We are then left with

$$\Delta S-\int\lambda dt$$
$$=\int dt \lim_{\tau\to 0,\mu\to 0}\frac{d}{d\tau}\frac{d}{d\mu}S(f_0+\mu g+\tau(L-D)f_0). \quad \text{(A9)}$$

We now introduce φ such that $g=f_0\varphi$, and X such that for arbitrary φ

$$(\varphi,X)\equiv\int dt \lim_{\tau\to 0,\mu\to 0}\frac{d}{d\tau}\frac{d}{d\mu}$$
$$\times S(f_0+\mu f_0\varphi+\tau(L-D)f_0), \quad \text{(A10)}$$

as well as $\mathfrak{R}$ such that

$$(\varphi\mathfrak{R}\varphi')=\int dt \lim_{\tau\to 0,\mu\to 0}\frac{d}{d\tau}\frac{d}{d\mu}$$
$$\times S(f_0+\mu f_0\varphi+\tau(L-D)f_0\varphi'). \quad \text{(A11)}$$

With these definitions of X and $\mathfrak{R}$, (III.14) and other formulas following it remain valid. In (III.18), we use for $\mathfrak{X}$

$$(\varphi\mathfrak{X}\varphi)=\lim_{\tau\to 0,\mu\to 0}\frac{1}{2}\frac{d^2}{d\mu^2}\frac{d}{d\tau}$$
$$\times S(f_0+\mu f_0\varphi+\tau(L-D')f_0). \quad \text{(A12)}$$

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Broken Symmetries and Massless Particles*

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The following generalization of a theorem conjectured by Goldstone is proven: In a theory admitting a continuous group of transformations, suppose a set of operators $\phi_i(x)$, transforming under an irreducible representation of the group, has the property that in the vacuum some expectation values $\langle\phi_i(x)\rangle\neq 0$ for $i=i'$. The theorem then asserts that $\tilde{D}_{ij}(p)$, the Fourier transform of the propagator of $\phi_i(x)$, is singular at $p^2=0$ for some $i\neq i'$. (The maximum number of $\langle\phi_i\rangle\neq 0$ is a property of the group representation. The further identification of the singularities as poles and their interpretation as massless particles depends on the usual apparatus of quantum field theory.)

The appropriate choice to be made for the field ϕ_i when it describes a boson excitation and when the Lagrangian contains only direct fermion-fermion coupling is discussed. It is suggested that such Fermi interaction theories may be renormalizable when expanded in terms of the coupling between fermions and the collective boson field.

The theorem is illustrated by the following models: (A) γ_5 gauge group (Nambu and Jona-Lasinio), where a massless pseudoscalar meson is predicted; (B) isospin group (Nambu and Jona-Lasinio) where massless charged mesons are predicted; (C) $SU(3)$ octet model (Baker and Glashow) where six or four massless mesons are predicted; (D) Lorentz group (Bjorken) where the massless photon is predicted. The limitations of the theorem are also discussed.

I. INTRODUCTION

A WIDESPREAD feature of many-body systems is the existence of collective modes of excitation for which the energy vanishes in the long-wavelength limit, these modes constituting the only low-energy excitations. Well-known examples are the spin waves in the Heisenberg model of ferromagnetism,[1] the phonons of superfluid helium,[2] and the phonons which presumably would be exhibited by a superconductor in the absence of Coulomb interactions.[3] The common feature of these systems is the appearance of a condensation or cooperative phenomenon; the theoretical description then requires, or at least is facilitated by, the introduction of a degenerate or symmetry-breaking ground state.

Thus, in ferromagnetism, where the Hamiltonian is invariant under spatial rotations, the ground state has a macroscopic spin proportional to the size of the system. For an infinite system, at least, the ground state is then not rotationally invariant. In this sense, the ground state is nonsymmetric or degenerate; the spin points in some direction which, because of the symmetry of the Hamiltonian, is arbitrary. Each particular choice of spin direction, however, defines a representation of the Hilbert space inequivalent to all the other possible choices.

The BCS model of superconductivity[4] can be formu-

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[1] F. Bloch, Z. Physik **61**, 206 (1930); **74**, 295 (1932).

[2] L. D. Landau and E. M. Lifschitz, *Statistical Physics*, (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1958), p. 198.

[3] P. W. Anderson, Phys. Rev. **110**, 827 (1958); **112**, 1900 (1958); G. Rickayzen, *ibid.* **115**, 195 (1959).

[4] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957). N. N. Bogoliubov, Zh. Eksperim. i Teor. Fiz. **34**, 58, 73 (1958) [translation: Soviet Phys.—JETP **7**, 41, 51 (1958)].

lated analogously.[5] Here Nambu views the conservation of charge or of fermion number as the invariance of the Hamiltonian under rotations about the 3 axis of a fictitious isospin space. The ground state is, however, characterized by a nonvanishing value of the isospin density in the 1,2 plane; this nonvanishing expectation value determines the energy gap. The invariance under rotations about the 3 axis is reflected by the arbitrariness in the choice of direction in the 1,2 plane for this nonvanishing component. The ground state is again degenerate and the symmetry broken in a sense precisely analogous to that explained for the ferromagnet. The superfluid boson system can be described similarly.

Nambu and Jona-Lasinio[6] have tried to extend these concepts to Lorentz covariant field theories and to found a theory of strong interactions on the assumption that the nucleon mass arises in analogy with the energy gap of superconductivity, the associated low-energy collective excitations being identified with the pion. In correspondence with the nonrelativistic examples, Nambu and Jona-Lasinio find that the pion energy should vanish with the momentum, i.e., that the pions should be massless.

The models of Nambu and Jona-Lasinio involve basic four-fermion interactions. The boson excitations then appear as collective modes of the fermion system. Goldstone[7] has, on the other hand, examined theories involving bosons as elementary fields. These elementary bosons transform by an irreducible representation of a continuous transformation group leaving the Lagrangian invariant. From these models, Goldstone conjectures that whenever the Lagrangian admits a continuous symmetry group, but the vacuum expectation value of some boson field is nonvanishing, some zero-mass boson states must exist. Goldstone, Salam, and Weinberg and, independently, Taylor,[8] then presented several proofs of Goldstone's conjecture.

