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Soluble Problems in the Scattering from Compound Systems*

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The problem of making an exact theory of the scattering of particles from composite targets is attacked by introducing elementary particles to represent the composite systems. It is shown that if the only couplings are those between the particle representing the compound and its constituents, soluble linear integral equations, reducible to the Lippmann-Schwinger equation, can be written for the scattering of one of the constituents of the composite system. As examples, an exactly soluble three-dimensional three-body problem based on nucleon-deuteron scattering and an exactly soluble three-dimensional model of deuteron stripping are presented. Each can be reduced to exact optical models. It is proven that these equations have solutions even when the singular limit which corresponds to an exact resemblance between the elementary and composite system is taken. The method for extending the equations to three-body problems with local interactions and the relation of the equations presented here to high-energy diffraction properties of amplitudes is discussed.

I. INTRODUCTION

WITH the exception of a few special cases, scattering experiments are usually performed with compound systems. That is, at least one of the particles involved in the scattering is a system capable of splitting into other particles either via a production mechanism or bound-state break-up. In spite of the vast body of experimental information assembled on these scatterings and reactions, the theory of them is rudimentary because any analysis goes immediately and essentially beyond the two-body problem. It is true that many ingenious approximate methods have been developed for treating the problem, for example the impulse approximation,[1] the optical model,[2] the distorted-wave Born approximation,[3] the strip approximation,[4] Regge poles,[5] and many others; and it is true than many of these methods work very well in some cases, but their range of validity can only be determined empirically at best and often their connection with more fundamental theory is unclear. The problem is that our inability to solve the three-body problem makes the finding of soluble examples difficult and the nonadiabatic nature of composite systems makes perturbation theory useless.[6] These two difficulties combine to make "exact" numerical computation impossible; that is, no one knows how to give a numerical program, the step by step execution of which can be made to come arbitrarily close to the exact amplitudes.

Some recent theoretical developments offer hope of surmounting some of these problems and this paper is a first foray in that direction. We shall concentrate on the nonrelativistic problem of two-body scattering reactions in which one of the incoming particles is compound. The recent development on which we shall lean most heavily is the substitution of "elementary" particles for composite systems.[7–9] This substitution gets us immediately over the nonadiabatic nature of the composite system by introducing it, or its equivalent elementary particle, into the theory from the beginning for all strength of interaction. What we shall see is that if no more interactions among particles are introduced than those required to couple the elementary particle to the other particles, soluble linear integral equations can be derived for the scattering amplitudes. The solutions of these equations represent exact three-dimensional soluble models of scattering and reactions involving production or break-up. In the limit in which the elementary particle represents a bound state, they yield an exact model of three-body scattering problems such as stripping. These equations can be cast into the form of Lippmann-Schwinger equations[10] or equivalent Schrödinger equations and, hence, are an exact optical model. We shall concentrate on deriving equations of this type, which involve amplitudes off the energy shell rather than the more fashionable equations involving on the energy-shell amplitudes only[11] since the latter involve unitarity and, hence, nonlinear conditions whereas the former are linear. Of course, the amplitudes we obtain will be unitary, as they are exact amplitudes. We wish only to point out that the price of staying on the energy shell is nonlinear relations.

The approach is to take some scattering or reaction amplitude involving compound systems and to introduce an elementary particle for each compound system.

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[1] G. F. Chew, Phys. Rev. **80**, 196 (1950); G. F. Chew and M. L. Goldberger, *ibid.* **87**, 778 (1952).
[2] S. Feshbach, C. E. Porter, and V. F. Weisskopf, Phys. Rev. **96**, 448 (1954).
[3] See W. Tobocman, Phys. Rev. **115**, 928 (1959).
[4] G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **5**, 580 (1960). R. Cutkosky, *ibid.* 4, 624 (1960).
[5] T. Regge, Nuovo Cimento **14**, 951 (1959); see G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **7**, 394 (1961).
[6] R. Aaron, R. D. Amado, and B. W. Lee, Phys. Rev. **121**, 319 (1961).
[7] M. Vaughn, R. Aaron, and R. D. Amado, Phys. Rev. **124**, 1258 (1961). Herein called VAA.
[8] This idea has been applied in R. D. Amado, Phys. Rev. **127**, 261 (1962).
[9] This method has also been proposed by S. Weinberg, Phys. Rev. **130**, 776 (1963).
[10] B. A. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950).
[11] See G. F. Chew, *The S-Matrix Theory of Strong Interactions* (W. A. Benjamin and Company, Inc., New York, 1961).

The minimum couplings between these particles and the others in the system are then introduced to make the reaction under study occur, but no extra interactions or couplings are allowed. This corresponds to constructing an elementary-particle theory which has the same Born approximation as the composite theory. The scattering integral equations are then derived for the simplified theory. They are not solved in this paper, but it is proven that they possess a solution even in the bound-state limit. Since they are no more complicated than the usual integral equations of ordinary potential scattering, the technique for their solution is straightforward. At this stage, the theory is a soluble example of scattering and reactions from compound systems with only the minimum couplings essential to the process. To go further than this requires the introduction into the theory of the interactions and couplings not directly responsible for the composite system but certainly present in physical problems. Since these "residual interactions" do not have bound states, and since all the composite, nonadiabatic effects are accounted for in our solution, it may be possible to treat these interactions as perturbations on the model solution presented here and, hence, develop a consistent, systematic approach to the three-body problem and more complex problems in quantum mechanics.[12] We hope to make an analysis of this possibility in a subsequent paper. In this one we present only the method of obtaining soluble, model three-body problems.

