High-Energy Behavior of Feynman Amplitudes. II. Nonplanar Graphs*†

GEORGE TIKTOPOULOS‡
The Enrico Fermi Institute for Nuclear Studies and the Department of Physics, The University of Chicago, Chicago, Illinois

(Received 26 April 1963)

A method previously developed by the author for obtaining the asymptotic behavior of Feynman integrals associated with planar graphs in the $\lambda\phi^3$ theory is extended for the study of nonplanar graphs. The results are discussed in connection with the high-energy behavior of amplitudes and the associated properties in the complex angular momentum plane.

I. INTRODUCTION

IN a previous publication[1] (which will be denoted by [I]) the author has studied the high-energy behavior of Feynman integrals motivated by recent interest in the high-energy properties of reaction amplitudes.[2] In the framework of the algebraically simple $\lambda\phi^3$ interaction a method was developed for obtaining the asymptotic form of the Feynman integral associated with a general nth order graph with two, three, or four external lines. Explicit formulas and a rule were given for reading off this form from the topology of a given graph. In the case of graphs with four external lines the method was restricted to *planar* graphs and it was realized that nonplanar graphs could behave asymptotically in an exceptional way.

The remarkable feature of planar graphs is that, if one assumes the validity of the double dispersion relations for the associated integrals, they can have *only one* nonvanishing spectral function. Similar analytic properties are exhibited in potential scattering (exchange terms correspond to "crossed" planar graphs). It is therefore reasonable to expect that the asymptotic properties of planar graphs are consistent with a perturbative expansion of a Regge formula. This point of view is supported by various field-theoretic approximations.[3,4] A summation of the leading asymptotic contributions from the series of simple ladder graphs was shown by Polkinghorne[2] to lead to a Regge asymptotic form.

Nonplanar graphs, having a nonvanishing third[5] spectral function, are specific for relativistic field theories and therefore an investigation of their asymptotic properties is of particular interest. Indeed, it is already known[6] that the λ-plane properties of an analytic amplitude are seriously complicated by the existence of the third[5] spectral function.

In the present paper the method of [I] is extended for the study of the asymptotic properties of nonplanar graphs. Explicit formulas based on the topology are derived in the case of graphs whose asymptotic behavior is not weaker than s^{-1} where s is the large variable.

In Sec. II we briefly review our previous results, give definitions, and establish the notation used in the subsequent discussion.

In Sec. III we discuss the asymptotic properties of nonplanar graphs and derive formulas for the contributions of various parts of the integration domain of the Feynman integral.

In Sec. IV we discuss the obtained results in connection with analyticity properties in the angular momentum plane. A certain set of nonplanar graphs exhibiting the exchange of a Regge pole is examined.

In Appendix A we derive the asymptotic form of certain types of integrals used in the text. In Appendix B it is shown that a graph is (essentially) planar if and only if the polynomial coefficient of the variable s in the Chisholm[7] form of the Feynman integral is definite (i.e., all its terms appear with the same sign).

II. DEFINITIONS AND PREVIOUS RESULTS

Although we will be explicitly concerned with the $\lambda\phi^3$ theory, most of our formulas and results can be easily rewritten for those graphs of the $\lambda\phi^3+g\phi^4$ theory in which no divergences appear.

Apart from unimportant numerical factors the Feynman integral for a strongly connected convergent graph for the process $p+k\to p'+k'$ with l-independent loops and $I=3l+1$ internal lines can be written in the Chisholm[7] form,

$$F(s,t)=\int_0^1\cdots\int_0^1\frac{[C(x)]^{l-1}}{[D(s,t,x)]^{l+1}}\delta(\sum_1^I x_j-1)dx_1\cdots dx_I\,,\quad(1)$$

where $x_1, x_2, \cdots, x_I$ are the Feynman parameters associated with the internal lines, $s=-(p+k)^2$ and $t=-(p-p)^2$ are the invariant kinematical variables, and $C(x)$ and $D(s,t,x)$ the familiar discriminants[8] of the graph.

* Part of a thesis submitted to the Department of Physics, the University of Chicago, in partial fulfillment of the requirements for the Ph.D. degree.

† Work supported in part by the U. S. Atomic Energy Commission.

‡ William Rainey Harper Fellow.

[1] G. Tiktopoulos, Phys. Rev. **131**, 480 (1963).

[2] Certain classes of Feynman graphs have been studied by J. C. Polkinghorne, J. Math. Phys. **4**, 503 (1963); and P. G. Federbush and M. T. Grisaru, Ann. Phys. (N. Y.) **22**, 263 and 299 (1963).

[3] B. W. Lee and R. F. Sawyer, Phys. Rev. **127**, 2266 (1962).

[4] L. Bertocchi, S. Fubini, and M. Tonin, Nuovo Cimento **25**, 626 (1962).

[5] We refer to the ρ_{su} spectral function if λ is the angular momentum in the t channel.

[6] R. Oehme and G. Tiktopoulos, Phys. Letters **2**, 86 (1962); V. N. Gribov and I. Pomeranchuk, *ibid.* **2**, 239 (1962).

[7] R. Chisholm, Proc. Cambridge Phil. Soc. **48**, 300 (1952).

[8] For a detailed study of the properties of the discriminants and related quantities, see R. J. Eden, Phys. Rev. **119**, 1763 (1960) and Brandeis Summer Institute lectures (1961).

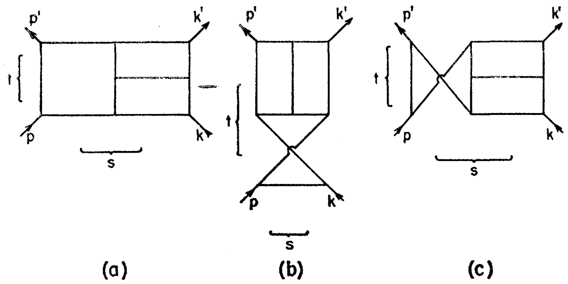

FIG. 1. (a) A planar graph. (b) A nonplanar graph which can be made planar by interchanging the p and k external lines. (c) A "crossed" planar graph. The reason for this nonsymmetric treatment of graphs (b) and (c) is that we have in mind s as the large variable and $f_c=-f_a$, whereas $f_b=f_a-g_a$.

