last equation in matrix form. It is lengthy, but not difficult, to show that if one introduces the spin-1 matrices

$$S_1=\begin{bmatrix}0&0&0\\0&0&-i\\0&i&0\end{bmatrix},\quad S_2=\begin{bmatrix}0&0&i\\0&0&0\\-i&0&0\end{bmatrix},\quad S_3=\begin{bmatrix}0&-i&0\\i&0&0\\0&0&0\end{bmatrix},$$

Eq. (2.31) can be written as

$$\begin{aligned}\frac{\partial\phi}{\partial t}=&e\varphi\phi+mU-m^{-1}(\mathbf{S}\cdot\boldsymbol{\pi})^2U-e\gamma m^{-1}(\mathbf{S}\cdot\mathbf{H})U\\&-e\gamma m^{-2}S_iS_jE_j\pi_i\phi+e\gamma m^{-2}(\mathbf{E}\cdot\boldsymbol{\pi})\phi\\&-e^2\gamma^2m^{-3}(\mathbf{S}\cdot\mathbf{E})^2U+e^2\gamma^2m^{-3}\mathbf{E}^2U\\&+e(\tfrac{1}{2}q)m^{-2}S_iS_j\partial_jE_i\phi-e(\tfrac{1}{2}q)m^{-2}(\boldsymbol{\nabla}\cdot\mathbf{E})\phi\end{aligned}\tag{2.32}$$

where

$$\phi=\begin{bmatrix}\phi_1\\\phi_2\\\phi_3\end{bmatrix},\quad U=\begin{bmatrix}U_1\\U_2\\U_3\end{bmatrix}.$$

Now Eq. (2.30) becomes

$$U_{4i}=\pi_4U_i-\pi_iU_4+ie(\tfrac{1}{2}q)m^{-2}U_j\partial_jF_{4i},$$

which can also be written in matrix form:

$$\begin{aligned}i\frac{\partial U}{\partial t}=&e\varphi U+m\phi+m^{-1}(\mathbf{S}\cdot\boldsymbol{\pi})^2\phi-(\boldsymbol{\pi}^2/m)\phi-em^{-1}(\mathbf{S}\cdot\mathbf{H})\phi\\&+e\gamma m^{-2}S_iS_j\pi_j(E_iU)-e\gamma m^{-2}\boldsymbol{\pi}\cdot(\mathbf{E}U)\\&+e(\tfrac{1}{2}q)m^{-2}S_iS_j(\partial_iE_j)U-e(\tfrac{1}{2}q)m^{-2}(\boldsymbol{\nabla}\cdot\mathbf{E})U.\end{aligned}\tag{2.33}$$

We now define a six-component wave function

$$\psi=(1/\sqrt{2})\begin{pmatrix}U+\phi\\-U+\phi\end{pmatrix}$$

so that Eqs. (2.32) and (2.33) take the Schrödinger form,

$$\begin{aligned}i\frac{\partial\psi}{\partial t}=\{&e\varphi+\rho_3m+i\rho_2(\mathbf{S}\cdot\boldsymbol{\pi})^2/m\\&-(\rho_3+i\rho_2)(\boldsymbol{\pi}^2+e\mathbf{S}\cdot\mathbf{H})/2m\\&-(\rho_3-i\rho_2)e\gamma(\mathbf{S}\cdot\mathbf{H})/2m\\&-(e\gamma/2m^2)(1+\rho_1)\\&\times[(\mathbf{S}\cdot\mathbf{E})(\mathbf{S}\cdot\boldsymbol{\pi})-i\mathbf{S}\cdot(\mathbf{E}\times\boldsymbol{\pi})-\mathbf{E}\cdot\boldsymbol{\pi}]\\&+(e\gamma/2m^2)(1-\rho_1)\\&\times[(\mathbf{S}\cdot\boldsymbol{\pi})(\mathbf{S}\cdot\mathbf{E})-i\mathbf{S}\cdot(\boldsymbol{\pi}\times\mathbf{E})-\boldsymbol{\pi}\cdot\mathbf{E}]\\&-(e^2\gamma^2/2m^3)(\rho_3-i\rho_2)[(\mathbf{S}\cdot\mathbf{E})^2-\mathbf{E}^2]\\&+(eq/4m^2)[Q_{ij}(\partial E_i/\partial x_j)-2(\partial E_i/\partial x_i)]\}\psi,\end{aligned}\tag{2.34}$$