In Sec. II we present a model of nucleon-deuteron scattering which has the same Born approximation as ordinary nucleon-deuteron scattering. We show there how treating the deuteron as elementary allows us to write down a perturbation expansion for the amplitude and then sum that expansion into an integral equation. This equation is cast into an exact optical model for the scattering. In Sec. III it is proved that, in the singular bound-state limit, the equation derived in II has a solution. Those with faith may omit Sec. III. Section IV treats a model of deuteron stripping by the methods of Sec. II. Soluble equations are obtained for elastic deuteron-nucleus scattering and for stripping. It should be noted that all these models are full three-dimensional models producing scattering in all partial waves. Section V discusses the results and points to a number of questions left open. In particular, it discusses further the question of the effect of the residual interactions and of the connection of these results with questions of analyticity of amplitudes and, in particular, of diffraction scattering at high energy. The derivation of some Green's functions is presented in Appendix I and a singular model for which the bound-state limit is difficult to take is presented in Appendix II.

[12] The idea of treating separately the nonadiabatic features and the rest in perturbation seems first to have been stressed by S. Tani, Phys. Rev. **117**, 252 (1960). This is also stressed by Weinberg, Ref. 9.

II. FORMULATION OF A SIMPLE EXAMPLE: NUCLEON-DEUTERON SCATTERING

As a simple example, we consider the scattering of a spinless particle by a composite of two such particles. One can think of this as a very simplified version of nucleon-deuteron scattering with only one sort of nucleon, assumed to be a spinless boson. Fermions could also be treated, but they would require the introduction of spin and the accompanying kinematical complications. In accordance with the resemblance, we call the particle n and the bound state D. Of course, if the n-n interaction is an ordinary local potential in which D is a bound state, we cannot reduce the equation for n-D scattering to simple form. The goal is to find a theory that can be handled exactly and the first approximation to which agrees with the first approximation of the local potential theory. This approximation, the first Born approximation for n-D scattering, is represented graphically in Fig. 1. It carries with it an amplitude

$$\frac{\gamma_0{}^2 f((\mathbf{n}'+\mathbf{n}/2)^2) f((\mathbf{n}+\mathbf{n}'/2)^2)}{E-\mathbf{n}^2-\mathbf{n}'^2-(\mathbf{n}+\mathbf{n}')^2}, \tag{1}$$

where we have taken units in which $\hbar = 2m$ (m is the n particle mass) $=1$. E is the total energy variable, it being recalled that we are, in general, allowing amplitudes to be off the energy shell. The vertex function $f(q^2)$ is related to the Fourier transform of the D bound-state wave function $\tilde{\phi}(q^2)$ by

$$\gamma_0 f(q^2) = (2q^2+\epsilon)\tilde{\phi}(q^2), \tag{2}$$

where ϵ is the D particle binding energy. f is normalized so that $f(q^2=-\epsilon/2)=1$, then $\frac{1}{2}\gamma_0{}^2$ is the residue of (1) at the pole of the denominator on the energy shell. γ_0 corresponds to the invariant strength or reduced width or coupling constant of the process $D\rightarrow 2n$. The factor of $\frac{1}{2}$ is kinematical.

The program is now to find a theory simple enough to be handled that has (1) as its first Born approximation. If we treat D as elementary, that is introduce an independent field for it, then a candidate for this simple model is one which contains only an interaction permitting $D \rightleftarrows 2n$. A discussion of the method for making this substitution and as much of its justification as is known has been given elsewhere.[7,9] So far it has been

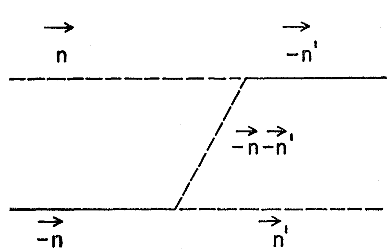

FIG. 1. The Born approximation to n-D scattering. The broken lines represent the n particles and the full lines the D's. The vector labels indicated momenta.

proven that this substitution can be made equivalent to an ordinary potential only for the n-n scattering channel.[13] However, the arguments given in VAA lead us to hope that the elementary-particle theory and bound-state theory are equivalent in all channels. We shall proceed on that assumption. It is most convenient to proceed in a second-quantized formalism. For the n particles we introduce momentum space field operators Ψ_{n} obeying the canonical boson commutation relations. For the D "particle" we introduce a renormalized momentum space field Φ_{D} which obeys the commutation relations

$$[\Phi_{\mathrm{D}},\Phi_{\mathrm{D}'}]=0\,,\quad [\Phi_{\mathrm{D}},\Phi_{\mathrm{D}'}{}^{\dagger}]=\delta_{\mathrm{D},\mathrm{D}'}/Z\,, \tag{3}$$

where Z is the wave function renormalization of the D particle. We assume its bare mass is always chosen so that its renormalized mass gives the proper D binding energy. We assume the fields are normalized in a unit box, but will later pass to the continuous limit. As an interaction Hamiltonian we take

$$\frac{\gamma}{2}\sum_{\mathbf{n},\mathbf{n}'} f\left(\left(\frac{\mathbf{n}-\mathbf{n}'}{2}\right)^2\right)[\Phi_{\mathbf{n}+\mathbf{n}'}\Psi_{\mathbf{n}}{}^{\dagger}\Psi_{\mathbf{n}'}{}^{\dagger}+\Phi_{\mathbf{n}+\mathbf{n}'}{}^{\dagger}\Psi_{\mathbf{n}}\Psi_{\mathbf{n}'}]. \tag{4}$$

γ is the renormalized coupling constant of the theory and the $\frac{1}{2}$ comes in because we take the convention

$$|n,n'\rangle=\frac{1}{\sqrt{2}}\Psi_n{}^{\dagger}|n'\rangle\,, \tag{5}$$

so that states are properly normalized. This interaction gives (1) as a Born approximation for n-D scattering, except that γ replaces γ_0. As outlined in VAA, the theory defined by (4) has meaning for all γ between 0 and γ_0. In the singular limit $\gamma=\gamma_0$, the theory yields the same predictions as a theory in which the D is a pure bound state in a potential between n particles separable in momentum space.[14] That is a theory in which the n-n potential gives the D bound state exactly but gives n-n scattering in S states only. In that limit we have $Z=0$. The point is not whether such a theory is a good approximation to actual nucleon-nucleon scattering, but rather whether we can solve n-D scattering in this model. If one wishes to study a local n-n potential, it is necessary to add to (4) the difference between this local potential and the separable potential. Since this difference has no bound state, it may be possible to develop a consistent perturbation expansion for n-D scattering in powers of that difference using the solutions with (4) alone as an unperturbed basis. We will discuss this point further in Sec. V, but the first order of business, either towards that end, or simply toward the goal of a soluble model, is the solution of n-D scattering with the interaction (4).