We have $D(s,t,x)=f(x)s+g(x)t+h(x)$, where f, g, and h are $(l+1)$th degree homogeneous polynomials in the x variables. f and g are linear in each x_j.

It is known[9] that if t is real and in the interval $(-4m^2, 4m^2)$ and $\mathrm{Im}s\neq 0$ the integral (1) is nonsingular and well defined without any deformation of the real integration contours. Accordingly, in our discussion we keep the contours real and obtain the asymptotic form of $F(s,t)$ as $s\to\infty$ under those conditions. It is reasonable to expect that the so obtained asymptotic form, having explicit analytic properties in t will also be valid for values of t to which it can be analytically continued.

Clearly, the contribution to $F(s,t)$ from the region in (real) x space where $|f(x)|>\epsilon$, where ϵ is some small positive number, behaves like s^{-l-1} at large values of $|s|$. It follows that the leading asymptotic contribution comes from an arbitrarily small neighborhood of the hypersurface defined by $f(x)=0$.

We shall need the following definitions:

Definition 1. We shall say that the polynomial $f(x)$ is *definite*, if all its monomial terms are of the same sign. Also a graph with a definite $f(x)$ will be called definite.

Definition 2. A graph is *planar* if it can be drawn on a plane so that no internal or external lines cross (without meeting), the latter being attached around the graph in the cyclic order p, k, k', p'.

The graph of Fig. 1(a) is an example of a planar graph. The graph of Fig. 1(b) is *not* planar although it would become one if we interchanged the p and k external lines.

In [I] we called "crossed" planar the graphs obtained from planar ones by interchanging the p and p' external lines. An example is given in Fig. 1(c). It is easily seen that crossing the lines p and p' just changes the over-all sign of $f(x)$ and so does not affect the hypersurface $f(x)=0$. We shall not refer to these "crossed" graphs explicitly in what follows.

In [I] we have shown that $f(x)$ is definite for planar graphs. In Appendix B of the present paper this is extended as follows: "$f(x)$ is definite if and only if the graph is *essentially planar*."

Definition 3. We shall call essentially planar or essentially nonplanar the graphs whose "skeleton" (obtained by shrinking all vertex parts to points and replacing all self energy parts by single lines) is planar or nonplanar, respectively.

A straightforward corollary is that for graphs with two or three external lines the function $f(x)$ (i.e., the coefficient of s in the Chisholm denominator of the Feynman integral) is always definite. This was first shown by Nambu.[10]

In [I] we have obtained the asymptotic form of definite graphs. For such graphs $f(x)$ vanishes only if all its monomial terms vanish individually. Thus, the hypersurface $f(x)=0$ consists of several branches of the form

$$x_1=x_2=\cdots=x_c=0.$$

Except for the case when certain singular subgraphs are present within the given graph, it was shown that the asymptotic behavior is determined by those branches of $f(x)=0$ which are of highest dimensionality (i.e., of minimum c) and for which the corresponding set of lines does not contain a complete loop of the graph. Such a set of lines we called a $\bar{t}$ path. It can be topologically characterized as follows.

Definition 4. A *t path* is a minimal (i.e., none of its subsets is a t path) connected set of lines which, if short-circuited, splits the entire graph in two parts having no common line and only one common vertex (to which the entire t path has been reduced), the p and p' external lines being attached to one part and the k and k' ones to the other. A *$\bar{t}$ path* is a t path of minimum "length," i.e., number of lines.

For example, in the graph of Fig. 2 the lines 7, 10, 11, 12 form a t path. There are three $\bar{t}$ paths: (4, 5, 6), (1, 2, 3), and (7, 8, 9).

The asymptotic behavior of definite[11] graphs was shown in [I] to be given essentially by the following rule:

"At large values of $|s|$ $F(s,t)$ behaves like $s^{-\rho}(\ln s)^{M-1}$, where ρ the (common) length of the $\bar{t}$ paths of the graph and M is the maximum number of $\bar{t}$ paths one can take in a sequence $P_1, P_2, \cdots, P_M$ so that (i) no loop is formed by lines belonging to $P_1, P_2, \cdots, P_M$; (ii) the lines of P_i are not *all* included in $P_{i+1}, P_{i+2}, \cdots, P_M$."

This rule is slightly complicated by the presence of certain "singular" subgraphs within the given graph.

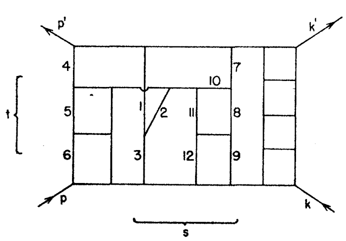

FIG. 2. This graph illustrates the t paths defined in text. For example (4,5,6), (1,2,3), (7,10,11,12), and (7,8,9) are t paths. There are only three $\bar{t}$ paths: (4,5,6), (1,2,3), and (7,8,9).

[9] T. T. Wu, Phys. Rev. **123**, 678 (1961).

[10] Y. Nambu, Nuovo Cimento **6**, 1064 (1957).

[11] In a recent paper by I. G. Halliday, University of Cambridge (unpublished), the case of planar graphs with nonoverlapping $\bar{t}$ paths is considered.

For the detailed rule and examples of its application the reader is referred to [I].