The primary purpose of this note is to prove a version of Goldstone's theorem generalized to Lagrangians admitting any continuous symmetry group and containing or not containing elementary boson fields. A secondary purpose is to apply the theorem, not only to the cases already cited, but also to more recent work[9–11] extending Nambu and Jona-Lasinio's program of broken symmetries. Our method is closest in spirit to that of Bjorken.[11]

In Sec. II, we prove the generalized Goldstone theorem, defining the conditions under which zero-mass excitations are predicted and relating the number of independent zero-mass excitations to the structure of the symmetry group. In Sec. III we apply the theorem to a number of models chosen to illustrate its wide scope. Most of these models have already been discussed in the literature.[5–11] A brief discussion of the limitations of our derivation of the theorem and of the theorem's physical implications concludes the paper.

II. FORMULATION AND PROOF OF THEOREM

A. Preliminaries

We consider a theory, defined by a set of field equations (or by a Lagrangian) and by an appropriate operator algebra, invariant under some continuous group of transformations. Let the set of operators ϕ_i transform according to an irreducible representation of the group and be described by an equation of the form

$$D_0^{-1}\phi_i = j_i\,, \tag{2.1}$$

where j_i is a current, constructed from the fundamental operators and transforming as ϕ_i.

For example, ϕ_i might be a fundamental spinless bose field of "bare mass" μ_0,

$$D_0^{-1} = -\partial^2 + \mu_0^2\,, \tag{2.2}$$

or designating the Fourier transform of D_0^{-1} by $\tilde{D}_0^{-1}$,

$$\tilde{D}_0^{-1}(p) = p^2 + \mu_0^2\,. \tag{2.3}$$

We shall also be concerned with examples where ϕ_i represents a Dirac particle or a vector boson and the D_0^{-1} are the associated well-known operators.

On the other hand, ϕ_i might not be an elementary field, but a synthetic object formed from other fields. For example, our theorem includes four-fermion theories of the Heisenberg[12] type. In this case $D_0 = F$ is simply a Fermi coupling constant with dimensions (mass)$^{-2}$. Our aim in every case is to define as "field" that object which classically describes the transmission of forces and quantum mechanically the propagation of particles.

We now assume that in the ground state (vacuum) of the system some components $i = i'$ of ϕ_i have nonvanishing expectation values,

$$\langle\phi_i\rangle \neq 0. \tag{2.4}$$

From Eq. (2.1) we have the conditions

$$\tilde{D}_0^{-1}(0)\langle\phi_i\rangle = \langle j_i\rangle\,, \tag{2.5}$$

to be fulfilled nontrivially. We refer to Eqs. (2.5) as the generalized Hartree conditions. These conditions bring new physical parameters, the $\langle\phi_i\rangle$, into the theory and

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[6] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961); **124**, 246 (1961). See also V. G. Vaks and A. I. Larkin, *Proceedings of the 1960 International Conference on High Energy Physics at Rochester*, edited by E. C. G. Sudarshan, J. H. Tinlot, and A. C. Melissions, (Interscience Publishers, Inc., New York, 1960), p. 871.

[7] J. Goldstone, Nuovo Cimento **19**, 154 (1961).

[8] J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. **127**, 965 (1962). J. C. Taylor, *Proceedings of the 1962 International Conference on High-Energy Physics at CERN* (CERN, Geneva, 1962), p. 670.

[9] M. Baker and S. L. Glashow, Phys. Rev. **128**, 2462 (1962).

[10] S. L. Glashow, Phys. Rev. **130**, 2132 (1963).

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distinguish theories of this kind from more conventional field theories.

It would appear at first sight that we must exclude the case $\tilde{D}_0^{-1}(0)=0$. If, however, $\langle j_i\rangle=0$ but $\langle\phi_i\rangle$ remains finite, the reasoning given below will remain valid providing the limit $\tilde{D}_0^{-1}(0)\to 0$ is sufficiently regular. We shall return to a discussion of this problem at the end of Sec. III.

Our approach is to calculate the Green's functions defined by the response of the quantum system to external sources J_i. We therefore modify Eq. (2.1) to

$$D_0^{-1}\phi_i=j_i+J_i. \tag{2.6}$$

This formalism is particularly suggestive here where we think of the vacuum as the ground state of a system (like a ferromagnet) which, because of some kind of dynamical instability, remains unsymmetric even in the limit of vanishing J_i. We know that, whether obtained by a summation of perturbation theory or directly from the action principle, these Green's functions are functionally related through the Dyson-Schwinger equations.

The theorem will be a statement about the propagator $D_{ij}(x-y)$ defined as the functional derivative

$$D_{ij}(x-y)=\delta\langle\phi_i(x)\rangle/\delta J_j(y)\,|_{J=0}. \tag{2.7}$$

For its Fourier transform, we obtain from (2.6) the well-known form

$$[\tilde{D}_0^{-1}(p)-\tilde{\pi}(p)]_{ik}\tilde{D}_{kj}(p)=\delta_{ij}, \tag{2.8}$$

where

$$\tilde{\pi}_{ik}(p)=\int e^{ip\cdot z}[\delta\langle j_i(0)\rangle/\delta\langle\phi_k(z)\rangle]_{J=0}d^4z. \tag{2.9}$$

In the limit $p=0$, this simplifies to

$$\tilde{\pi}_{ik}(0)=\partial\langle j_i\rangle/\partial\varphi_k\,|_{J=0}, \tag{2.10}$$

where we have defined $\langle\phi_i\rangle\equiv\varphi_i$. For this case, we thus have

$$\begin{aligned}[\tilde{D}^{-1}(0)]_{ik}&=[\tilde{D}_0^{-1}(0)-\tilde{\pi}(0)]_{ik}\\&=[\tilde{D}_0^{-1}(0)]_{ik}-\partial\langle j_i\rangle/\partial\varphi_k\,|_{J=0}.\end{aligned} \tag{2.11}$$

The theorem can finally be stated as an assertion about $\tilde{D}^{-1}(0)$. Let the representation ϕ_i be of dimension N and let there be N' nonvanishing Hartree conditions. Then we shall prove that $\tilde{D}_{ij}^{-1}(0)=0$ for some i or $j\neq i'$. (The Hartree conditions define an N'-dimensional subspace in the N-dimensional representation space. The maximum value $N'_{\max}\equiv\nu$ that N' may have depends on the group representation. Until subsection 3 we will suppose $N'=N'_{\max}$. Of course, how large N' is, i.e., how much the symmetry is broken in the vacuum, depends upon the dynamics.)