where $Q_{ij}=S_iS_j+S_jS_i$. For $\gamma=q=0$, Eq. (2.34) reduces to the Sakata-Taketani[6] equation. The charge matrices ρ_1, ρ_2, ρ_3 are the usual 2 by 2 Pauli matrices:

$$\rho_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad i\rho_2=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\quad \rho_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

D. Nonrelativistic Limit of the Vector-Meson Theory

To find the nonrelativistic limit of Eq. (2.34) we use the Foldy-Wouthuysen method[7] of successive unitary transformations. The free-particle Hamiltonian [$e=0$ in Eq. (2.34)] is diagonalized by the unitary transformation

$$\exp(\tfrac{1}{2}i\rho_1\phi),$$

where

$$\tan(\tfrac{1}{2}\phi)=[2i/(E^2+m^2)][\tfrac{1}{2}\mathbf{P}^2-(\mathbf{S}\cdot\mathbf{P})^2],\tag{2.35}$$

so that we have

$$U=\begin{bmatrix}(E+m)/2(mE)^{1/2} & -[\tfrac{1}{2}P^2-(S\cdot P^2)]/(E+m)(mE)^{1/2}\\ -[\tfrac{1}{2}P^2-(S\cdot P^2)]/(E+m)(mE)^{1/2} & (E+m)/2(mE)^{1/2}\end{bmatrix}.\tag{2.36}$$

Thus, in the noninteracting case, $H'=U^{-1}HU=\rho_3E$, so that each sign of the charge can be represented by a three-component wave function.

In the interacting Hamiltonian of Eq. (2.34) we define "even" operators as those containing ρ_3 or 1, and "odd" operators as those containing ρ_2 or ρ_1. For the nonrelativistic limit we require that H be free of odd operators up to some order in the inverse mass. Successive canonical transformations U, where $U=e^{iS}$, $S=i\rho_3O/2m$, and the O are odd operators of the Hamiltonian, will eliminate O from the Hamiltonian. An example of such an O is $i\rho_2(\mathbf{S}\cdot\boldsymbol{\pi})^2/m$. The resulting wave equation is

$$i\partial\psi/\partial t=(H_0+H_1)\psi;\tag{2.37}$$

and

$$H_0=e\varphi+m+\boldsymbol{\pi}^2/2m-(\boldsymbol{\pi}^2)^2/8m^3$$

$$H_1=-\frac{e}{2mc}\mathbf{S}\cdot\left[g\mathbf{H}+\frac{g-1}{2mc}(\mathbf{E}\times\boldsymbol{\pi}-\boldsymbol{\pi}\times\mathbf{E})\right]-(eQ/4)Q_{ij}\partial E_i/\partial x_j+e(\tfrac{1}{2}Q)\boldsymbol{\nabla}\cdot\mathbf{E}+O(m^{-3}),$$

where $\boldsymbol{\pi}=\mathbf{P}-e\mathbf{A}$ and $Q=-(g-1+q)(\hbar/mc)^2$. The three terms in H_1 are identified as a magnetic-moment spin-orbit coupling term, an electric-quadrupole coupling term, and a Darwin term. Except for this last term, the same Hamiltonian H_0+H_1 is also obtained for spin-0 ($S_i=Q_{ij}=0$) and for spin-$\frac{1}{2}$ particles (that is,

$S_i=\frac{1}{2}\sigma_i$, $Q_{ij}=0$) of arbitrary gyromagnetic ratio. The Darwin term is zero for spin 0 and $[e\hbar/2(2mc)^2]\nabla\cdot\mathbf{E}$ for spin $\frac{1}{2}$. Except for these Darwin terms, which vanish in the classical ($\hbar=0$) limit, particles of different spin are, thus, found to obey the same nonrelativistic wave equation (2.37), once allowance is made for the possibility of arbitrary magnetic dipole and electric quadrupole moments in the higher spin cases. This result suggests that, except for the specifically quantum-mechanical Darwin term, the nonrelativistic wave equation is actually spin-independent and that its form depends on classical invariance arguments only.