In our discussion we shall also make use of the following two "operations" on a graph G and the associated polynomial $f(x)$: (i) By removing a line with parameter x_j we obtain a graph (denoted by $G-x_j$) whose f polynomial is $\partial f/\partial x_j$.[12] We shall say that $G-x_j$ is a derivative graph of G. (ii) By shortcircuiting a line with parameter x_j we obtain a graph whose f polynomial is $f(x)|_{x_j=0}$.

III. ASYMPTOTIC BEHAVIOR OF ESSENTIALLY NONPLANAR GRAPHS

In Appendix B we show that for graphs with four external lines definiteness is equivalent to essential planarity. We also know that all graphs with two or three external lines are definite. Therefore, the rule given in [I] and outlined at the end of the previous section applies to all essentially planar graphs with four external lines and to all graphs with two or three external lines.

We now turn to the case of essentially nonplanar graphs with four external lines. For these graphs $f(x)$ contains terms of both signs and, thus, can vanish without all its monomial terms vanishing. We shall now discuss the asymptotic contributions from the various parts of the $(I-1)$ dimensional hypersurface $f(x)=0$ in the I-dimensional cube $\{0\leq x_j\leq 1;\, j=1, 2, \cdots, I\}$.

(A) Regular Part of $f(x)=0$ in the Interior of the Cube

The regular points of $f(x)=0$ are those at which at least one of the first derivatives of $f(x)$ does not vanish. In the neighborhood of regular points we can introduce the distance ξ from the hypersurface as a new variable. It is convenient to first introduce a dummy variable z

$$dx_1dx_2\cdots dx_I=\delta(z)dzdx_1dx_2\cdots dx_I$$

and make the change of variables

$$z,\, x_1,\, x_2,\, \cdots,\, x_I \rightarrow \xi,\, x_1,\, x_2,\, \cdots,\, x_I$$

according to the equations

$$x_i=x_i'+\xi\frac{\partial f}{\partial x_i}\bigg|_{x'}\cdot\left[\sum_{j=1}^{I}\left(\frac{\partial f}{\partial x_j}\bigg|_{x'}\right)^2\right]^{-1};\quad i=1, 2, \cdots, I.$$

$$z=f(x').$$

We expand

$$f(x)=f(x')+\sum(x_i-x_i')\frac{\partial f}{\partial x_i}\bigg|_{x'}+\cdots$$

$$=f(x')+\xi+\xi^2B(\xi,x'),$$

where $B(\xi,x)$ is bounded. It is also easy to see that the Jacobian is

$$J=1+O(\xi),$$

so that we have the contribution

$$\int\frac{[C(x)]^{l-1}\delta[f(x')]J}{[\xi(1+\xi B)s+gt+h]^{l+1}}\delta(\sum_{j=1}^{I}x_j-1)d\xi dx_1'\cdots dx_I'.$$

Since B is bounded, taking $|\xi|<\epsilon<|\sup B|^{-1}$ guarantees that $1+\xi B$ does not vanish. Therefore, the same behavior will be exhibited by the integral

$$\int_0^1\cdots\int_0^1\int_{-\epsilon}^{\epsilon}\frac{[C(x)]^{l-1}\delta(f(x))}{[\xi s+g(x)t+h(x)]^{l+1}}\times\delta(\sum_{j=1}^{I}x_j-1)d\xi dx_1\cdots dx_I,$$

whose asymptotic form at large values of $|s|$ is simply

$$s^{-l-1}\int_{-\epsilon}^{\epsilon}\xi^{-l-1}d\xi\int_0^1\cdots\int_0^1[C(x)]^{l-1}\delta(f(x))\times\delta(\sum_{j=1}^{I}x_j-1)dx_1\cdots dx_I,$$

where the ξ-integration is done *around* the pole at $\xi=0$. This contribution is of the same order as the one coming from the region $|f(x)|>\epsilon$ and cannot compete with that of the boundary of the cube which is at least as strong as given by the rule for planar graphs.

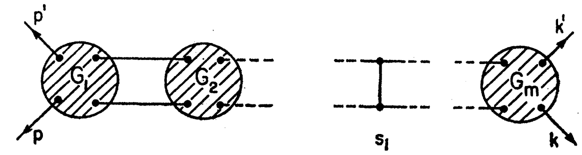

FIG. 3. A graph reducible into the component subgraphs G_1, G_2, $\cdots S_i$, $\cdots$ connected by pairs of lines in the t channel.

(B) $(I-2)$-Dimensional Singular Manifold of $f(x)=0$ in the Interior of the Cube

In other words all first derivatives of $f(x)$ vanish on an $(I-2)$-dimensional submanifold of $f(x)=0$. It is easy to see that this is the case, where $f(x)=f_1(x)f_2(x)$, where f_1 and f_2 are both nondefinite.

In order to treat this case we first decompose $f(x)$ into factors *completely*:

$$f(x)=f_1f_2\cdots f_ax_1x_2\cdots x_bf_{a+b+1}\cdots f_m;\; a>1$$

where f_1, f_2, $\cdots$, f_a are nondefinite and $f_{a+b+1}\cdots f_m$ definite (irreducible) polynomials. Topologically this means that the given graph consists of a chain[13] of sub-

[12] This result is due to R. J. Eden (Ref. 8).

[13] This can be shown by a simple induction argument. The converse statement, i.e., that such a chain of subgraphs has an $f(x)$ polynomial equal to the product of those of the subgraphs can be obtained by means of the theorem given in [I].

graphs $G_1, \cdots, G_a, S_1\cdots S_b, G_{a+b+1}, \cdots G_m$ connected by pairs of lines as shown in Fig. 3. $f(x)$ is just the product of the polynomials associated with these subgraphs. $G_1, \cdots, G_a$ are essentially nonplanar, $S_1, \cdots, S_b$ are 2nd order parts whose lines are labeled by $x_1, x_2, \cdots, x_b$, respectively, and $G_{a+b+1}, \cdots, G_m$ are essentially planar. The particular order in which they are connected does not affect our argument.