The dispersion methods used previously[8] can be used here to derive integral equation for the n-D scattering amplitude, but a direct derivation, via a diagrammatic perturbation expansion is simpler. Such a derivation in the bound-state limit is highly suspect, but the assumption is that γ in (4) can be made arbitrarily small, so that an expansion is valid and then, when the series is resummed to give an integral equation, γ can be made large. The first Born approximation for the amplitude is represented graphically in Fig. (1). Further approximations can easily be written down recalling that all that can happen in this theory is $D\rightleftarrows 2n$, so that any internal D line must first split into two n's. One of these n's can go across and form a D with the third n, giving a "rung" in a "ladder" graph, or the n can recombine with the original n, giving a "bubble" in the D propagator. With this in mind, we may write for the n-D amplitude the sum of graphs shown in Fig. 2. The top line of the figure represents the sum of all ladder-type interactions, under each of which we indicate the sum of all possible bubble-type insertions on internal D lines. These bubbles can be summed into a full D propagator represented by a heavy line as in Fig. 3. Putting this into Fig. 2 we get just a standard sum of ladders, with each internal D propagator represented by a heavy line. This can be summed to a standard integral equation, which is a kind of exact Bethe-Salpeter[15] equation for n-D scattering, as in Fig. 4. This figure

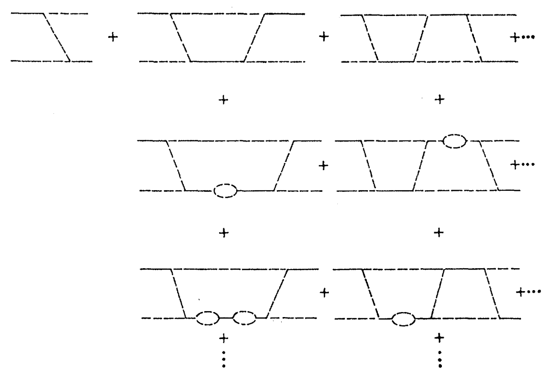

FIG. 2. The sum of graphs for n-D scattering, broken lines for n's and full lines for D. The external lines are indicated only to show what comes in and goes out, but are not included in the definition of the amplitude.

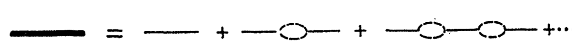

FIG. 3. The sum of "bubbles" for the full D propagator.

[13] This equivalence is proven in general in the Appendix of VAA. It is also the content of the equivalence theorem of Weinberg, Ref. 9.

[14] The fact that some three-body problems, particularly stationary state problems, are soluble with separable potentials has been exploited by A. N. Mitra, Nucl. Phys. **32**, 529 (1962); Phys. Rev. **127**, 1342 (1962).

[15] E. E. Salpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951).

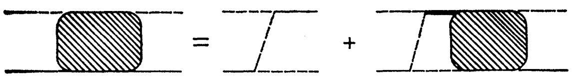

FIG. 4. The integral equation for the n-D amplitude, represented by the cross-hatched "black box."

represents a simple linear integral equation for the amplitude. It is linear since we are allowing the amplitude to go off the energy shell.

The form of the equation is made more substantial if we call the scattering amplitude in the center of mass $(\mathbf{n}'|t(E)|\mathbf{n})$ and the Born term (1), $(\mathbf{n}'|B(E)|\mathbf{n})$. The propagator we need is for an n particle of momentum $\mathbf{n}''$ and energy n''^2 and for a D, with its bubbles, of momentum $-\mathbf{n}''$ and energy $D_{\mathbf{n}''}$. This propagator we write as

$$(E-n''^2-D_{\mathbf{n}''})^{-1}S(E-n''^2-D_{\mathbf{n}''})=G_0S, \qquad (6)$$

where S is the known effect of the bubbles and is normalized so that $S(0)=1$. Its precise form is derived in Appendix I. The energy E is understood to have a small imaginary part of the appropriate sign. Translated to these terms, the diagrammatic equation of Fig. 4 becomes

$$(\mathbf{n}'|t(E)|\mathbf{n})=(\mathbf{n}'|B(E)|\mathbf{n})+\frac{1}{(2\pi)^3}\times\int\frac{d^3n''(\mathbf{n}'|B(E)|\mathbf{n}'')S(E-n''^2-D_{\mathbf{n}''})(\mathbf{n}''|t(E)|\mathbf{n})}{E-n''^2-D_{\mathbf{n}''}}, \qquad (7)$$

which is a linear integral equation for t very similar to the Lippmann-Schwinger equation.[10] In fact, we may transform it to that equation. Let us write the equation formally as

$$t=B+BG_0St. \qquad (8)$$

Since G_0 and S commute, we may write

$$t=B+BS^{1/2}G_0S^{1/2}t. \qquad (9)$$

If we now define

$$t'=S^{1/2}tS^{1/2} \qquad (10)$$

and

$$B'=S^{1/2}BS^{1/2}, \qquad (11)$$

we have

$$t'=B'+B'G_0t', \qquad (12)$$

which is just the Lippmann-Schwinger equation with B' playing the role of the potential. Since $S(0)=1$, $t'=t$ on the energy shell.

Equation (12), or the corresponding Schrödinger equation into which it can be cast, may be thought of as the exact optical model for n-D scattering. It is an optical model since n-D collisions can lead to D breakup, or to production in the language of an elementary D, and this is exactly taken account in the equation. B is purely real, but S becomes complex at the production threshold. This is to be expected since the production possibility comes from the virtual $D\rightarrow 2n$ process, which is summed in S. The potential B' is nonlocal and energy-dependent and, hence, the solution of the Lippmann-Schwinger equation, or of the corresponding Schrödinger equation is not simple, but it can be found by standard methods, particularly as B' is spherically symmetric, so that a partial-wave decomposition is permitted. That is it can be found provided it exists. It is easy to see that it does for the elementary particle case, but since $S(\infty)=1/Z$, closer attention must be paid to the convergence of integrals, etc., in the bound-state limit for which $Z=0$. The next section is devoted to that problem.