It is worth noting that a vector particle could have, except for $g=1$, a quadrupole-moment interaction proportional to the "anomalous moment" $g-1$, even if the specific form (2.27) had not been introduced. Unless there are reasons (unknown) for preferring $g=1$ theory, a term (2.27) is not to be excluded. As we shall see later, such a q term apparently does not lead to any more divergent a form of electromagnetic interaction than does the γ term itself.

The factor $\frac{1}{4}$ has been introduced before Q in H_1 in order to make our normalization of the quadrupole moment strength conform to that conventionalized by Ramsey.[10] Consider the meson to have its spin along the positive z axis, and also take as a very weak electric field

$$E_1=-(\tfrac{1}{2}k)x\,,\quad E_2=-(\tfrac{1}{2}k)y\,,\quad E_3=kz\,,$$

where k is a small constant. For a meson with spin up,

$$\psi=1/\sqrt{2}\begin{pmatrix}1\\ i\\ 0\end{pmatrix},$$

so that we write

$$\left\langle\uparrow\left|\frac{-e}{4}QQ_{ij}\frac{\partial E_i}{\partial X_j}\right|\uparrow\right\rangle=\frac{-eQ}{4}k.$$

Ramsey defines the energy E of an electric-quadrupole moment q as

$$E=-\tfrac{1}{4}q(\partial E_3/\partial z)_{z=0}$$

for particles with spin along the positive z axis. The quadrupole moment is usually divided by the charge and given in units cm^2, and so the vector meson has quadrupole moment $Q=-(g-1+q)(\hbar/mc)^2\ \mathrm{cm}^2$. If we consider the spin projection along the z axis to be 0, then we have

$$\psi=\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}$$

and

$$\langle S_3=0|-\frac{eQ}{4}\frac{\partial E_i}{\partial X_j}Q_{ij}|S_3=0\rangle=\frac{eQ}{2}k\,,$$

to give $-2Q$, in agreement with the group theoretical result

$$Q(m)=Q[3m^2-S(S+1)]/S(2S-1)\,,$$

where S is the particle spin and m the projection of the spin along the z axis. The charge distribution can be considered as having the shape of an ellipsoid of revolution centered at the origin, and thus $Q=\frac{4}{5}\eta R^2$, where $\eta=(C^2-a^2)/(C^2+a^2)$, $R=\frac{1}{2}(a^2+C^2)$ is the mean square radius, C is the axis of the ellipsoid in the z direction, and a is the axis perpendicular to the z direction. A positive quadrupole moment corresponds to a cigar-shaped charge distribution, and a negative quadrupole moment corresponds to a pancake-shaped charge distribution.

For $g=1$, $q=0$, our result (2.37) reduces to that obtained by Case.[11]

E. Classical Spin Equations of Motion

In the preceding section we noted that spinning particles of the same gyromagnetic ratio have (except for the Darwin term) the same Hamiltonian, at least to order $1/m^2$. This suggests the possibility of a classical spin-independent description of the magnetic-moment precession. Bargmann, Michel, and Telegdi[12] have recently given such a description, using a four-vector s_μ for the spin or magnetic moment. In quantum mechanics the spin has, however, more often been described as part of the angular momentum antisymmetric tensor $S_{\mu\nu}$. We will here derive covariant classical equations of motion in terms of the more familiar $S_{\mu\nu}$. While the equations (2.40) we obtain are apparently quite different from the equations (2.42) obtained by Bargmann, Michel, and Telegdi, the two sets of equations are actually the same when s_μ and $S_{\mu\nu}$ are related as they have to be. This will show then that covariant spin-precession equations equivalent to those of Bargmann, Michel, and Telegdi can be derived from classical invariance arguments by using the more familiar $S_{\mu\nu}$ formulation for the spin angular momentum.