Introducing a ξ variable for each nondefinite factor as above we have

$$\int_0^1\cdots\int_{-\epsilon_i'}^{\epsilon_i}\cdots\frac{[C(x)]^{l-1}\prod_1^\alpha \delta(f_i(x))\delta(\sum_1^I x_j-1)d\xi_1\cdots d\xi_a}{(\xi_1\xi_2\cdots\xi_a x_1\cdots x_b f_{a+b+1}\cdots f_m s+gt+h)^{l+1}}dx_1\cdots dx_I\,,$$

whose asymptotic form at large values of $|s|$ is (see Appendix A)

$$\pm 2\pi i s^{-1}\frac{(\ln s)^{a+b-2}}{(a+b-2)!}\frac{2^{a-2}}{l}\int_0^1\cdots\frac{[C_0(x)]^{l-1}\prod_1^\alpha \delta(f_i(x))}{f_{a+b+1}\cdots f_m(g_0t+h_0)^l}\delta(\sum_1^I x_j-1)dx_{b+1}\cdots dx_I \tag{2}$$

for $\mathrm{Im}s\gtrless 0$. The subscript zero denotes evaluation at

$$x_1=x_2=\cdots=x_b=0.$$

The introduction of the ξ_i variables is strictly possible only at regular points of the surfaces $f_i(x)=0$. However, it can be verified that the singular parts of these surfaces cannot produce a stronger behavior, because they are of a dimension lower than $I-2$ (f_i are *irreducible* factors).

(C) Singular Manifolds of Dimension Less than $I-2$

In Appendix A it is shown that as $s\to\infty$ the real contours of the integral

$$\int_0^1\int_0^1\frac{dx_1dx_2}{[(ax_1x_2+b_1x_1+b_2x_2+c)s+d]^M}\,,$$

are pinched by a pair of poles provided

$$ax_1x_2+b_1x_1+b_2x_2+c=a^{-1}(ax_1+b_2)(ax_2+b_1)\,,$$

and each factor changes sign in the integration domain. In any case we can replace the real contour by a complex one plus the contribution of one of the poles which is

$$\pm\frac{2\pi i}{s}\frac{a^{M-2}}{(M-1)[(ac-b_1b_2)s+ad]^{M-1}};\quad \mathrm{Im}s\gtrless 0.$$

Writing $f(x)=ax_1x_2+b_1x_1+b_2x_2+c$ we see that the contribution from the interior of the cube is weaker than s^{-1} unless $ac-b_1b_2\equiv 0$ (identically) namely $f(x)$ is factorizable into two nondefinite polynomial factors. This is the previously considered case. It follows that the singular parts of $f(x)=0$ of dimension $I-3$ or lower will always have a contribution weaker than s^{-1}. For example, if $f(x)=\varphi_1\varphi_2+\varphi_3\varphi_4$, where the φ_i's are nondefinite, we could have an s^{-2} contribution.[14] Another example is $f(x)=\varphi_1\varphi_2+\varphi_2\varphi_3+\varphi_3\varphi_1$ with a possible $s^{-3/2}$ behavior.[14] However, the topological meaning of such forms of $f(x)$ is not transparent and one is tempted to think that they are not possible in actual Feynman graphs. If they exist at all, they must be associated with essentially nonplanar graphs of a fairly high order. We hope to report on this problem in a future publication.

(D) Boundary of the Cube

An $(I-d)$-dimensional "side" of the cube is specified by a set of equations like

$$x_1=x_2=\cdots=x_d=0.$$

We write

$$f(x)=\varphi(x)+f_0(x);\quad f(x)=f(x)|_{x_1=x_2=\cdots=x_d=0}.$$

f_0 is the polynomial associated with the graph obtained by shortcircuiting the lines $1, 2, \cdots, d$. If $f_0=0$, then the lines, $1, 2, \cdots, d$ form a t path and the corresponding asymptotic contribution is obtained as described in [I]. We are here interested in the case where $f_0\not\equiv 0$ and is nondefinite. The regular parts of the surface $f_0(x)=0$ are again easily seen to give an s^{-l-1} contribution. However, if by shortcircuiting the lines $1, 2, \cdots, d$ our graph is split into two essentially nonplanar subgraphs joined only through two vertices as shown in the example of Fig. 4, then $f_0(x)$ is the product of the f's of these two subgraphs

$$f_0(x)=f_{01}(x)\cdot f_{02}(x)\,.$$

If that is the case we shall say that the lines $1, 2, \cdots, d$ form a t_2 *path* of length d.

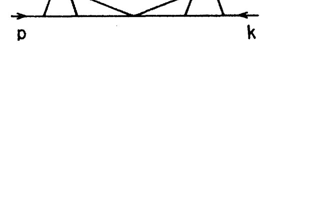

FIG. 4. In (a) the lines 1 and 2 form a t_2 path. In (b) these lines have been shortcircuited and the resulting graph consists of two nonplanar subgraphs connected through only two vertices.

[14] Clearly, for such behavior, it is also necessary that $\delta(\varphi_1)\delta(\varphi_2)\cdots\neq 0$.