The propagator $\tilde{D}_{ij}(0)$ is singular for some M' directions in the subspace normal to the N'-dimensional space defined above. Thus, the number of massless mesons $M'\leq N-N'$ depends on both the group representation and the dynamics. The proof itself will provide the means for counting M' in individual cases.

B. Proof

The Hartree conditions (2.5) represent a set of numerical relations covariant under the group representation. We consider the effect of an infinitesimal transformation

$$\bar{\delta}\varphi_i=X_{ij}{}^{\alpha}\lambda_\alpha\varphi_j, \tag{2.12}$$

where the set φ_j is the solution of (2.5) in a given coordinate system and the λ_α infinitesimal parameters of the group. From (2.5) we derive

$$\{[\tilde{D}_0^{-1}(0)]_{ik}-\bar{\delta}\langle j_i\rangle/\bar{\delta}\varphi_k\}\bar{\delta}\varphi_k=0, \tag{2.13}$$

where the use of the symbol $\bar{\delta}$ for differentiation of $\langle j_i\rangle$ (not to be confused with functional derivative) serves to remind the reader that this is the special value of the derivative for those changes which are group transformations. We refer to this derivative as the kinematical derivative.

We note that the quantity in curly brackets in (2.13) bears a strong resemblance to $[\tilde{D}_0^{-1}(0)]_{ik}$, Eq. (2.11). We have to examine the conditions under which the two expressions are indeed identical.

1. *A Simple Example*

We start with an example that serves to illustrate the type of result obtainable, by considering φ_i to transform under the fundamental representation of $O(N)$, the real orthogonal group in N dimensions. The general form for $\langle j_i\rangle$ in Eq. (2.5) is then simply

$$\langle j_i\rangle=\varphi_i\Theta(\varphi^2), \tag{2.14}$$

with Θ an invariant function of $\varphi^2=\varphi_i\varphi_i$, and the Hartree condition reduces to a single equation

$$\tilde{D}_0^{-1}(0)=\Theta, \tag{2.15}$$

assuming that φ_i has at least one nonvanishing component. Without loss of generality, we can adapt the coordinate system so that $\varphi_1\neq 0$ and $\varphi_i=0$, $i\neq 1$. From (2.14) we have immediately for the kinematical derivative,

$$\bar{\delta}\langle j_i\rangle/\bar{\delta}\varphi_k=\delta_{ik}\Theta, \tag{2.16}$$

because of the invariance of Θ.

To compare this with the dynamical derivative in (2.11) we note that, in the presence of a uniform external source J_i, the *form* (2.14) is still correct; we thus obtain

$$\partial\langle j_i\rangle/\partial\varphi_k=\delta_{ik}\Theta+2\varphi_i\varphi_k\Theta',\quad \Theta'\equiv\partial\Theta/\partial\varphi^2. \tag{2.17}$$

Setting $J_i=0$, which is the same as setting $\varphi_k=0$, $k\neq 1$, tells us that

$$\bar{\delta}\langle j_i\rangle/\bar{\delta}\varphi_k=\partial\langle j_i\rangle/\partial\varphi_k\,|_{J=0}=\delta_{ik}\Theta,\quad i,k\neq 1, \tag{2.18}$$

whereas for $i=1$

$$\partial j_1/\partial\varphi_1|_{J=0}=\Theta+2\varphi_1{}^2\Theta' \tag{2.19}$$

bears no simple relation to the kinematical derivative.

We note further that the $\bar{\delta}\varphi_k$ in (2.13) are only restricted by the condition of invariance for φ^2,

$$\varphi_k\bar{\delta}\varphi_k=0\,, \tag{2.20}$$

which is satisfied by $\bar{\delta}\varphi_1=0$, $\bar{\delta}\varphi_k$ arbitrary, $k\neq 1$. For the group chosen, no new restrictions on the $\bar{\delta}\varphi_k$ emerge from (2.12), since the number of parameters is, in $O(N)$ for $N\geq 2$, greater than or equal to the number of nonvanishing $\bar{\delta}\varphi_k$, once (2.20) is satisfied.

From (2.13) and (2.18), we thus conclude that

$$\tilde{D}_{ii}{}^{-1}(0)=0 \text{ (no sum on } i), \quad i=2,3\cdots N. \tag{2.21}$$

2. Generalization to More Complicated Group Representations

The defining representation of $O(N)$ is especially simple in that the Hartree condition defines a direction, which by operations of the group can be made to point along the 1 axis. In general, however, a group representation can, by operations of the group, only be brought to a canonical form in which ν of the $\langle\phi_i\rangle$ are nonvanishing. This number ν, the minimum number of components necessary to specify an N representation after suitable orientation of axes, which is also $N'_{\max}$, the maximum number of independent Hartree conditions that can be imposed, will be called the canonical number. The basis in which the N representation achieves "principal-axis form" will be called the canonical basis.

That generally $\nu>1$ can be pictured by regarding the N representation of the group in question as constrained to transform as some subgroup of $O(N)$ considered above. Equivalently, from an arbitrary vector $\varphi(i)$, we can construct ν independent algebraic invariants. These are of the form

$$\begin{aligned}I_2&=[i_1i_2]\varphi(i_1)\varphi(i_2)\,,\\ I_3&=[i_1i_2i_3]\varphi(i_1)\varphi(i_2)\varphi(i_3)\,,\\ &\vdots\\ I_{\nu+1}&=[i_1\cdots i_{\nu+1}]\varphi(i_1)\cdots\varphi(i_{\nu+1})\,,\end{aligned} \tag{2.22}$$

where $[i_1i_2]=\delta(i_1i_2)$ and $[i_1\cdots i_p]$ is a symmetric invariant numerical tensor under the group. For instance, for the adjoint or regular representation of $SU(n)$, $\nu=n-1$ and the invariants have been constructed explicitly.[13] Associated with each invariant I_p is a vector

$$V_p(i)=[ii_2\cdots i_p]\varphi(i_2)\cdots\varphi(i_p)\,. \tag{2.23}$$

The canonical number ν is, thus, the number of algebraically independent invariants that can be constructed from the N representation.

In addition to their form and number, we require but a single additional property of the invariants: These invariants are adapted to a choice of canonical basis such that when any one of the indices of $[i_1\cdots i_p]$ is outside the canonical subspace and the remaining indices are inside, the symbol vanishes.