III. THE PROPAGATORS AND SOLUTIONS OF THE EQUATION

We wish to discuss as much as we can of the several properties of the solutions of (7) and, in particular, the question of the existence of these solutions in the bound-state limit. This latter question is most easily studied for the standard equation (7), symmetrization of the kernel, as in Eq. (12), or turning the problem to a differential Schrödinger equation, neither affects nor sheds much new light on this question. From the standard theory of integral equations, we know that we may apply the Fredholm method to (7) and, hence, obtain a solution, so long as the kernel of the equation is square integrable.[16] The kernel of (7) is

$$(\mathbf{n}'|K(E)|\mathbf{n}'')=(\mathbf{n}'|B(E)|\mathbf{n}'')S(E-n''^2-D_{\mathbf{n}''})/(E-n''^2-D_{\mathbf{n}''}), \qquad (13)$$

with $(\mathbf{n}'|B(E)|\mathbf{n}'')$ given in (1) but γ_0^2 replaced by γ^2 and

$$S(x)=\left[1-\frac{\gamma^2x}{2(2\pi)^3}\int\frac{d^3n f^2(n^2)}{(\epsilon+2n^2)^2(x-\epsilon-2n^2)}\right]^{-1}, \qquad (14)$$

as we show in Appendix I. In discussing the boundedness and square integrability of the kernel we need not be concerned with the pole of the propagator since we can give E a finite imaginary part and push the pole off the integration path. Since the kernel then is bounded for all finite argument, the question of square integra-

[16] See R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Interscience Publishing Company, New York, 1953), 1st English ed., pp. 112–153.

bility comes down to the question of the behavior of the kernel for large argument. This would correspond, in the Schrödinger equation formulation, to a study of the singularity of the potential for small distances in configuration space. Since $S(\infty)=1/Z$, the behavior of the kernel for large argument will be very different for an "elementary" $D(Z\neq 0)$ and in the bound-state limit $(Z=0)$. So long as $Z\neq 0$ and f is bounded at infinity, it is easy to see that the kernel is square integrable. Hence, the question is only whether the equation has a solution in the singular limit.

To study this we shall need to know more about the behavior of f for large argument. As we have seen, if we wish D to represent an S-wave bound state with wave function in momentum space $\tilde{\phi}(q^2)$, the appropriate choice for f is given by Eq. (2). The behavior of f for large argument will depend on $\phi(r)$, the bound-state wave function in configuration space, for small r. If the bound state is a state in a potential $V(r)$; and if $rV(r)$ is analytic near $r=0$, then it follows from the general theory of differential equations that $\phi(r)\sim r^s$ $s=0,\ -1$ near $r=0$.[17] $s=-1$ is ruled out as being too singular.[18] Thus, $\phi(r)$ tends to a constant as $r\to 0$, and, of course, decays exponentially for large r. For the Fourier transform we have

$$\tilde{\phi}(q^2)=\frac{4\pi}{q}\int_0^\infty r dr\, \phi(r)\sin qr$$
$$=\frac{4\pi}{q^2}\left[\int_0^\infty dr\,\phi(r)\cos qr+\int_0^\infty r dr\frac{d\phi(r)}{dr}\cos qr\right], \quad (15)$$

where we have integrated once by parts and used $r\phi(r)|_0^\infty=0$. If we assume $\int_0^\infty \phi(r)dr$ exists and that $\phi(r)$ and $d\phi/dr$ have bounded variation, it follows from the Riemann-Lebesgue lemma[19] that the integrals in the second line of (15) are at least of order $1/q$ for large q. Hence, we have that

$$\tilde{\phi}(q^2)=C/q^{3+\eta}, \quad \eta\geqslant 0 \quad (16)$$

for large q, C a constant. For a Yukawa potential, this result can be proved more directly using the representation for the bound-state wave function in a Yukawa potential given by Blankenbecler and Cook.[20]

In general, $\tilde{\phi}(q^2)$ will have a branch point at infinity and (16) is the strongest result we can obtain. If in some special case $\tilde{\phi}(q^2)$ is analytic at infinity, then from (16) and the fact that $\tilde{\phi}(q^2)$ is a function of q^2 only we can get that $\tilde{\phi}(q^2)$ is of order $1/q^4$ for large q^2. An example of this is the Hulthén wave function[21] which has Fourier transform

$$\tilde{\phi}(q^2)=c/(q^2+\alpha^2)(q^2+\beta^2),$$

where c, α^2, and β^2 are constants. For our purposes (16) is sufficiently strong. It implies for f that

$$f(q^2)=C/q^{1+\eta}, \quad \eta\geqslant 0, \quad q \text{ large}. \quad (17)$$

In order to study the kernel at large argument, we must discover the rate at which $Z\to 0$. In the bound-state limit we can write

$$S^{-1}(x)=\frac{\gamma_0^2}{2(2\pi)^3}\int\frac{d^3n\, f^2(n^2)}{(\epsilon+2n^2)(x-\epsilon-2n^2)}. \quad (18)$$

For large x this will go like C/x if f decreases sufficiently rapidly so that

$$\int d^3n\, f^2(n^2)/(\epsilon+2n^2) \quad (19)$$

exists. The bound given in (17) is sufficient to make (19) exist even with $\eta=0$. Hence, for large argument $S(x)$ tends to Cx. The kernel, for large n', n'' then tends to $C(\mathbf{n}'|B|\mathbf{n}'')$, which because of (17) is square integrable even with $\eta=0$. This establishes that Eq. (7) has meaningful solutions even in the bound-state limit, and that they can be discovered, for example, by use of Fredholm methods. It is clear that if the kernel of (7) is sufficiently regular to admit the usual solutions, the potential in the corresponding Schrödinger equation derivable from (12) will also be sufficiently regular. All of this depends on the large argument behavior of the source function. A singular example where we put $f(x)=1$ all x, is discussed in Appendix II.