We introduce ξ variables for f_{01} and f_{02} and apply a λ transformation[1] on $x_1, x_2, \cdots, x_d$. We have ($\varphi=\lambda\tilde{\varphi}$)

$$\int_0^1\cdots\int_0^1\int_0^\eta\int_{-\epsilon_1'}^{\epsilon_1}\int_{-\epsilon_2'}^{\epsilon_2}\frac{[C(x)]^{l-1}\delta(f_{01})\delta(f_{02})\lambda^{d-1}d\lambda}{[(\lambda\tilde{\varphi}+\xi_1\xi_2)s+gt+h]^{l+1}}\times d\xi_1 d\xi_2\delta(\sum_1^d x_j'-1)\delta(\sum_{d+1}^I x_j-1)dx'\cdots dx\cdots,$$

whose asymptotic form is (see Appendix A)

$$\pm 2\pi i s^{-d-1}\frac{(d-1)!(l-d-1)!}{l!}\int_0^1\cdots\times\int_0^1\frac{[C_0(x)]^{l-1}\delta(f_{01})\delta(f_{02})}{\tilde{\varphi}_0{}^d\cdot(g_0t+h_0)^{l-d}}\times\delta(\sum_1^d x_j'-1)\delta(\sum_{d+1}^I x_j-1)dx'\cdots dx\cdots, \quad (3)$$

for $\mathrm{Im}s\gtrless 0$. The subscript zero denotes evaluation $\lambda=0$.

From singular manifolds of $f_0(x)=0$ of dimension $I-3$ or lower we might have an s^{-d-2} contribution or weaker. For example for $f_0(x)=\varphi_1\varphi_2+\varphi_3\varphi_4$ with nondefinite φ's we could[14] have an s^{-d-2} behavior.

We can summarize the results of this section as follows: As far as its asymptotic behavior is concerned, an essentially nonplanar graph G belongs to one of the following two classes.

(1) G consists of a chain of subgraphs $G_1, \cdots, G_a, G_{a+1}, \cdots, G_{a+b}, G_{a+b+1}, \cdots, G_m$ connected by pairs of lines as shown in Fig. 3. $G_1, G_2, \cdots, G_a$ are essentially nonplanar. $G_{a+1}, \cdots, G_{a+b}$ are second-order parts. $G_{a+b+1}, \cdots, G_m$ are essentially planar of order >2.

If $a\geq 2$, the asymptotic behavior is $s^{-1}(\ln s)^{a+b-2}$ as given by formula (2).

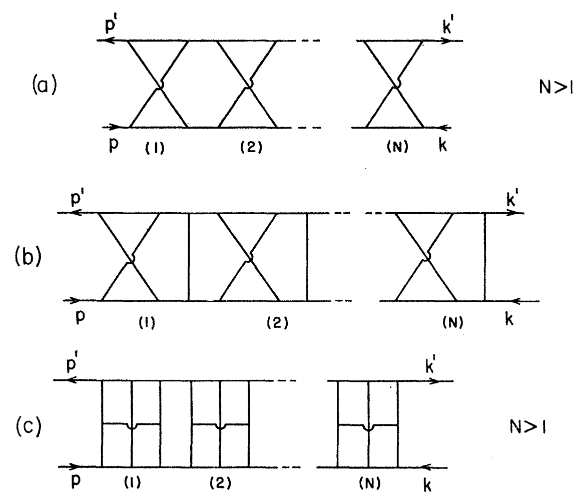

FIG. 5. Examples of the determination of the asymptotic behavior of nonplanar graphs belonging to class 1 (see end of Sec. III): (a) for $N<1$ we have $a=N\geq 2$ and $b=0$; the behavior is $S^{-1}(\ln s)^{N-2}$ (b) for $N=1$ we have $a=1$, $b=1$ and the behavior is S^{-1}; for $N>1$ we have $a=N\geq 2$, $b=N$ and the behavior is $S^{-1}(\ln s)^{2N-2}$ (c) for $N>1$ we have $a=N$, $b=0$ so that the integral behaves like $S^{-1}(\ln s)^{N-2}$.

If $a=1$, $b\geq 1$ the asymptotic behavior is $s^{-1}(\ln s)^{b-1}$ as given in [I].

This rule is applied to some examples in Fig. 5.

(2) G is not reducible into subgraphs as above. Then it is asymptotically weaker than s^{-1}. Although we have sketched a method applicable to such cases, explicit formulas were not given because of the possibility of singular parts of $f(x)=0$ of dimension $I-3$ or lower. Nevertheless specific contributions from the boundary of the integration cube were explicitly given and may serve as lower bounds for the leading asymptotic behavior.

IV. DISCUSSION AND APPLICATIONS

Let us consider the case of a graph which is reducible into essentially nonplanar subgraphs $G_1, G_2, \cdots$ as shown in Fig. 3. The associated Feynman integral is a real analytic function which for real values of s can be written

$$F(s\pm i0)=D(s,t)\pm iA(s,t),$$

in terms of the "dispersive" and "absorptive" parts. The leading contribution for $s\to\infty$ is given by formula (2), so that the absorptive part dominates, i.e.,

$$\lim_{s\to\infty}D(s,t)/A(s,t)=0.$$

In contrast, planar graphs have no left-hand cut in s, i.e., $F(s+i0)=F(s-i0)$ for s real and negative, so that as $s\to-\infty$ along the real axis we obtain an explicitly real asymptotic form. If this form is valid for all ways of approaching infinity, we always have (for essentially planar graphs)

$$\lim_{s\to\infty}A(s,t)/D(s,t)=0,$$

which, of course, is also obvious from the formulas given in [I]. We note that this behavior of the perturbation amplitudes is compatible with the asymptotic form obtained from Regge poles.[15]

An interesting problem is posed by formula (2) for $b=0$. In that case the coefficient of t in the denominator is $g(x)$ namely the same as in $F(0,t)$ and one might expect that the asymptotic form of $F(s,t)$ and therefore of $A(s,t)$ for $s\to\infty$ has a left-hand cut in t. This would disprove the Mandelstam representation for $F(s,t)$ according to which

$$A(s,t)=\frac{1}{\pi}\int\frac{\rho_{ts}(t',s)}{t'-t}dt'+\frac{1}{\pi}\int\frac{\rho_{us}(u',s)}{u'-4m^2+s+t}du'$$

so that the left-hand cut starts at $t_0=4m^2-u'(s)-s$ and therefore recedes[16] to $-\infty$ as $s\to\infty$. However, the delta function constraints require

$$f_1(x)=f_2(x)=\cdots=f_a(x)=0,$$

[15] P. G. O. Freund and R. Oehme, Phys. Rev. Letters **10**, 199 (1963).