Returning to the proof, the Hartree condition is now equivalent to at most ν coupled equations for the ν invariants $I_2,\cdots I_{\nu+1}$, or equivalently ν equations for a vector φ_i in its canonical coordinate system where $\varphi_i\neq\nu$, $i=1\cdots\nu$. We suppose a nontrivial solution to exist. Replacing Eq. (2.14), we have the form

$$\begin{aligned}\langle j_i\rangle=\varphi(i)\Theta_1+[ii_1i_2]\varphi(i_1)\varphi(i_2)\Theta_2+\cdots\\ +[ii_1\cdots i_\nu]\varphi(i_1)\cdots\varphi(i_\nu)\Theta_\nu\,,\end{aligned} \tag{2.24}$$

where

$$\Theta_i=\Theta_i(I_2,\cdots I_{\nu+1})\,. \tag{2.25}$$

The properties of the numerical tensors [ ] now yield, from (2.24), upon varying about the canonical coordinate system,

$$\bar{\delta}\langle j_i\rangle/\bar{\delta}\varphi_k=\partial\langle j_i\rangle/\partial\varphi_k|_{J=0}\,,\quad i,k=\nu+1,\cdots N. \tag{2.26}$$

i.e., the kinematical and dynamical derivatives are again equal in the subspace orthogonal to the first ν components, i.e., to the canonical subspace. Since these derivatives are also symmetric in their two indices, they can be taken as diagonal.

If the $\bar{\delta}\varphi_k$, $k=\nu+1,\cdots N$ can be chosen independently and nonzero, we shall then have proven that

$$\tilde{D}_{ii}{}^{-1}(0)=0\,,\quad i=\nu+1,\cdots N. \tag{2.27}$$

This will certainly be the case when $N-\nu$ is less than or equal to the number N_λ of parameters λ_α of Eq. (2.12), since then the only condition on $\bar{\delta}\varphi$, that it be orthogonal to the vectors V_p of Eq. (2.23) [as follows from (2.22)] is precisely satisfied by choosing $\bar{\delta}\varphi_k\neq 0$, $k=\nu+1,\cdots N$. These restrictions are satisfied by all the examples of Sec. III and in particular by the adjoint representation of $SU(n)$.

The maximum number of mesons $M'_{\max}=N-\nu$ never exceeds N_λ, the number of parameters. This is obviously so when N does not exceed N_λ, i.e., for the most interesting representations of small dimensionality. When $N>N_\lambda$, we have precisely $N-\nu=N_\lambda$ and our proof still applies. This is true because by definition $N-\nu$ is the number of components of φ_i that can be chosen without loss of generality to vanish. But given φ_i, in an arbitrary coordinate system, we can determine a canonical coordinate system by fixing N_λ of the components in the new coordinate system to vanish, thus fixing the N_λ parameters. This statement is simply a generalization of the familiar idea that in the higher representations of the rotation group, three components of an arbitrary tensor transforming under the representation serve merely to orient the geometrical form associated with the tensor with respect to an arbitrary (space-fixed) coordinate system.

[13] L. C. Biedenharn, Phys. Letters **3**, 69 (1962); A. Klein, J. Math. Phys. (to be published).

3. Less than Maximum Breaking of Symmetry

The theorem as proved above depends on the assumption that the symmetry is broken maximally. This is because from the structure of the group and the meaning of canonical basis, if $\varphi_j \neq 0$, $j=1,\cdots \nu$, then from (2.12) there exist λ_α and $X_{ij}{}^\alpha$ such that $\bar{\delta}\varphi_i \neq 0$, $j=\nu+1,\cdots N$. If one or more of the $\varphi_i=0$ so that some of the symmetry is preserved, the number of conditions (2.27) will be *reduced*, either because some of the $\bar{\delta}\varphi_i$ are now necessarily zero or because there are linear relations among them. Whereas the first possibility can be deduced from study of (2.12), the second comes about because the vectors V_p are no longer all linearly independent.

C. Physical Picture

A simple physical picture, extending that of Goldstone,[7] may be applied to the results of this section. Given a symmetric theory in the ground state of which, however, a constant field $\boldsymbol{\varphi}$ points in a certain direction, consider infinitesimal oscillations $\delta\boldsymbol{\varphi}=\boldsymbol{\phi}-\boldsymbol{\varphi}$ about $\boldsymbol{\varphi}$ which alter its direction but not its magnitude, i.e., such that $\delta\boldsymbol{\varphi}\cdot\boldsymbol{\varphi}=0$. (Such occur, of course, only if the symmetry is that of a continuous group.) The infinite wavelength ($p=0$) oscillations are then constant overall rotations of the system which, precisely because of the symmetry, do not alter the energy. Thus, the significance of the "massless bosons" is the vanishing of the excitation energies for $p=0$ for the modes of oscillation perpendicular to $\boldsymbol{\varphi}$. Whether such modes occur or not is a question of whether or not the system supports, in its state of minimum energy, a nonvanishing field expectation value.

The above picture also applies when the N representation φ_i is not simply an N vector in $O(N)$, i.e., when there are other invariants besides φ_i^2. The results are modified in detail only because not all directions are equivalent.

III. SPECIFIC EXAMPLES

In this section we illustrate the theorem proven by considering four examples of broken symmetries: (A) an elementary bose field transforming under the real orthogonal group $O(N)$. This is the case which suggested the general occurrence of massless bosons when the ground state is asymmetric.[7] (B) A Heisenberg-type theory invariant under $SU(n)$. The definition of the boson field operator when no boson occurs in the original Lagrangian is discussed. (C) The broken symmetry is that of space-time, and a massless photon is produced. (D) Finally we give a concise derivation of the origin of massless phonons in a superconductor with short-range interactions, and discuss the reason for the breakdown of the theorem in the presence of Coulomb interactions.

The cases (A)–(C) have been discussed in the literature.[6–11] Our contribution is to give a unified treatment specifically relating the number of massless particles predicted to the extent by which the symmetry is broken. We also discuss the question of renormalizability in a new light.