IV. STRIPPING EXAMPLE

In the previous sections we discussed a simple example of a three-body problem and a method for its analysis. In this section we take our example from a more complex situation—deuteron stripping. Once again, we shall use the real names of the particles, even though they appear in the theory as mere shadows of their true selves.

We consider a typical stripping reaction on a complex nucleus A, $d+A\to p+B$. The Born approximation may be represented graphically as in Fig. 5. As before, our method is to treat all particles as elementary and intro-

[17] See E. T. Whittaker and G. N. Watson, *Modern Analysis* (The Macmillan Company, New York, 1948), American ed., pp. 194–200.
[18] See P. A. M. Dirac, *The Principles of Quantum Mechanics* (Oxford University Press, Oxford, England, 1958), 4th ed., pp. 155–156.
[19] See Ref. 17, pp. 172–174.
[20] R. Blankenbecler and L. F. Cook, Phys. Rev. **119**, 1745 (1960).
[21] L. Hulthén and M. Sugawara, in *Encyclopedia of Physics*, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. 34.

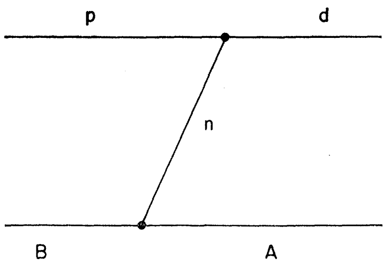

FIG. 5. The Born approximation for $A(d,p)B$.

duce only those interactions among fields necessary for Fig. 5 to occur. These clearly are a coupling which allows $B \rightleftarrows n+A$ and one which allows $d \rightleftarrows n+p$. With the former we associate a renormalized coupling constant Γ and source function $F(n^2)$ and with the latter a renormalized coupling constant γ and source function $f(n^2)$. For kinematical clarity we assume B and A to be fixed at the origin; it is simple to relax that requirement. In terms of these quantities, the amplitude associated with Fig. 5 is

$$\frac{\gamma\Gamma F(n^2)f([(\mathbf{n}-\mathbf{p})/2]^2)}{E-p^2-n^2-\epsilon_A}, \tag{20}$$

where we have put $2m_p=2m_n=\hbar=1$, and where ϵ_A is the energy of nucleus A. It should be noted that we assume no p-A coupling. Introduction of such a coupling seems to complicate the problem to a point where it is no longer possible to write a Schrödinger or Lippmann–Schwinger-like equation for the amplitudes. It should, however, be possible to introduce the p-A interaction in some perturbation sense in a more refined theory so long as there is no p-A bound state. As before, we remind the reader that the point is not to make a good theory of deuteron stripping, or at least not yet; but rather to make an exact theory which has the essential Born approximation of stripping.

We may now write down the equations for the stripping amplitude. The arguments proceed just as Sec. II, and are again most easily presented graphically. The analogy of Fig. 4 in this case is Fig. 6(a). We see that it is not an integral equation for stripping, but rather relates the stripping amplitude to the elastic d-A scattering amplitude. Studying this elastic amplitude in the same way leads to Fig. 6(b). This relates elastic d-A scattering back to stripping. We may eliminate one for the other and get Fig. 7(a) for the stripping amplitude or Fig. 7(b) for elastic d-A scattering. Both can be combined in a "matrix equation" as shown in Fig. 7(c).

As a specific example we take the elastic scattering amplitude. Given this amplitude, the stripping amplitude may be computed as an integral over it by use of the equation implicit in Fig. 6(a). Figure 7(b) may be written

$$(\mathbf{k}|T(E)|\mathbf{k}')=(\mathbf{k}|I(E)|\mathbf{k}')+\frac{1}{(2\pi)^3}\int\frac{d^3k''(\mathbf{k}|I(E)|\mathbf{k}'')S_d(E-\epsilon_A-D_{\mathbf{k}''})(\mathbf{k}''|T(E)|\mathbf{k}')}{E-\epsilon_A-D_{\mathbf{k}''}}, \tag{21}$$

where $\mathbf{k}$ and $\mathbf{k}'$ are the momenta of the in and outgoing deuterons, $D_{\mathbf{k}}$ is the energy of a deuteron of momentum $\mathbf{k}$, ϵ_A the energy of A, and E the total energy variable. S_d is the sum of bubbles for the deuteron and is given by

$$S_d(x)=\left[1-\frac{\gamma^2x}{(2\pi)^3}\int\frac{d^3n\, f^2(n^2)}{(\epsilon_d+2n^2)^2(x-2n^2-\epsilon_d)}\right]^{-1}, \tag{22}$$

where ϵ_d is the deuteron binding energy. The expression (22) may be derived by the methods in Appendix I. The inhomogeneous term $(\mathbf{k}|I(E)|\mathbf{k}')$ is given by

$$(\mathbf{k}|I(E)|\mathbf{k}')=\frac{\gamma^2\Gamma^2}{(2\pi)^3}\int\frac{d^3p\, f((\mathbf{p}-\mathbf{k}/2)^2)f((\mathbf{p}-\mathbf{k}'/2)^2)F((\mathbf{k}-\mathbf{p})^2)F((\mathbf{k}'-\mathbf{p})^2)}{(E-p^2-(\mathbf{k}-\mathbf{p})^2-\epsilon_A)(E-p^2-(\mathbf{k}'-\mathbf{p})^2-\epsilon_A)}\frac{S_B(E-p^2-\epsilon_B)}{E-p^2-\epsilon_B}, \tag{23}$$

where ϵ_B is the energy of a B particle and S_B is the sum of bubbles for a B particle. The appropriate form in this case is

$$S_B(x)=\left[1-\frac{\Gamma^2x}{(2\pi)^3}\int\frac{d^3nF^2(n^2)}{(\epsilon_B-\epsilon_A-n^2)^2(x-n^2+\epsilon_B-\epsilon_A)}\right]^{-1}. \tag{24}$$