[16] This is the reason for the absence of the left-hand branch cuts in the Regge pole trajectories $\alpha(s)$ and the associated residues $\beta(s)$ for an amplitude satisfying the Mandelstam representation. R. Oehme and G. Tiktopoulos, Phys. Letters **2**, 86 (1962).

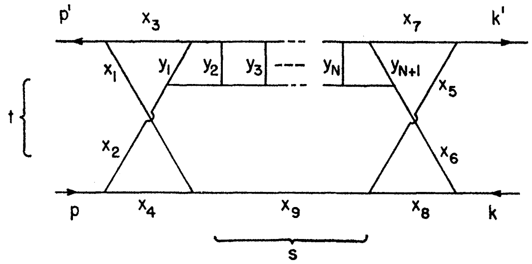

FIG. 6. A class of nonplanar graphs discussed in the text in connection with moving branch points in the λ plane. The lines referred to in the text are labeled by their respective Feynman parameters.

and may remove the left-hand cut from our asymptotic form. Indeed, we have checked by direct computation in the case of the eight-order graph [Fig. 5(a), $N=2$] that we always have $g(x)\leq 0$ under the constraints $f_1(x)=f_2(x)=0$. It seems possible that a general proof can be given based on an induction argument.

Another peculiarity of formula (2) for $b=0$ is that the integrand is not related to any s-independent reduced graph (like the "P-reduced" graph appearing in connection with planar graphs in [I]) and so the integral does not seem to be factorizable by a technique analogous to the one used by Polkinghorne.[2] This means, for example, that the leading asymptotic contributions from the graphs of Fig. 5(a) for $N=2, 3, \cdots$ do not sum up to a simple Regge form in spite of the increasing logarithmic powers. This behavior might be related to the fact that these graphs are generated by the iteration of a "unit" graph having a nonvanishing ρ_{su} spectral function. Furthermore, since in the interval $4m^2<t<9m^2$ this series of graphs satisfies the elastic unitarity condition *exactly* (in the t channel) one certainly expects a manifestation of the λ-plane properties already noticed by Gribov and Pomeranchuk.[5]

Finally, let us consider a certain set of graphs (Fig. 6) which could possibly give rise to moving (t-dependent) branch points in the angular momentum plane and which will provide an application of the technique described in this paper. In [I] we have shown that the iteration of ladder graphs in the s channel does not produce the moving cut in the λ plane that one might expect from the combination of Regge pole terms by means of the elastic unitarity condition.[17] Yet, one might think that this cancellation of the cut is a special feature of planar graphs. The idea is that, loosely speaking, graphs with Regge particles in intermediate states might produce moving branch points in the λ plane (or λ-dependent branch points in the t plane) just as in ordinary dispersion theory the position of many particle-threshold branch points depends on the associated masses.[18,19]

[17] D. Amati, S. Fubini, and A. Stanghellini, Nuovo Cimento **26**, 896 (1962).

[18] R. Oehme and G. Tiktopoulos, Phys. Letters **2**, 86 (1962); and R. Oehme, in *Proceedings of the 1962 Conference on High-Energy Physics at CERN* (CERN, Geneva, 1962), p. 564.

[19] R. Oehme, Nuovo Cimento (to be published) and Phys. Rev. **130**, 424 (1963).

We first notice that the hypersurface $f(x)=0$ has no irregular points in the interior of the integration cube since

$$\partial f/\partial x_9=-(x_1+x_4)y_1y_2\cdots y_{N+1}(x_5+x_8),$$

which vanishes only on the boundary of the cube. On the boundary the shortest t_2 paths (see Sec. III D) consist of just one line namely any one of the y_i's (rungs of the ladder). We write

$$f=(\partial f/\partial y_1)y_1+(x_1x_2-x_3x_4)A(x_5,\cdots,x_9,y_2,y_3,\cdots,y_{N+1}).$$

Introducing ξ variables for the nondefinite factors $x_1x_2-x_3x_4$ and A we obtain a contribution from the pinching of the corresponding contours (see Appendix) of the form

$$\pm\frac{2\pi i}{s}\int\frac{[C(x)]^{l-1}\delta(x_1x_2-x_3x_4)\delta(A)}{[(\partial f/\partial y_1)y_1s+gt+h]^l}\times\delta(\textstyle\sum x_j-1)dx_1dx_2\cdots dy_1dy_2\cdots.$$

On the hypersurface $A=0$ we notice that $\partial f/\partial y_1$ vanishes for $y_2=0$ *or* $y_3=0\cdots$ *or* $y_{N+1}=0$. In fact, one can explicitly use the condition $A=0$ to express $\partial f/\partial y_1$ in the form $x_1x_5y_2y_3\cdots y_{N+1}\psi$. According to the formulas given in the Appendix we obtain an $s^{-2}(\ln s)^{N+2}$ behavior. There remains the interesting question concerning the character of the singularity in the angular momentum plane which corresponds to the sum of these $s^{-2}(\ln s)^{N+2}$ terms. This requires a further study of the t-dependent coefficients.

ACKNOWLEDGMENTS

I wish to express my gratitude to Professor Reinhard Oehme for his help and criticism during the course of this work. I would also like to thank Dr. A. P. Balachandran for interesting discussions.