A. A Real Boson Field $O(n)$

Consider a theory containing an elementary Hermitian field operator ϕ_i. Since the defining representation has a unique invariant $\phi_i\phi_i$, the theorem asserts that if

$$\langle\phi_1\rangle\neq 0\,, \tag{3.1}$$

the Fourier transform of $\langle T(\phi_i(x)\phi_i(y))\rangle$ (no sum on i) is singular at $p^2=0$ for $i=2\cdots n$. This covers all the applications considered in detail by Goldstone[7] and by Goldstone *et al.*[8]

B. Heisenberg-Type Theories

1. Definition of Boson Field Variables

We now consider the case where ϕ_i is coupled to a fermion current density j_i, and ϕ_i and j_i each transform as an N-dimensional representation of $SU(n)$. If ϕ_i were an elementary field, the situation would be essentially that considered in the previous illustration. Instead we shall consider the case

$$f\phi_i(x)=Fj_i(x)\,, \tag{3.2}$$

where F is a dimensional coupling constant and f is a dimensionless number. Under these circumstances, where no boson appears in the original Lagrangian, how should the effective boson field be understood?

Suppose the Lagrangian is

$$\mathcal{L}(x)=-\bar{\psi}(x)(\gamma^\mu\partial_\mu+m_0)\psi(x)+\tfrac{1}{2}Fj_i(x)j_i(x)+Fj_i(x)J_i(x)\,, \tag{3.3}$$

where m_0 is a possible bare fermion mass,

$$j_i(x)=\tfrac{1}{2}[\bar{\psi}(x),T_i\psi(x)] \tag{3.4}$$

is (with T_i the coupling matrices in the N representation) the unitary current density, and $J_i(x)$ is an external source which is ultimately allowed to vanish.

With the conventional definition of the Green's function,

$$G(x,x')=i\langle T(\psi(x)\bar{\psi}(x'))\rangle\,, \tag{3.5}$$

and with the help of the formula, valid for an arbitrary operator $\langle O\rangle$ and for the Lagrangian (3.3),

$$\{F\langle j_i(x)\rangle-\delta/\delta J_i(x)\}\langle O\rangle=F\langle T(j_i(x)O)\rangle, \tag{3.6}$$

we find the equation

$$\{\gamma^\mu\partial_\mu+m_0-F[\langle j_i(x)\rangle+J_i(x)]T_i+iT_i(\delta/\delta J_i(x))\}\times G(x,x')=\delta(x-x')\,. \tag{3.7}$$

The form of (3.7) suggests the definition of an effective meson field $\varphi_i(x)$

$$f\varphi_i(x)=F[\langle j_i(x)\rangle+J_i(x)]\,. \tag{3.8}$$

Rewriting Eq. (3.7) in the matrix form

$$\left\{\gamma^\mu\partial_\mu+m_0 - \int d^4\xi\, T_i(\xi)[f\varphi_i(\xi)-i(\delta/\delta J_i(\xi))]\right\}G=1\,, \quad (3.9)$$

we can, by introducing the vertex operator

$$\Gamma_i(\xi)=-\delta G^{-1}/\delta f\varphi_i(\xi)\,, \quad (3.10)$$

and the boson propagator

$$D_{ij}(\xi,\xi')=\delta\varphi_i(\xi)/\delta f J_j(\xi')\,, \quad (3.11)$$

recast (3.9) into the form

$$\left\{\gamma^\mu\partial_\mu+m_0-f\int d^4\xi\, T_i(\xi)\varphi_i(\xi) +if^2\int d^4\xi d^4\xi'\, T_i(\xi)D_{ij}(\xi_1\xi')G\Gamma_j\right\}G=1\,. \quad (3.12)$$

This is completely reminiscent of the equation for the Green's function in a theory with Yukawa coupling and suggests that some Fermi theories may be renormalizable when expanded in terms of the effective boson propagator D rather than in terms of the Fermi coupling constant.[11] This is the meaning we assign to the kind of excitations under consideration.

2. *Nambu's Original Model: Broken U(1) Symmetry.*

Nambu and Jona-Lasinio[6] consider the Heisenberg-type Lagrangian

$$\mathcal{L}=-\bar\psi\gamma^\mu\partial_\mu\psi+\tfrac{1}{2}F\mathbf{j}\cdot\mathbf{j}\,, \quad (3.13)$$

where ψ is an ordinary four-component (massless) spinor field and

$$\mathbf{j}=(j_0,j_5)=(\bar\psi\psi,i\bar\psi\gamma_5\psi) \quad (3.14)$$

is a vector in a two-dimensional "parity space." Because of the masslessness of the spinor field, this Lagrangian is invariant under the γ_5 gauge transformation

$$\begin{aligned}\psi &\to \psi'=\exp(\tfrac{1}{2}i\alpha\gamma_5)\psi\,,\\ \bar\psi &\to \bar\psi'=\bar\psi\exp(\tfrac{1}{2}i\alpha\gamma_5)\,,\end{aligned} \quad (3.15)$$

which rotates the vector $\mathbf{j}$

$$\begin{aligned}j_0 &\to j_0'=\cos\alpha j_0+\sin\alpha j_5\,,\\ j_5 &\to j_5'=\sin\alpha j_0+\cos\alpha j_5\,,\end{aligned} \quad (3.16)$$

keeping $\mathbf{j}\cdot\mathbf{j}$ invariant.

Suppose now that the γ_5 gauge symmetry is broken because there exists a self-consistent solution in which $\mathbf{j}$ has a nonvanishing vacuum expectation value along some direction in the parity space. Choose a basis so that this direction is the 0 direction,

$$\langle j_0\rangle\neq 0,\quad \langle j_5\rangle=0\,. \quad (3.17)$$

That $\langle j\rangle$ has this direction is purely a matter of convention. Goldstone's theorem then shows that the propagator

$$\tilde D_{55}^{-1}(p^2)=(f^2/F)-\tilde\pi_{55}(p^2) \quad (3.18)$$

has a zero at $p^2=0$. These excitations of the Fermi system are identified as massless pseudoscalar neutral mesons.