We wish to generalize to an arbitrary Lorentz frame the equation of spin precession

$$d\mathbf{s}/dt=(eg/2m)\mathbf{s}\times\mathbf{H}\,,\qquad(2.38)$$

which holds in a rest frame, by using an antisymmetric tensor $S_{\mu\nu}$. The tensor $S_{\mu\nu}$ must have only three independent components, which in a rest frame are s_1, s_2, s_3. This condition is expressed covariantly by the constraint

$$S_{\mu\nu}u_\nu=0\,,\qquad(2.39)$$

where u_ν is the four-velocity ($u^2=-1$). It is readily confirmed that the unique expression for the time variation of $S_{\mu\nu}$ consistent with the particle equation of motion $du_\mu/d\tau=(e/m)F_{\mu\nu}u_\nu$ and reducing to the form

[10] N. F. Ramsey, *Nuclear Moments* (John Wiley & Sons, Inc., New York, 1953).

[11] K. M. Case, Phys. Rev. **95**, 1323 (1954).

[12] V. Bargmann, L. Michel, and V. L. Telegdi, Phys. Rev. Letters **2**, 435 (1959).

(2.38) in a rest frame is[13]

$$dS_{\mu\nu}/d\tau = -(eg/2m)[S_{\mu\alpha}F_{\alpha\nu} - S_{\nu\alpha}F_{\alpha\mu}] - [e(g-2)/2m][u_\mu S_{\beta\nu} - u_\nu S_{\beta\mu}]F_{\beta\alpha}u_\alpha. \quad (2.40)$$

Here τ is the eigentime.

Define a four-vector s_α by the relation

$$s_\alpha = -(i/2)\epsilon_{\alpha\mu\nu\beta}S_{\mu\nu}u_\beta, \quad (2.41)$$

which then also satisfies a supplementary condition

$$s_\mu u_\mu = 0.$$

The time variation of s_α can be obtained from Eqs. (2.40) and (2.21):

$$\begin{aligned} ds_\alpha/d\tau &= -(i/2)\epsilon_{\alpha\mu\nu\beta}[\dot{u}_\beta S_{\mu\nu} + u_\beta \dot{S}_{\mu\nu}] \\ &= (ie/4m)\epsilon_{\alpha\mu\nu\beta}\{gu_\beta(S_{\mu\lambda}F_{\lambda\nu} - S_{\nu\lambda}F_{\lambda\mu}) \\ &\quad + (g-2)u_\lambda F_{\rho\lambda}u_\beta[u_\mu S_{\rho\nu} - u_\nu S_{\rho\mu}]\} \\ &\quad - (ie/2m)\epsilon_{\alpha\mu\nu\beta}S_{\mu\nu}F_{\beta\lambda}u_\lambda, \end{aligned}$$

where $\dot{A} \equiv dA/d\tau$. Now use the two relations

$$S_{\mu\nu} = i\epsilon_{\mu\nu\alpha\beta}u_\alpha s_\beta,$$

$$\epsilon_{\mu\alpha\beta\nu}\epsilon_{\mu\lambda\rho\sigma} = [\delta_{\alpha\lambda}\delta_{\beta\rho}\delta_{\sigma\nu} - \delta_{\alpha\lambda}\delta_{\beta\sigma}\delta_{\rho\nu} + \delta_{\alpha\rho}\delta_{\nu\lambda}\delta_{\sigma\beta} - \delta_{\alpha\rho}\delta_{\beta\lambda}\delta_{\sigma\nu} + \delta_{\alpha\sigma}\delta_{\beta\lambda}\delta_{\nu\rho} - \delta_{\alpha\sigma}\delta_{\beta\rho}\delta_{\lambda\nu}]$$

to obtain

$$ds_\alpha/d\tau = (e/m)[(\tfrac{1}{2}g)F_{\alpha\nu}s_\nu - (\tfrac{1}{2}g-1)s_\nu F_{\nu\mu}u_\mu u_\alpha]. \quad (2.42)$$

This is the result obtained by Bargmann, Michel, and Telegdi.[12]