APPENDIX A

(1) The "technical" reason for the exceptional asymptotic behavior of essentially nonplanar graphs is exhibited by the integral

$$G_2(s)=\int_0^1\int_0^1\frac{dx_1dx_2}{[(ax_1x_2+b_1x_1+b_2x_2+c)s+d]^2}.$$

After the x_2 integration we obtain

$$\int_0^1\frac{dx_1}{[(a+b_1)sx_1+(b_2+c)s+d][b_1sx_1+cs+d]}.$$

The integrand has two poles at

$$x_1'=-\frac{b_2+c}{a+b_1}-\frac{1}{s}\frac{d}{a+b_1},$$

$$x_1''=-\frac{c}{b_1}-\frac{1}{s}\frac{d}{b_1}.$$

Clearly, as $|s|\to\infty$ these two poles pinch the real x_1 contour if and only if

$$-1<\frac{b_2+c}{a+b_1}=\frac{c}{b_1}<0 \quad\text{and}\quad (a+b_1)b_1<0.$$

This condition is equivalent to

$$ax_1x_2+b_1x_1+b_2x_2+c=a^{-1}(ax_1+b_2)(ax_2+b_1),$$

where both factors change sign in the interior of the integration square. We can always replace the real contour by a complex one crossing over one of the poles plus the contribution of that pole:

$$\pm\frac{2\pi i}{s}\frac{1}{(ac-b_1b_2)s+ad};\quad \text{Im}s\gtrless 0.$$

By differentiating $G_2(s)M-2$ times with respect to d we have

(Pole contrib. to $G_M(s)$)

$$=\pm\frac{2\pi i}{s}\frac{a^{M-2}}{(M-1)[(ac-b_1b_2)s+ad]^{M-1}}.$$

(2) We shall obtain the asymptotic form of integrals of the form,

$$I_n^{(M)}(s)=\int_{-\epsilon_1'}^{\epsilon_1}\cdots\int_{-\epsilon_n'}^{\epsilon_n}\frac{dx_1dx_2\cdots dx_n}{(x_1x_2\cdots x_ns+A)^M}$$

for $M>1$, $\epsilon_j>0$, and $\epsilon_j'\geq 0$. We have a recursion formula by writing

$$I_n^{(M)}(s)=s^{-1}\int_{-\epsilon_1'}^{\epsilon_1}\cdots\int_{-\epsilon_{n-1}'}^{\epsilon_{n-1}}\int_{-\epsilon_n's}^{\epsilon_ns}\frac{dx_1dx_2\cdots dx_{n-1}dz}{(x_1x_2\cdots x_{n-1}z+A)^M},$$

whence,

$$\frac{d}{ds}[sI_n^{(M)}(s)]=\epsilon_nI_{n-1}^{(M)}(\epsilon_ns)+\epsilon_n'I_{n-1}^{(M)}(-\epsilon_n's).$$

We distinguish the following cases:

(a) $\epsilon_1'=\epsilon_2'=\cdots=\epsilon_n'=0$. Explicit integration gives

$$I_1^{(M)}(s)\underset{s\to\infty}{\longrightarrow}1/(M-1)A^{M-1}s.$$

Using induction and the L'Hospital rule we obtain

$$I_n^{(M)}(s)\to s^{-1}\frac{(\ln s)^{n-1}}{(n-1)!}\frac{1}{(M-1)A^{M-1}}.$$

This was obtained also in [I].

(b) $\epsilon_1'\neq 0$, $\epsilon_2'\neq 0$. $I_2^{(2)}(s)$ is a special case of the integral discussed in the beginning of this Appendix. We have (the pinching condition is satisfied)

$$I_2^{(2)}(s)\underset{s\to\infty}{\longrightarrow}\pm\frac{2\pi i}{s}\frac{1}{A};\quad \text{Im}s\gtrless 0.$$

Repeated differentiation with respect to A yields

$$I_2^{(M)}(s)\underset{s\to\infty}{\longrightarrow}\pm\frac{2\pi i}{s}\frac{1}{(M-1)A^{M-1}};\quad \text{Im}s\gtrless 0.$$

By means of the recursion formula we have

$$\frac{d}{ds}[sI_3^{(M)}(s)]=\epsilon_3I_2^{(M)}(\epsilon_3s)+\epsilon_3'I_2^{(M)}(-\epsilon_3's)\underset{s\to\infty}{\longrightarrow}\pm\frac{2\pi i}{s}\frac{2}{(M-1)A^{M-1}}.$$

L'Hospital's rule gives

$$I_3^{(M)}(s)\underset{s\to\infty}{\longrightarrow}\pm\frac{2\pi i}{s}\ln s\frac{2}{(M-1)A^{M-1}}.$$

Proceeding by induction we finally obtain

$$I_n^{(M)}(s)\underset{s\to\infty}{\longrightarrow}\pm\frac{2\pi i}{s}\frac{(\ln s)^{n-2}}{(n-2)!}\frac{2^{\alpha-2}}{(M-1)A^{M-1}},$$

where α is the number of ϵ''s different from zero.

(c) $\epsilon_1'\neq 0$, $\epsilon_2'=\epsilon_3'=\cdots=\epsilon_n'=0$, $n>1$. There is no pinching. Proceeding as in (a), we find

$$I_n^{(M)}(s)\underset{s\to\infty}{\longrightarrow}s^{-1}\frac{(\ln s)^{n-2}}{(n-2)!}\frac{1}{(M-1)A^{M-1}}\int_{-\epsilon_1'}^{\epsilon_1}\frac{dx_1}{x_1},$$

where the x_1 integration is done *around* the pole at $x_1=0$ (above or below $x_1=0$ for $\text{Im}s\gtrless 0$).

(d) $\epsilon_1'\neq 0$, $n=1$. There is no pinching. As $s\to\infty$ we have

$$I(s)=\int_{-\epsilon_1}^{\epsilon_1}\frac{dx}{(xs+A)^M}\to s^{-M}\int_{-\epsilon_1}^{\epsilon_1}\frac{dx}{x^M},$$

where the integration is done above or below the pole at $x=0$.