3. *Broken Isospin Symmetry*

We consider the Lagrangian

$$\mathcal{L}=-\bar\psi(\gamma^\mu\partial_\mu+m_0)\psi+\tfrac{1}{2}F\mathbf{j}\cdot\mathbf{j}\,, \quad (3.19)$$

where

$$\mathbf{j}=\tfrac{1}{2}[\bar\psi,\boldsymbol{\tau}\psi]\,. \quad (3.20)$$

The ψ are two-component isospinors, transforming as

$$\begin{aligned}\psi &\to \psi'=\exp(i\boldsymbol{\alpha}\cdot\boldsymbol{\tau})\psi\,,\\ \bar\psi &\to \bar\psi'=\bar\psi\exp(-i\boldsymbol{\alpha}\cdot\boldsymbol{\tau})\,,\end{aligned} \quad (3.21)$$

so that $\mathbf{j}$ transforms as an isovector and $\mathcal{L}$ is isospin invariant. The bare mass m_0 need not vanish. [We could consider symmetry under the direct product of these isospin rotations and the parity rotations (3.15); a Lagrangian[6] invariant under this group would need to have $m_0=0$. Such a model, which is reducible from the group theoretical point of view, will not be considered here.]

Assume that the isospin symmetry is broken because there exists a self-consistent solution in which $\mathbf{j}$ has nonvanishing vacuum expectation value along a direction called the 3-axis in isospin space

$$\langle j_3\rangle\neq 0\,. \quad (3.22)$$

Goldstone's theorem shows that $\tilde D_{ij}^{-1}(p^2)$ vanishes at $p^2=0$ for i and $j=1$ or 2. This asserts the presence of massless scalar mesons positively and negatively charged.

It was Baker and Glashow[9] who pointed out that a neutron-proton mass difference could originate in such a spontaneous breakdown of isospin symmetry because of some kind of dynamical instability. We would emphasize, however, that this may happen whether or not *all* of the nucleon mass is of dynamical origin.

4. *Broken Octet Symmetry*

Consider now the Lagrangian

$$\mathcal{L}=\mathcal{L}_0+\tfrac{1}{2}F\mathbf{j}\cdot\mathbf{j}\,, \quad (3.23)$$

where $\mathcal{L}_0$ is the free-particle Lagrangian including the mass m_0, and ψ is an eight-component unitary vector transforming according to

$$\begin{aligned}\psi &\to \exp(i\boldsymbol{\alpha}\cdot\boldsymbol{\Lambda})\psi\,,\\ \bar\psi &\to \bar\psi\exp(-i\boldsymbol{\alpha}\cdot\boldsymbol{\Lambda})\,,\end{aligned} \quad (3.24)$$

where the $\lambda_a(a=1\cdots 8)$ are the infinitesimal generators of $SU(3)$ in the 8 representation and

$$j_a = \tfrac{1}{2}[\bar{\psi}, \lambda_a \psi]. \quad (3.25)$$

(Ordinary spin indices on ψ and the additional covariants this leads to will be suppressed here.)

We wish to study the consequences of assuming that $\langle\bar{\psi}\lambda\psi\rangle \neq 0$. Two of the λ_a, conventionally taken to be λ_3 and λ_8 and linearly related to a charge and hypercharge, are simultaneously diagonalizable. The eight basis vectors ψ_a are distinguished, besides by the eigenvalues of λ_3 and λ_8, by an additional label conventionally identified with the isotopic spin. Using this basis, the most general way of breaking the symmetry is to allow solutions for which

$$\langle\bar{\psi}\lambda_3\psi\rangle \neq 0 \neq \langle\bar{\psi}\lambda_8\psi\rangle. \quad (3.26)$$

Our theorem now asserts that six massless bosons ensue, interpretable as π^+, π^-, K^+, K^0, K^-, $\bar{K}^0$, associated conventionally with the 1, 2, 4, 5, 6, 7 axes.

The solution (3.26) breaks the isospin symmetry. To preserve the latter we consider a solution in which only $\langle j_8\rangle \neq 0$. Examination of the infinitesimal transformations of the 8 representation, in this case a table of structure constants of $SU(3)$,[14] shows that in this case we are left with four massless mesons ($K^+, K^0, K^-, \bar{K}^0$). This is in accord with the geometric interpretation of the theorem, since in the 8 representation, a vector originally pointing in the 8 direction can be acquired by infinitesimal transformations of the group components along the axes 4, 5, 6, 7 but not along the 1, 2, 3 axes.

C. The Lorentz Group: Quantum Electrodynamics

Following Bjorken[11] and Bialynicki-Birula,[11] we begin with a Heisenberg model containing the interaction of the conserved current

$$j_\mu(x) = \tfrac{1}{2} i[\bar{\psi}, \gamma_\mu \psi] \quad (3.27)$$

with itself. The equations for the one-fermion Green's function and associated functions are Eq. (3.8)–(3.12) with $T_i \to i\gamma_\mu$ and generally the index $i \to \mu$.

We introduce an external current $J_\mu(x)$, which is also assumed to be conserved

$$\partial^\mu j_\mu(x) = \partial^\mu J_\mu(x) = 0. \quad (3.28)$$

As a consequence only $J_\mu{}^T$, the transverse part of J_μ, is coupled to j_μ.

For the "photon" propagator we obtain from Eq. (3.8) and (3.11)

$$\mu^2 D_{\mu\nu}(x-y) = \int d^4z\, \pi_{\mu\lambda}(x-z) D_{\lambda\nu}(z-y) + \delta_{\mu\nu}{}^T \delta(x-y), \quad (3.29)$$

where $\mu^2 \equiv (f^2/F)$,

$$\delta_{\mu\nu}{}^T = \delta_{\mu\nu} - \partial_\mu \partial_\nu / \partial^2, \quad (3.30)$$

and

$$\pi_{\mu\nu}(x-z) = \delta f \langle j_\mu(x)\rangle / \delta\varphi_\nu(z). \quad (3.31)$$

From its definition $\Pi_{\mu\nu}$ is divergenceless. In momentum space, where Eq. (3.29) reads

$$\mu^2 \tilde{D}_{\mu\nu}(p) = \tilde{\pi}_{\mu\lambda}(p) \tilde{D}_{\lambda\nu}(p) + \tilde{\delta}_{\mu\nu}{}^T(p), \quad (3.32)$$

we may write

$$\tilde{\pi}_{\mu\nu}(p) = \tilde{\delta}_{\mu\nu}{}^T(p) \tilde{\pi}(p^2), \quad p^2 \neq 0, \quad (3.33)$$

and similarly

$$\tilde{D}_{\mu\nu}(p) = \tilde{\delta}_{\mu\nu}{}^T(p) \tilde{D}(p^2), \quad p^2 \neq 0. \quad (3.34)$$