We may write (21) formally as

$$T=I+IG_0S_dT, \tag{25}$$

which can again be cast into the form of a Lippmann–Schwinger equation by defining $T'=S_d^{1/2}TS_d^{1/2}$ and $I'=S_d^{1/2}IS_d^{1/2}$ giving

$$T'=I'+I'G_0T'. \tag{26}$$

In this example now I' plays the role of the exact optical potential for the theory. It is more complicated than the corresponding potential in the n-D example. In particular, the richness of channels is reflected by the fact that I' can go complex in many more ways; by the possibility of stripping reflected in the propagator in I, by the possibility of $B \to n+A$ reflected in S_B in I, and by the possibility of $d \to n+p$ reflected in S_d in I'. The appropriate thresholds may be read off of (23), (24), and (22). Of course, the potential defined by (23) is nonlocal and energy-dependent, but it is spherically symmetric. The proof that solutions of (21) exist even in the bound-state limit follows as in Sec. III and again depends on the large argument behavior of the source functions f and F.

V. DISCUSSION

We have seen that by treating composite systems as elementary, we are able to derive linear integral equations for the scattering of particles off these systems, which equations have solutions, even in the singular limit in which the elementary particle represents the composite system exactly. These equations can be cast into the form of a two-body Schrödinger equation in which the three-body effects are exactly taken into account in a nonlocal, but spherically symmetric, optical potential. In this model, in which the only interactions are those which are needed to form the composite systems, the equations allow an exact solution of a three-dimensional three-body problem and as such are a useful model of a number of physical situations.

The success of the method lies essentially in the fact that by introducing an elementary particle for the composite system, we are explicitly taking account of the nonadiabatic effects of the interaction. These nonadiabatic effects are seen most clearly in an attempt to

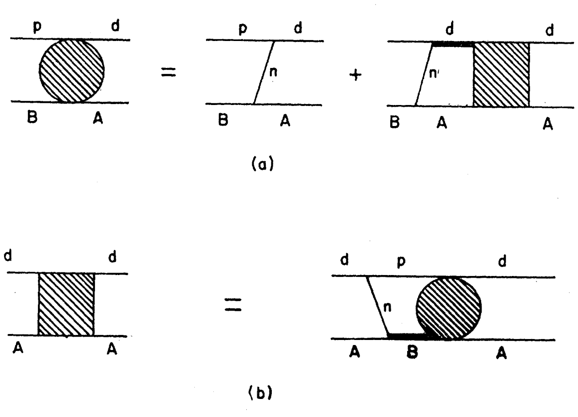

FIG. 6. (a) The relation between the stripping amplitude, represented by the round "black box," and elastic d-A scattering, the square "black box." (b) The relation of d-A scattering back to stripping. The thick lines represent the full d and B propagators including bubbles.

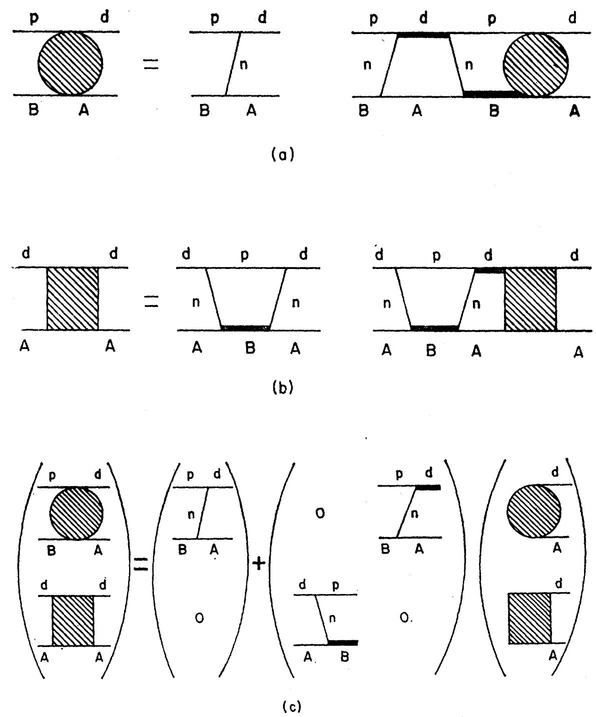

FIG. 7. The integral equation for (a) stripping, and (b) elastic scattering, (c) a combined "matrix" equation for the amplitudes.

apply ordinary perturbation theory to the problem. The existence of an expansion in powers of the potential for the scattering amplitude means that the amplitude changes smoothly (is analytic) as the potential is "turned off." This clearly is not the case if one of the incident particles is a bound state in that potential and hence the Born series must fail. This nonadiabatic behavior of the amplitude is avoided by treating the composite system separately and exactly. One mechanism for doing this is to introduce an elementary particle to substitute for it. The remaining interactions then are the difference between the full interaction and the separable potential responsible for the composite system. This difference has no point spectrum and, hence, may perhaps be legitimately treated adiabatically, that is as a perturbation on a zero-order nuclear state which takes the composite system into account exactly. We only indicate how to solve the later part of that program in this paper, namely the finding of the exact solution for zero-residual interaction. This solution may be viewed as a model, as an approximation to the actual world, or as a first and necessary step to including the residual interaction. We hope to investigate its inclusion in a later paper. The idea of splitting the interaction into a bound-state part and residual part was presented previously by Tani using projection operators.[12] It is clear that the difference between the two-body interaction and the separable potential which has the same bound state is a projection operator off the bound-state manifold. Tani argues, but does not prove, that this can be treated as a perturbation so long as the projection on the bound state is treated exactly but does not show how to construct the operators. The introduction of an elementary particle gives an explicit construction of the projection operators. It is not clear what leeway one has in constructing them, nor is it proven that they in fact represent the bound state in all channels, although this seems intuitively clear.