(3) Finally, we shall obtain the asymptotic form of the integral

$$I(s)=\int_0^1\int_{-\epsilon_1}^{\epsilon_1}\int_{-\epsilon_2}^{\epsilon_2}\frac{\lambda^{d-1}d\lambda d\xi_1d\xi_2}{[(\lambda a+\xi_1\xi_2)s+A]^M}.$$

Clearly, the leading contribution comes from the pinching of the ξ_1 and ξ_2 contours. According to the discussion given at the beginning of this Appendix the corresponding pole contribution is

$$\pm\frac{2\pi i}{s}\frac{1}{M-1}\int_0^1\frac{\lambda^{d-1}d\lambda}{(\lambda as+A)^{M-1}}.$$

By differentiating

$$\int_0^1 \frac{d\lambda}{(\lambda as+A)^{M-1}} \underset{s\to\infty}{\longrightarrow} s^{-1}\frac{1}{(M-2)aA^{M-2}}$$

$(d-1)$ times with respect to a we obtain

$$I(s) \underset{s\to\infty}{\longrightarrow} \pm\frac{2\pi i}{s^{d+1}}\frac{(d-1)!(M-d-2)!}{(M-1)!}\frac{1}{a^dA^{M-d-1}}.$$

APPENDIX B

We shall show that graphs with 4 external lines are definite [i.e., have a definite $f(x)$] if and only if they are essentially planar.

It is easy to see that essentially planar graphs are definite. The nonvanishing $(l-1)$th derivatives of $f(x)$ correspond to graphs with one loop obtained by striking out $l-1$ lines from the given graph. But all these one-loop graphs are planar if our original graph is essentially planar and so they have definite f's of the same (negative) sign. Therefore, $f(x)$ must be definite.

It is interesting to realize that the same rather trivial argument proves directly that *all* graphs with 3 or 2 external lines are definite simply because the one-loop graphs with 3 or 2 external lines are planar.

We now turn to the converse. We shall need the following lemma.

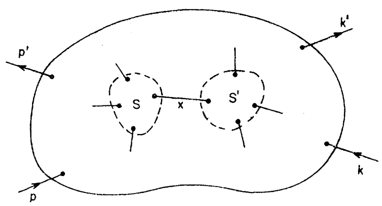

FIG. 7. A schematic representation of the skeleton graph of the lemma.

Lemma. A skeleton graph always contains a line which is *not* one of the external lines of some subgraph with 4 external lines and nonzero number of loops.

Proof. The lemma is evidently true for skeleton graphs with $l=1$. We proceed by induction. We assume that the lemma is true for skeleton graphs whose number of loops is less than l. Let G be a skeleton graph with l loops and x one of its lines such that $G-x$ is not weakly connected; in other words, x does not belong to any pair of lines representing a two-particle intermediate state in any channel. Of course $G-x$ may not be a skeleton graph. Let the maximal vertex parts in $G-x$ be S and S' as shown in Fig. 7. If both S and S' have zero number of loops, then $G-x$ is a skeleton graph and x is the required line. If this is not the case, the skeleton of $G-x$ is obtained by shrinking S and S' to points. Since $G-x$ is not weakly connected, its skeleton must have at least one loop and so it contains at least one line, y say, (not coinciding with any of the external lines of S or S') with

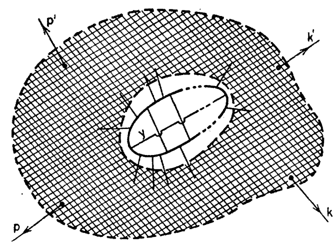

FIG. 8. A schematic representation of the skeleton graph of the theorem. The shaded part is strictly planar.

the required property. It is straightforward to verify that y retains this property in G.

Theorem. Definite graphs are essentially planar.

Proof. It suffices to prove it for skeleton graphs. We use induction again. The theorem is true for graphs with one loop. We assume that it is true for graphs whose number of loops is less than l. Let G be a definite skeleton graph with l loops. Consider a line y with the property of the lemma. $G-y$ is definite and has $l-1$ loops; therefore, it is essentially planar. Furthermore, because of our choice of y, $G-y$ must be *strictly* planar. Thus G is obtained by adding a line to a strictly planar graph. If G were nonplanar, it would be of the form shown in Fig. 8, where the shaded part is strictly planar and y is assumed to be drawn with the least number of crossings. We distinguish two cases:

(i) The loop circumscribing the crossed lines includes all four external vertices; clearly, by removing a number of lines from such a graph we obtain a nondefinite graph. Therefore, this case is excluded since G is definite.

(ii) One of the external vertices p' say does not lie on the loop circumscribing the crossed lines. Let a and b be the two internal lines meeting at that vertex and S_a and S_b the maximal vertex parts in $G-a$ and $G-b$, respectively, [Fig. 9(a)]. One can readily verify that S_a and S_b are disjoint because G is a skeleton graph. Thus, y must be disjoint from at least one of these vertex parts S_a say. In that case the skeleton of $G-a$ (obtained by removing a and replacing S_a by a point vertex) is planar only if y can be drawn as shown in Fig. 9(b) by a dashed line not crossing any line (a is removed). This means that only *one* line is crossed by y in G which is impossible because then S_a would not be maximal. We conclude that G is planar and the induction is completed.

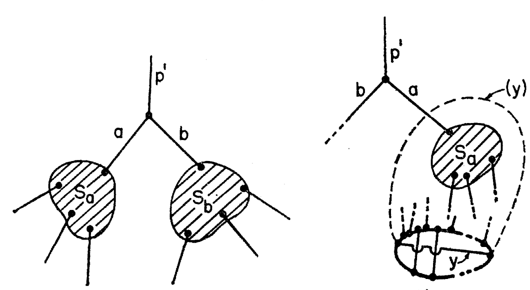

FIG. 9. Illustration of two steps of the proof of the theorem.