We assume that $j_\mu{}^L$, the time-like component of j_μ, has a nonvanishing vacuum expectation value.

$$f\varphi_\mu{}^L = F\langle j_\mu{}^L\rangle \neq 0. \quad (3.35)$$

For $p^2 = 0$, $\tilde{\delta}_{\mu\nu}{}^T(0)$ is defined by

$$\tilde{\delta}_{\mu\nu}{}^T(0) = \delta_{\mu\nu} - \eta_\mu{}^L \eta_\nu{}^L, \quad (3.36)$$

where $\eta_\mu{}^L$ is a unit vector in the direction of $\langle j_\mu{}^L\rangle$. Following the reasoning of Sec. II, we find

$$\mu^2 \tilde{\delta}_{\mu\nu}{}^T(0) = \tilde{\pi}_{\mu\nu}{}^T(0) = \tilde{\delta}_{\mu\nu}{}^T(0) \tilde{\pi}(0) \quad (3.37)$$

or

$$\mu^2 = \tilde{\pi}(0). \quad (3.38)$$

If we write

$$\tilde{\pi}(p^2) - \tilde{\pi}(0) \equiv -p^2 \tilde{\pi}_1(p^2), \quad (3.39)$$

Eq. (3.32) becomes

$$\tilde{\delta}_{\mu\nu}{}^T(p) p^2 \tilde{\pi}_1(p^2) D(p^2) = \tilde{\delta}_{\mu\nu}{}^T(p), \quad (3.40)$$

exhibiting the singularity at $p^2 = 0$ for the transverse excitations orthogonal to the direction of the vacuum expectation value $\varphi_\mu{}^L$. The latter plays a role analogous to that of a constant vector potential in conventional quantum electrodynamics.

Bjorken goes on to establish the full equivalence of this Heisenberg-type theory to conventional quantum electrodynamics. In our formulation the proof consists of several observations. (1) The Dyson-Schwinger equations obtained are formally the same as in quantum electrodynamics, except that the Green's function $D_{\mu\nu}$ is defined as above. This is equally true of the equations after renormalization. (2) For Eq. (3.40), renormalization consists in the replacement

$$\tilde{D}(p^2) \to \tilde{D}_r(p^2)/\tilde{\pi}_1(0). \quad (3.41)$$

One then verifies that $\tilde{\pi}_1(p^2)/\tilde{\pi}_1(0)$ is the same renormalized "sum of bubbles" as found in the conventional theory. In other words, the Heisenberg theory differs from the conventional electrodynamics only in the "values" of the renormalization constants.

D. Model for Superconductivity

It is worth remarking finally that our theorem provides a concise new proof of zero-mass excitations for a

[14] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962).

Fermi system with short-range attractive interactions. As a Lagrangian we choose in analogy with the work of Nambu[5]

$$\mathcal{L}(x)=\Psi^\dagger(x)[i(\partial/\partial t)+\tau_3(p^2/2m)]\Psi(x) + V_0\Psi^\dagger(x)\tau_3\Psi(x)\Psi^\dagger(x)\tau_3\Psi(x)+\text{const}\,, \quad (3.42)$$

where, if $\psi_{1,2}(x)$ are the two spin components of the usual electron operator,

$$\Psi(x)=\begin{pmatrix}\psi_1(x)\\ \psi_2{}^\dagger(x)\end{pmatrix}. \quad (3.43)$$

The theory is invariant under rotations about the third axis in the isospace defined by (3.54), where

$$\Psi\to\exp(i\tau_3\alpha)\Psi\,,\quad \Psi^\dagger\to\Psi^\dagger\exp(-i\tau_3\alpha)\,. \quad (3.44)$$

For $V_0>0$, corresponding to an attractive interaction, we expect a solution (occurrence of an energy gap) such that $\langle j_1(0)\rangle=\langle\Psi^\dagger(0)\tau_1\Psi(0)\rangle\neq 0$. The theorem of Sec. II then assures us that the propagator $\langle\tau(j_2(x)j_2(y))\rangle$ has a singular Fourier transform when $\mathbf{p}=p_0=0$, corresponding to the onset of a phonon spectrum.

It has frequently been observed that the phonons do not occur when the long-range interaction between charged particles is included. This result is only apparently in conflict with Goldstone's theorem. We have assumed throughout that the various matrix elements such as $\langle j_i\rangle$, are continuous, differentiable functionals of the collective fields φ_i. In the case of a Coulomb field, for $i=3$, the appropriate definition of φ_3 is

$$\tilde{\varphi}_3(\mathbf{p})=p^{-1}\tilde{j}_3(\mathbf{p})\,. \quad (3.45)$$

By adding a uniform background of positive charge, $\tilde{\varphi}_3(\mathbf{p})$ is defined to vanish for $\mathbf{p}=0$, but becomes quite singular for small $\mathbf{p}$. Since $\langle\tilde{j}_{1,2}(\mathbf{p})\rangle$ are functionals of $\tilde{\varphi}_3(\mathbf{p})$, we encounter a situation where the limit of the functions as $p\to 0$ is not their value at the limit. Thus, the theorem predicting zero-mass particles does not apply. We are unable to say whether it is possible to construct a relativistic model with analogous properties.

IV. CONCLUDING DISCUSSION

In this section we wish to discuss the significance of the generalized Hartree conditions (2.5), the lacunae in the derivation of the Goldstone theorem from these conditions, and the physical implications of the theorem.

A. Asymmetric Vacuum

Theories of the kind we have been discussing are distinguished from conventional field theories by the presence of generalized Hartree conditions (2.5), which introduce into the theory new physical parameters, the $\langle\phi_i\rangle$. Together with the Dyson-Schwinger equations, these conditions contain information on the nonvanishing particle masses. Nambu and Jona-Lasinio[6] and then Baker and Glashow[9,10] have, for example, restricted themselves to Heisenberg-type theories and used the Hartree approximation in which the momentum dependence of the fermion mass operator is neglected. The Hartree conditions can then be solved to obtain the fermion masses. We wish to emphasize that our proof of the existence of the boson singularity at $p^2=0$ depends only on the (rigorous) existence of the generalized Hartree conditions, and not on any such Hartree approximation.

Although well defined in the nonrelativistic situation, the $\langle\phi_i\rangle$ are in relativistic models the most divergent quantities in the theory. Though we have maniuplated them formally, the $\langle\phi_i\rangle$ are well-defined only after a cutoff is introduced.