In our derivations, discussion, and nomenclature we have concentrated on the problem of the scattering from composite systems which we represent by elementary particles. It is clear that the method works just as well, if not better, if we are interested in the scattering from a "real" elementary particle coupled to the other particles as in (4). The method presented here will allow that problem also to be reduced to an exact optical model taking production, etc. into account. In fact, it makes observable predictions about the difference between the elementary particle case ($Z\neq 0$) and the composite particle $Z=0$. Because of the fact that $S(\infty)=1/Z$, the high-energy behavior of the amplitudes satisfying the equations should be quite different if $Z\neq 0$ or if $Z=0$. For example, in the large E limit in the n-D case, the potential B' defined by (11) tends to B/Z which is the unrenormalized Born approximation. Hence, in the high-energy limit when $Z\neq 0$, the potential tends to zero like $1/E$, and probably the first Born approximation, that is the inhomogeneous term in (7) dominates. This term does not contain the factor of $1/Z$ since $S=1$ on the energy shell. If this is true, the scattering from an elementary D particle will tend to the renormalized Born approximation at high energy, even though the potential tends to the unrenormalized Born approximation, but both tend to zero. In the $Z=0$ limit however, the situation is quite different. The first Born approximation on the energy shell still decreases like $1/E$, but the potential does not vanish, since $S\sim E$ for large E when $Z=0$. It is not fair to assume, however, that the potential now approaches

$$\lim_{E\to\infty}[S(E-n^2-D_{\mathbf{n}})]^{1/2}(\mathbf{n}|B(E)|\mathbf{n}')[S(E-n'^2-D_{\mathbf{n}'})]^{1/2} = Cf((\mathbf{n}-\tfrac{1}{2}\mathbf{n}')^2)f((\mathbf{n}'-\tfrac{1}{2}\mathbf{n})^2), \quad (27)$$

where C is a constant, since n'^2 and n^2 are not necessarily small compared with E in the kernel. The precise high-energy limit of the amplitude remains to be elucidated, but it is clear that this limit will be strikingly different if $Z=0$ from the $Z\neq 0$ case. In the $Z\neq 0$ case, the amplitude almost certainly goes to zero at high energies, just as well-behaved potential scattering amplitudes are expected to do. If $Z=0$, however, things are quite different and it is tempting to suppose that some sort of diffraction behavior will set in. The fact that the potential in momentum space does not go to zero at high energy opens the possibility that there may be no shrinking of the diffraction peak.[22] The singular example discussed in Appendix II is an even better candidate for this behavior. Unfortunately, the nonlocal nature of the potential makes straightforward application of the eikonal method difficult, but we hope, in a subsequent paper, to study the high-energy behavior formally or numerically or both. It has been suggested from many points of view that the "fundamental" particles all have $Z=0$, or are in some sense purely composite.[23] This is, in fact, the simplest interpretation one can give to the idea of interaction of maximal strength.[24] It is tempting to hope that light can be cast on the high-energy behavior of the scattering amplitudes of these particles from the analysis presented here.

Another possible extension of these methods lies in the study of the scattering from unstable particles. For example, suppose one wishes to study π-ρ scattering and uses as a model the theory in which the only coupling is one allowing $\rho\rightleftarrows 2\pi$. This is essentially the n-D theory of Sec. II. One can study the equations derived there and analytically continue the mass of the ρ above the 2π threshold so as to make the ρ unstable. Finally, one could study the differences in the π-ρ system for an elementary, unstable ρ and a composite, unstable ρ.

It is not difficult to see that many more questions are opened by the equations presented here. For example, one can ask about the analyticity of the amplitudes derived here. Do they, for example, satisfy a Mandelstam representation?[25] What is the status of subtractions? One can ask about iterative expansions. The Born series presumably does not converge,[6] although a full proof of that is still absent, but perhaps the Neumann series for (7) will converge now that the nonadiabatic effects are summed in the propagators, or at least it might converge for large enough energies. For low energies it probably will be necessary to insert three-particle bound states as well if they exist. The question of their existence can of course be studied by the methods presented here. There is also the question of the convergence of the expansion in the residual interactions already alluded to. If that expansion converges, we will have a systematic approximation scheme for calculation of the three-body problem. Since we have an exact multichannel model, we can also analyze these analyticity questions in the multichannel framework, in particular, with respect to multichannel generalizations of Levinson's theorem and questions relevant to the usefulness of eigenamplitudes, etc. Since we have here an exact model of multiparticle events, we also have the chance to check and comment on approximation

[22] Y. Nambu and M. Sugawara, Phys. Rev. Letters **10**, 304 1963).
[23] This was apparently first suggested privately by R. Feynman to G. F. Chew. See Chew and Frautschi, Ref. 5.
[24] G. F. Chew and S. C. Frautschi, Phys. Rev. **123**, 1478 (1961). See also note in proof of VAA.
[25] S. Mandelstam. Phys. Rev. **112**, 1344 (1958).

schemes. In direct interactions such as stripping, for example, we may be able to explain the amazing success of the distorted-wave Born approximation[3] and to discover the theoretical limits of its applicability.

APPENDIX I. CALCULATION OF PROPAGATORS

We wish to derive (14) for the sum of bubbles. We shall do so in a straightforward, if inelegant, manner by summing the perturbation expansion for the propagation of a D and an n represented graphically in Fig. 8. The unrenormalized propagator for this sum is

$$P^{(u)}=\frac{1}{E-D_{\mathbf{n}}^{(0)}-n^2}+\frac{1}{E-D_{\mathbf{n}}^{(0)}-n^2}I\frac{1}{E-D_{\mathbf{n}}^{(0)}-n^2}+\frac{1}{E-D_{\mathbf{n}}^{(0)}-n^2}I\frac{1}{E-D_{\mathbf{n}}^{(0)}-n^2}\times I\frac{1}{E-D_{\mathbf{n}}^{(0)}-n^2}+\cdots, \quad \text{(A1)}$$

where we assume that E has an appropriate imaginary part and where $D_{\mathbf{n}}^{(0)}$ is the bare energy of a D of momentum $\mathbf{n}$. In terms of the bare "binding energy" $\epsilon^{(0)}$ it is given by $D_n^{(0)}=\frac{1}{2}n^2-\epsilon^{(0)}$. The integral I is