We now show, in particular, that Eqs. (2.40) and (2.42) both lead to the same coupling (spin-orbit coupling) between spin and momentum in an electric field, to order $1/m^2$. For this purpose we express both equations in three-vector form and keep terms linear in the velocity $\mathbf{v}$. From Eq. (2.40) we have

$$\begin{aligned} d\mathbf{s}/dt &= -(eg/2m)[-\mathbf{s}\times\mathbf{H} + (\mathbf{s}\times\mathbf{v})\times\mathbf{E}] \\ &\quad - [e(g-2)/2m](\mathbf{s}(\mathbf{v}\cdot\mathbf{E}) - \mathbf{E}(\mathbf{s}\cdot\mathbf{v})] \\ &= (eg/2m)\mathbf{s}\times\mathbf{H} + [e(g-2)/2m]\mathbf{s} \\ &\quad \times(\mathbf{E}\times\mathbf{v}) + (e/m)\mathbf{E}\times(\mathbf{s}\times\mathbf{v}), \end{aligned}$$

but

$$\mathbf{E}\times(\mathbf{s}\times\mathbf{v}) = \tfrac{1}{2}\mathbf{s}\times(\mathbf{E}\times\mathbf{v}) + (m/2e)d\mathbf{v}'/dt,$$

where $\mathbf{v}' = \mathbf{s}v^2 - \mathbf{v}(\mathbf{s}\cdot\mathbf{v})$; and we have used $d\mathbf{v}/dt = (e/m)\mathbf{E}$, so that we write $d\mathbf{s}/dt = (eg/2m)\mathbf{s}\times\mathbf{H} + [e(g-1)/2m]\mathbf{s}\times(\mathbf{E}\times\mathbf{v}) + (m/2e)d\mathbf{v}'/dt$ to terms linear in $\mathbf{v}$. Now consider the case in which the spin changes slowly compared with the velocity, and the velocity periodically takes on the same values, so that we can drop the last term. The spin precession result to order m^{-2} then becomes

$$d\mathbf{s}/dt = (eg/2m)\mathbf{s}\times\mathbf{H} + [e(g-1)/2m^2]\mathbf{s}\times(\mathbf{E}\times\mathbf{p}) \quad (2.43)$$

for particles with a positive charge. Equation (2.42) expressed in the same way becomes

$$\begin{aligned} d\mathbf{s}/dt &= (e/m)[(g/2)\mathbf{s}\times\mathbf{H} + (g/2)\mathbf{E}(\mathbf{s}\cdot\mathbf{v}) \\ &\quad - (g/2-1)\mathbf{v}(\mathbf{s}\cdot\mathbf{E})] \\ &= (eg/2m)\mathbf{s}\times\mathbf{H} + [e(g-1)/2m]\mathbf{s}\times(\mathbf{E}\times\mathbf{v}) \\ &\quad + (m/2e)d\mathbf{v}''/dt, \end{aligned}$$

where $\mathbf{v}'' = +\mathbf{v}(\mathbf{s}\cdot\mathbf{v})$. Thus, by dropping the last term in exactly the same way as we arrived at Eq. (2.43), we obtain the same result. It is easily shown that (2.43) is identical with the result obtained from the Hamiltonian Eq. (2.37) through the relation $d\mathbf{s}/dt = i[H,\mathbf{s}]$.

III. APPLICATION TO DECAY: $\mu^\pm \to e^\pm + \gamma$

A. ($\mu \to e\gamma$) Matrix Element

The Feynman diagrams for the process $\mu \to e+\gamma$ are given in Fig. 1; the matrix element for the process $\mu \to e$ with emission of a real or virtual photon is given by the expression[14]

$$\mathfrak{M} = ie\bar{u}_e(1-\gamma_5)\Gamma_\nu u_\mu A_\nu, \quad (3.1)$$

where u_e, u_μ are the electron and muon spinors, respectively, and

$$\Gamma_\mu = -i(2\pi)^{-3}\{if_0(\gamma_\mu k_\nu - \gamma_\nu k_\mu)k^{-2} + f_1\sigma_{\mu\nu}/\mu\}k_\nu.$$

Thus,

$$i\Gamma_\mu A_\mu = (2\pi)^{-3}\{f_0\gamma_\mu j_\mu^{\text{ext}}/k^2 + (f_1/2\mu)\sigma_{\mu\nu}F_{\mu\nu}\}. \quad (3.2)$$