The significance of these Hartree conditions or of condition (2.4) is that in the physical vacuum some "direction" or subspace in the symmetry space is preferred. Because of the initial symmetry in the Lagrangian, which particular subspace is taken is conventional and serves to establish a labeling of one-particle states. Now consider the components $i\neq i'$ for which

$$\langle\phi_i\rangle=0\,,\quad i\neq i'\,.$$

The meaning of this equation is that ϕ_i operates on a vacuum state $|0>$ to produce an orthogonal state $\phi_i|0>$ that, since it is equivalent to $|0>$ is degenerate with $|0>$ in energy. The different vacuum states are distinguished by the presence of different numbers of massless low-energy bosons created by $\phi_i(i\neq i')$.

It is worth emphasizing that the vacuum is degenerate only in a description in which the particle number, or other conserved quantity, is not a good quantum number. In the conventional treatment of superconductivity, for example, the particle number is not definite. In a system of finite volume V, however, the superconducting ground state is not, strictly speaking, degenerate, but is one of a large number of equivalent states separated by an energy $\sim V^{-1}$. Likewise, in Nambu's theory of elementary particles, a cutoff Λ is introduced. For any finite value of the cutoff, the ground state is nondegenerate but separated from many other equivalent states by an energy difference $\sim\Lambda^{-1}$.

Any state of definite particle number (or charge, etc.) is built on a particular one of these equivalent vacua. The degenerate and orthogonal vacua are distinguished by the presence of different numbers of particles of zero four-momentum.

Once this formal nature of the degenerate vacuum treatment is recognized,[15] this kind of degeneracy would appear to offer no obstacle to the derivation of axiomatic field theory results such as the spin-statistics theorem.

B. Limitations of Our Proof

Whereas the degenerate vacuum by itself offers no genuine conceptual difficulties, the theorem issuing from

[15] R. Haag, Nuovo Cimento **25**, 287 (1962).

this source of symmetry breakdown appears to demand the existence of certain collective states of zero energy momentum.

We should emphasize, however, that the formal group-theoretical argument has shown only that certain $\tilde{D}_{ij}(p)$ are singular at $p^2=0$. In fact it is not certain that $\tilde{D}_{ij}(p)$ exists for $p^2\neq 0$. Only if $\tilde{D}_{ij}^{-1}(p)$ is analytic near $p^2=0$ is the singularity obtained that of a massless particle. To establish such analyticity properties requires more than broken symmetry alone.

Of course, when the singularity obtained is precisely at $p^2=0$, questions of asymptotic condition and particle interpretation arise. Nevertheless, when dealing with a theory that is renormalizeable in the conventional perturbative sense, the singularity we have found establishes a "particle" of zero mass in the same sense that, in conventional field theory, the massless photon is a "particle."

We have also emphasized, at the end of Sec. III, that our treatment may be inapplicable when zero-mass fields are originally present in the Lagrangian. We have in mind nonrelativistic situations where gauge invariance calls forth the existence of massless phonons, but the long-range Coulomb interactions turn these into massive plasmon modes.[16] For the relativistic case, however, the value of the bare mass of a particle may be irrelevant. The question deserves further study.

C. Massless Particles

The main effect of the generalized Hartree conditions in an originally symmetric system has been (subject to the above qualifications) to give a dynamical reason for the existence of some zero-mass particles. In the case of the photon (and possibly of the neutrino) this result may be welcome. In the domain of strong interactions for which the Heisenberg and Nambu theories were originally proposed, however, no massless particles are known. Before concluding that this invalidates the original program of spontaneous breakdown of strong-interaction symmetries, we should observe

(1) Extended gauge invariance also seems to demand the existence of massless gauge particles but does not dictate the renormalized coupling strength with which these particles must be coupled. The coupling strength of the massless bosons predicted by the Goldstone theorem is likewise not dictated by the theorem.

It is true that in any actual calculation like Nambu's or Bjorken's that produces a propagator with a singularity at $p^2=0$, this singularity has a residue of order of magnitude unity. The value of the residue calculated is, however, cutoff-dependent and decreases to zero (logarithmically slowly) with increasing cutoff.

Massless bosons also may, precisely because of their long-range interaction, more or less completely screen away their renormalized coupling to their sources.

(2) If the currents involved in weak decays are the same as those involved in strong interactions, then this boson strong-coupling constant is inversely proportional to the observed boson decay rate and therefore can certainly not vanish. Indeed, this Goldberger-Treiman proportionality was the original reason Nambu[17] and Gell-Mann and collaborators[18] had for considering asymptotically conserved currents. In these situations, precisely because the current is only asymptotically conserved, the invoked symmetry is only approximate. Thus, the pion decay amplitude is proportional, not only to the reciprocal of the pion coupling constant, but also to the pion mass.

From a puristic point of view this might appear to be an argument against introducing symmetries which are, to begin with, actually approximate. Remarkably enough, however, Nambu and Jona-Lasinio[6] showed that only a small breaking of γ_5 invariance was necessary to give the observed pion mass. The broken-symmetry mechanism then serves to introduce bosons with the requisite quantum numbers even though their masslessness is not taken seriously.

The currents involved in weak-decay processes may, on the other hand, *not* be directly related to the observed strong-interaction currents. Despite the success of the Goldberger-Treiman equality in predicting the pion decay rate, this may be the implication to be drawn from the low rate observed for strangeness-changing β and μ decays and from the appearance of $\Delta Q=-\Delta S$ processes.

The lesson that we would draw is that the Goldstone theorem shows how short-range interactions can lead to long-range effects. (This relationship is reciprocal. Long-range interactions, if sufficiently strong, may screen themselves out into short-range interactions.) When perturbation theory is inapplicable, it is simply not manifest whether a given Lagrangian will lead to a symmetric or asymmetric ground state and to short-range or long-range interactions.

[16] P. W. Anderson, Phys. Rev. **130**, 439 (1962).

[17] Y. Nambu, Phys. Rev. Letters **4**, 380 (1960).

[18] M. Gell-Mann and M. Levy, Nuovo Cimento **16**, 705 (1960); J. Bernstein, M. Gell-Mann, and L. Michel, *ibid.* **16**, 560 (1960); J. Bernstein, S. Fubini, M. Gell-Mann, and W. Thirring, *ibid.* **17**, 757 (1960).