$$I=\frac{\gamma^{(u)2}}{2(2\pi)^3}\int\frac{d^3n'\; f^2(n'^2)}{E-n^2-(\mathbf{n}'-\frac{1}{2}\mathbf{n})^2-(\mathbf{n}'+\frac{1}{2}\mathbf{n})^2}, \quad \text{(A2)}$$

where $\gamma^{(u)}$ is the unrenormalized coupling constant. Summing (A1) we have

$$P^{(u)}=(E-D_{\mathbf{n}}^{(0)}-n^2-I)^{-1}, \quad \text{(A3)}$$

the condition that $P^{(u)}$ have a pole at $E=n^2+D_{\mathbf{n}}=n^2+\frac{1}{2}n^2-\epsilon$ that is at the physical D with renormalized binding energy ϵ, gives for $\epsilon^{(0)}$

$$\epsilon^{(0)}=\epsilon-\frac{\gamma^{(u)2}}{2(2\pi)^3}\int\frac{d^3n'\; f^2(n'^2)}{\epsilon+2n'^2}. \quad \text{(A4)}$$

The renormalized propagator must have unit residue at that pole. From the commutation relation (3) we see that the renormalized propagator $P^{(r)}$ is related to $P^{(u)}$ by $P^{(r)}=P^{(u)}/Z$. Combining (A3) and (A4), we may write

$$P^{(r)}=\left[(E-\tfrac{3}{2}n^2+\epsilon)\left(Z-\frac{\gamma^2}{2(2\pi)^3}\times\int\frac{d^3n'\; f^2(n'^2)}{(\epsilon+2n'^2)(E-D_{\mathbf{n}}-n^2-\epsilon-2n'^2)}\right)\right]^{-1}, \quad \text{(A5)}$$

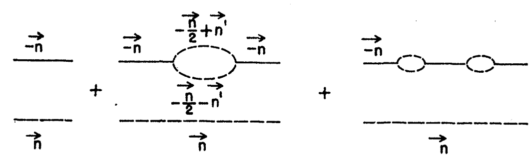

FIG. 8. The sum of graphs for the propagation of an n of momentum $\mathbf{n}$ and a D of momentum $-\mathbf{n}$ including bubbles.

where we have defined the renormalized coupling constant by $\gamma^2=\gamma^{(u)2}Z$. From the condition that the residue be 1, we get

$$Z=1-\frac{\gamma^2}{2(2\pi)^3}\int\frac{f^2(n'^2)d^3n'}{(\epsilon+2n'^2)^2}. \quad \text{(A6)}$$

This condition and the condition $0\leq Z\leq 1$ place the allowable limits on γ. Substituting (A6) into (A5) and noting that $S(x)$ is defined by

$$P^{(r)}(x)=(1/x)S(x), \quad \text{(A7)}$$

we get (14). It also follows from this that $S(\infty)=1/Z$.

APPENDIX II. SINGULAR EXAMPLE

We have seen in Sec. III that the existence of solutions of (7) depends on the behavior of the source function for large argument. In this Appendix, we wish to explore the observation that even if we put $f\equiv 1$, the integrals defining the D particle propagator (14) do not diverge. This is different from the case in relativistic theories or in theories with relativistic kinematics such as the Lee model[26] in which the local limit, $f=1$, is divergent. The difference arises from the fact that the energy goes like the momentum squared nonrelativistically but is linear in the momentum for large momentum relativistically.

The integral for S in (14) is easily done for $f\equiv 1$, and we get

$$[S(x)]^{-1}=1-\Gamma\left(1-\frac{2\epsilon}{x}+\frac{2\epsilon}{x}\left(1-\frac{x}{\epsilon}\right)^{1/2}\right), \quad \text{(A8)}$$

where we have defined $\Gamma=\gamma^2/32\pi(2\epsilon)^{1/2}$ in terms of which the limit $Z\rightarrow 0$ is $\Gamma\rightarrow 1$. In this limit S becomes

$$S(x)=\tfrac{1}{2}[1+(1-x/\epsilon)^{1/2}] \quad \text{(A9)}$$

and the kernel (13) goes to

$$\tfrac{1}{2}\gamma^2\frac{1+[1-(E-n'^2-D_{\mathbf{n}'})/\epsilon]^{1/2}}{[E-n^2-n'^2-(\mathbf{n}+\mathbf{n}')^2](E-n'^2-D_{\mathbf{n}'})}, \quad \text{(A10)}$$

[26] T. D. Lee, Phys. Rev. **95**, 1329 (1954).

which is easily seen not to be square integrable. Of course, if we had not taken the $Z=0$ limit it would be. Thus, when $Z\neq 0$, a solution by Fredholm method is possible, whereas when $Z=0$, it is not. Just what the status of the integral equation and its solution is in that limit is unclear. It would seem "physically" that a scattering amplitude should exist in the bound-state limit, but it may be that the only way to solve (7) with this kernel is to solve the equation with $Z\neq 0$ and then take the limit $Z\rightarrow 0$ in the solution. This point may not be relevant to the ordinary nonrelativistic applications of the theory because of the results of Sec. III, but it may be relevant to more singular applications in relativistic field theory.

Another way of stating the problem of the singularity is in terms of the optical potential defined in (11). When $Z\neq 0$ the potential is perfectly well-behaved, but in the limit $Z\rightarrow 0$ with the form (A9) for S, the potential becomes singular. That is, it is possible to find a class of normalizable wave functions for which the expectation of the Hamiltonian becomes negatively infinite. This arises from the divergence of the momentum space integrals over the potential and corresponds to a potential that is too singular at short distances. This is the case so long as the energy variable which occurs in the potential is finite. It is seen from (A9) and (11), however, that for very large values of this variable, the potential goes like $E^{-1/2}$. Since the potential depends on E, solutions of the problem must be self-consistent. The E in the potential must be the same as the "eigenvalue" of the Hamiltonian. A negatively unbounded E does not satisfy this criterion and, hence, may be excluded, but just how is unclear.

ACKNOWLEDGMENT

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