Here k is the photon momentum, μ the muon mass, and

$$F_{\mu\nu} = i(k_\nu A_\mu - k_\mu A_\nu),$$

$$j_\mu^{\text{ext}} = ik_\nu F_{\mu\nu}.$$

The form factors f_0 and f_1, which are functions of k^2, are responsible for monopole radiation (in the Coulomb field of a nucleus) and dipole radiation, respectively. The rate for $\mu \to e+\gamma$ with emission of a real photon is proportional to $|f_1(0)|^2$, and the rate for the coherent process μ+nucleus $\to e$+nucleus is proportional to $[f_0(\mu^2)+f_1(\mu^2)]^2$.

B. Branching Ratio $\omega_{\mu\to e+\gamma}/\omega_{\mu\to e+\nu+\bar{\nu}}$

If the $\mu \to e$ conversion proceeds through $\mu \to \nu+B$ and $\nu+B \to e$, then the branching ratio between the unobserved decay $\mu \to e+\gamma$ and the normal decay can be written as

$$\rho = \omega_{\mu\to e+\gamma}/\omega_{\mu\to e+\nu+\bar{\nu}} = (3\alpha/8\pi)N^2, \quad (3.3)$$

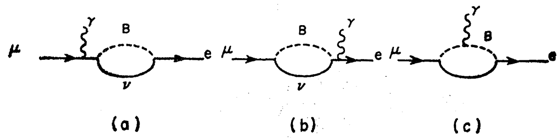

FIG. 1. Diagrams for decay $\mu \to e+\gamma$.

[13] Mrs. H. Hartmann, Lawrence Radiation Laboratory, Berkeley (private communication). The authors are also indebted to Mrs. Hartmann for an independent calculation of N in Eqs. (3.4).

[14] S. Weinberg and G. Feinberg, Phys. Rev. Letters 3, 111 (1959).

where α is the fine-structure constant, and N is a number independent of the weak-coupling constant. The amplitude N generally diverges logarithmically with Λ/m, the ratio of cutoff to the B-meson mass. Feinberg[2] and Gell-Mann[3] found (tacitly assuming unit magnetic moment for the vector meson), for $\Lambda\approx$ nucleon mass, and $m\approx K$-meson mass, $N\approx 1$. This value for N gives $\rho\approx 10^{-3}$, which is 10^3 times the experimentally measured upper limit for ρ.[4]

Aside from the mild cutoff dependence, there are two reasons in a one-neutrino theory as to why the above-calculated ρ need not be taken as evidence against the B meson. We have already pointed out that there is an infinity of free-particle B-meson Lagrangians which differ in their definition of "normal" magnetic moment. Also, if the B meson exists it must have a large mass (greater than the K-meson mass), and yet the gauge-invariance type of argument for its presence[15] indicates that it should have a vanishing mass. This implies that the B meson must have a rather complicated structure, so that one should keep an open mind with regard to its electromagnetic properties.

We have recalculated the $\mu e\gamma$ vertex as a function of magnetic moment $(1+\gamma)e\hbar/2mc$ and electric quadrupole moment $Q=-(\gamma+q)\cdot(\hbar/mc)^2$, with the interaction Lagrangian given by Eq. (2.28). After a lengthy calculation, the value of N obtained[13] is

$$\begin{aligned}N=&(1-\gamma-q\mu^2/8m^2)I_0'+(1+2\gamma+q\mu^2/4m^2)I_1'\\&+(3-\gamma\mu^2/2m^2+11\mu^2/6m^2)I_2'\\&+(22/3+4\gamma)(\mu^2/m^2)I_3'+10\mu^2/m^2I_4',\end{aligned}\tag{3.4}$$

where

$$I_n'=+\left(\frac{im^{2n}}{\pi^2}\right)\int\left(\frac{d^4q}{q^2-m^2}\right)^{n+2}.$$

This result is correct to order μ^2/m^2, terms of order $(\mu/m)^4$ have been dropped, and the electron mass has been set equal to zero. The expression (3.4) for N is consistent with that obtained by Meyer and Salzman[16] and by Ebel and Ernst,[17] who, however, did not calculate terms in μ^2/m^2 or q. Because q was originally defined divided by the square of the boson mass m^2, and the muon mass is the only other quantity of dimensions of mass in our calculation, q always appears in N multiplied by μ^2/m^2.

C. Discussion of N

In our calculation of N, γ and q appear only in the combination

$$\gamma'=\gamma+q\mu^2/8m^2=(g-1)(1-\mu^2/8m^2)-Q\mu^2/8.\tag{3.5}$$

This means that the rate for $\mu\rightarrow e+\gamma$ depends only on this combination of moments. This result is apparently fortuitous, since in the monopole form factor f_0 this particular combination does not occur.[13]

1. Finite N

The integral I_0' is logarithmically divergent so that, except for $\gamma'=1$, N is formally divergent. Since we have

$$I_n'=(-)^n/n(n+1),\tag{3.6}$$

for $\gamma'=1$, we obtain

$$N=1+2\mu^2/9m^2,\tag{3.7}$$

which for any value of the boson mass leads to a branching ratio $\rho>10^{-3}$. The cutoff-independent calculation of N is, thus, in definite disagreement with experiment.

2. Logarithmically Divergent N

N can be made vanishingly small by retaining the integral I_0', making it finite by the formal device of a covariant cutoff Λm. Consistency then requires that all integrals I_n be calculated with the same kind of cutoff. With the Feynman cutoff factor $-\Lambda^2m^2/(q^2-\Lambda^2m^2)$ we obtain the integrals

$$I_n=\left(\frac{-im^{2n}}{\pi^2}\right)\int\left[\left(\frac{d^4q}{q^2-m^2}\right)^{n+2}\right]\times[\Lambda^2m^2/(q^2-\Lambda^2m^2)],\tag{3.8}$$

or

$$I_0=[\Lambda^2/(1-\Lambda^2)^2][1-\Lambda^2+\Lambda^2\ln\Lambda^2]$$

and

$$I_n=(-1)^{n+1}\Lambda^2/n(n+1)(1-\Lambda^2)-[1/(1-\Lambda^2)]I_{n-1},\quad\text{for}\quad n\geqslant 1.$$

By defining γ_0' as that value of γ which makes N vanish we find

$$\gamma_0'=A+B\epsilon,\tag{3.9}$$

where

$$A=(I_0+I_1+3I_2)/(I_0-2I_1),$$

$$\begin{aligned}B=&(I_0-2I_1)^{-1}[(11/6)I_2+(22/3)I_3+10I_4]\\&-(\tfrac{1}{2}I_2-4I_3)(I_0+I_1+3I_2)(I_0-2I_1)^{-2},\end{aligned}$$

and $\epsilon=(\mu/m)^2\ll 1$; in fact, we expect the upper limit for ϵ to be 1/25, since m must be greater than the K-meson mass. For two representative values of Λ, say, $\Lambda=1$, $\Lambda=2$, we have

	I_0	I_1	I_2	I_3	I_4	A	B
$\Lambda=1$	0.5000	−0.167	0.084	−0.050	−0.033	0.700	−0.91
$\Lambda=2$	1.13	−0.296	0.125	−0.070	−0.044	0.702	−0.67

This table shows that γ_0' is insensitive to both the cutoff Λ and the square of the ratio of the masses ϵ (as long as ϵ is small), With $\epsilon=1/25$, then for $\Lambda=1$, $\gamma_0'=0.698$ and for $\Lambda=2$, $\gamma_0'=0.703$. In the expression (3.4) for N, it is evident that we can write

$$N=R(1-\gamma'/\gamma_0'),$$

[15] S. A. Bludman, Nuovo Cimento **9**, 442 (1958).
[16] P. Meyer and G. Salzman, Nuovo Cimento **14**, 1310 (1959).
[17] M. E. Ebel and F. J. Ernst, Nuovo Cimento **15**, 173 (1960).

where

$$R=I_0+I_1+3I_2+\epsilon[(11/6)I_2+(22/3)I_3+10I_4].$$

The term proportional to ϵ in R will always be small in comparison with the other terms, so that in R we can neglect ϵ to obtain

$$R=[\Lambda^2/2(1-\Lambda^2)^4]\times[2\Lambda^2(\Lambda^4-\Lambda^2+3)\ln\Lambda^2+(1-\Lambda^2)(2\Lambda^4+\Lambda^2+3)].$$

The branching ratio ρ then becomes

$$\rho=(3\alpha/8\pi)R^2(1-\gamma'/\gamma_0')^2$$

The quantity $3\alpha/8\pi R^2$ has been plotted by Ebel and Ernst, and varies from 10^{-4} to 10^{-12} as Λ varies from 1 to 10.

The branching ratio ρ, when it does not vanish (i.e., for $\gamma'\neq\gamma_0'$), is sensitive to the value of Λ. The combination of γ and q necessary to forbid the $\mu\rightarrow e+\gamma$ decay is thus, certainly *ad hoc*. On the other hand, we know of no criterion for fixing on a choice of γ and q *a priori*.

Now only one combination of the two parameters γ and q is involved in choosing γ' to forbid the process $\mu\rightarrow e+\gamma$. Another different combination of γ and q will determine the rate of the coherent process $\mu+$nucleus $\rightarrow e+$nucleus. In other words, we expect to be able to choose γ and q so that $f_1(0)^2$ and $[f_1(\mu^2)+f_0(\mu^2)]^2$ are both small enough not to exclude the vector-meson hypothesis.

D. Two-Neutrino Hypothesis

Another explanation for the absence of $\mu\rightarrow e$ conversion consists in the assumption[18] that two different neutrinos ν and ν' are involved in μ decay, ν being coupled to the electron, and ν' to the muon. Since these neutrinos are different, they are not capable of annihilating each other, and thus any $\mu\rightarrow e$ processes are strictly forbidden. The implications of this alternative are not pursued here.

Note added in proof. Since this paper was written, the experiment of G. Danby, J-M. Gaillard, K. Goulianos *et. al*, Phys. Rev. Letters **9**, 36 (1962) has established that ν and ν' are different particles, or at least different states of one four-component neutrino, incapable of annihilating each other. A method of calculation for vector meson electrodynamics has also been proposed [T. D. Lee and C. N. Yang, Phys. Rev. **128**, 885 (1962)] which yields finite results for some heretofore divergent quantities. For $\nu=\nu'$ and $q=0$, Lee obtains $N=(\gamma-1)\ln\alpha\gamma^2$, i.e., the divergent logarithm $\ln\Lambda^2$ is replaced by the finity quantity $\ln\alpha\gamma^2$. See T. D. Lee, Phys. Rev. **128**, 899 (1962).

IV. CONCLUSION

We have shown that the various charged vector-meson formalisms are equivalent and describe in the general case a particle of arbitrary magnetic-dipole and electric-quadrupole moments. The quadrupole-moment interaction is no more divergent than an anomalous magnetic-moment interaction. Indeed, when, to the normal interaction, an anomalous moment $\gamma e\hbar/2mc$ is added, this itself introduces a quadrupole moment $\gamma(\hbar/mc)^2$.

A first-order Stückelberg formalism has been developed in order to ensure internal consistency between the subsidiary condition and the other equations of motion in the presence of electromagnetic interaction. The nonrelativistic equations of motion of a spin-1-particle of arbitrary magnetic moment, like those of a spin-$\frac{1}{2}$ particle, agree with the classical equations of motion derived on invariance grounds.

Because of the absence of criteria fixing its magnetic-dipole and electric-quadrupole moments, the electromagnetic interactions of charged vector mesons is ambiguous enough that the absence of $\mu\rightarrow e$ conversion processes cannot, by themselves, be a proof of the nonexistence of intermediary mesons in the weak interactions.

[18] J. Schwinger, Ann. Phys. **2**, 407 (1957); S. A. Bludman, Bull. Am. Phys. Soc. **4**, 80 (1959); B. Pontecorvo, Zh. Eksperim. i Teor. Fiz. **37**, 1751 (1959) [translation: Soviet Phys.—JETP **10**, 1236 (1960)]; K. Nishijima, Phys. Rev. **108**, 907 (1957).