

APPENDIX

Test for Rate Dependence

The data of the present measurement are plotted versus π^+ stopping rate in Fig. 3. For convenience in display, all runs falling in a given range of abscissa have been statistically weighted and assigned to the center of the interval. A straight-line fit (to the actual data rather than the lumped data mentioned) has been made using,

$$T = aR + T_0,$$

where R is the instantaneous stopping rate in $10^8/\text{sec}$; T_0 is the intercept at zero rate. The results of this calculation are,

$$a = -0.00068 \pm 0.00057$$

$$T_0 = 2.2003 \pm 0.0031.$$

It is interesting to note that if one applies this test to the Chicago data,¹³ one obtains

$$a = +0.00125 \pm 0.00038,$$

$$T_0 = 2.1991 \pm 0.0015.$$

Proton-Proton Bremsstrahlung and the Off-Energy-Shell Behavior of the p - p Interaction*

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The cross section for proton-proton bremsstrahlung at 160-MeV incident energy is calculated using the Yale and the Brueckner-Gammel-Thaler phenomenological nucleon-nucleon potentials. The cross section depends strongly on the off-energy-shell behavior of the T matrix for these potentials. Results for these two potentials differ by a factor of two to three, indicating that bremsstrahlung experiments should be able to distinguish between them.

I. INTRODUCTION

IN recent years extensive experimental effort has gone into obtaining precise measurements of the high-energy proton-proton differential cross section, polarization, and triple scattering parameters, and a number of phenomenological potentials have been found to fit these data.¹ However, scattering experiments of this kind can only determine the asymptotic behavior of the p - p wave function and hence only the behavior of the p - p interaction on the energy shell. Thus different phenomenological potentials which are equally successful in fitting the p - p scattering data can be expected to give quite different results for processes which depend on their off-energy-shell behavior.

The nuclear matter problem is an important example of a problem which requires knowledge of the nucleon-nucleon interaction off the energy shell. Using various phenomenological potentials, Brueckner and Masterson²

obtained quite different values for the binding energy and equilibrium density of nuclear matter, depending on the potential used. This work shows that we are still far from a complete determination of the nucleon-nucleon interaction.

For such a determination it will probably be necessary to investigate more closely those simple processes in which off-energy-shell effects are expected to be important. Everett^{3,4} has studied off-energy-shell effects in inelastic quasifree proton-deuteron scattering. However just at the quasifree peak the kinematics are such as to minimize the amount of energy nonconservation in the scattering matrix.⁵ Thus he found the effect was small and essentially masked by uncertainties in the calculation of multiple scattering effects.

The natural process to investigate for off-energy-shell effects is nucleon-nucleon bremsstrahlung, since multiple scattering corrections are expected to be small compared to off-energy-shell effects. In this paper calculations are made of the p - p bremsstrahlung cross section using two of the potentials studied by Brueckner

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¹ B. P. Nigam, *Rev. Mod. Phys.* **35**, 117 (1963). This is the most recent review article on the status of the nucleon-nucleon interaction and references to the literature can be found here.

² K. A. Brueckner and K. S. Masterson, Jr., *Phys. Rev.* **128**, 2267 (1962).

³ A. Everett, Ph.D. thesis, Harvard University, 1960 (unpublished).

⁴ A. Everett, *Phys. Rev.* **126**, 831 (1962).

⁵ A. Cromer, *Phys. Rev.* **129**, 1680 (1963).

and Masterson: the Yale potential⁶ and the Brueckner-Gammel-Thaler (BGT) potential.⁷ Ashkin and Marshak⁸ originally found that the p - p bremsstrahlung cross section is identically zero if the following approximations are made: the nuclear potential is assumed to depend only on total spin and parity and is treated in Born approximation, the gamma-ray momentum is neglected relative to the proton momentum transfer, and the nuclear recoil is neglected in the energy denominators. In the present calculation none of these approximations are made. The p - p interaction is treated using both tensor and spin-orbit terms and the nuclear recoil is not neglected. There is still a large amount of cancellation between some of the terms which enter into the cross section, so that the p - p bremsstrahlung cross section is probably smaller than the corresponding n - p cross section (see Section IIIC).

In spite of this smaller cross section the presence of two charged particles should make it at least as feasible to measure the p - p cross section as it would be to measure the n - p cross section. Gottschalk⁹ has proposed an experiment using two proton telescopes, placed symmetrically at an angle θ on either side of the incident beam direction, to detect the scattered and recoil proton in coincidence. Each telescope has a detector which measures the energy of each proton. For elastic scattering of 160-MeV protons the angle between the outgoing protons is always 87.5° . If the telescopes are arranged so that $2\theta < 87.5^\circ$, no elastic events will be recorded. In this arrangement the kinematics of a p - p bremsstrahlung event are *overdetermined* (see Sec. IIIA) so that a clean separation of true events from background should be possible.

The calculations in this paper are done with this experiment in mind. The Yale and BGT potentials are found to give values for the cross section which differ by a factor of 2 to 3. It is felt that the eventual measurement of this cross section can provide important information concerning the off-energy-shell behavior of the p - p interaction and enable a distinction to be made between various potentials on this basis.

II. THEORY

The Hamiltonian for two protons with momenta p_1 and p_2 , respectively is

$$H_N = H_0 + V_N = (p_1^2/2M) + (p_2^2/2M) + V_N. \quad (1)$$

Here V_N represents one of the phenomenological nucleon-nucleon potentials which will be used in this calculation. A nonrelativistic Hamiltonian is used here because these V_N have all been determined by use of

⁶ K. E. Lassila, M. H. Hull, Jr., H. M. Ruppel, F. A. McDonald, and G. Breit, *Phys. Rev.* **126**, 881 (1962).

⁷ K. A. Brueckner and J. L. Gammel, *Phys. Rev.* **109**, 1023 (1958).

⁸ J. Ashkin and R. E. Marshak, *Phys. Rev.* **76**, 989 (1949), *ibid.* **76**, 58 (1949).

⁹ B. Gottschalk (private communication).

the nonrelativistic Schrödinger equation. The perturbation which produces a gamma ray of momentum \mathbf{K} , energy ω , and polarization \mathbf{e} , is described by the vector potential $\mathbf{A} = a\mathbf{e} \exp(i\mathbf{K} \cdot \mathbf{r} - i\omega t)$.¹⁰ The normalization factor a is given by $(8\pi c^2/\omega)^{1/2}$ and comes from the requirement that the total field energy be ω . The total Hamiltonian is now

$$H = \{[\mathbf{p}_1 - (e/c)\mathbf{A}(\mathbf{r}_1, t)]^2/2M\} + \{[\mathbf{p}_2 - (e/c)\mathbf{A}(\mathbf{r}_2, t)]^2/2M\} + V_N - (e/2Mc)\mu_p\{\sigma_1 \cdot [\nabla_1 \times \mathbf{A}(\mathbf{r}_1, t)] + \sigma_2 \cdot [\nabla_2 \times \mathbf{A}(\mathbf{r}_2, t)]\} = H_0 + V_{em} + V_N, \quad (2)$$

where

$$V_{em} = -(e/Mc)\{\mathbf{p}_1 \cdot \mathbf{A}(\mathbf{r}_1, t) + \frac{1}{2}i\mu_p\sigma_1 \cdot [\mathbf{K} \times \mathbf{A}(\mathbf{r}_1, t)] + \mathbf{p}_2 \cdot \mathbf{A}(\mathbf{r}_2, t) + \frac{1}{2}i\mu_p\sigma_2 \cdot [\mathbf{K} \times \mathbf{A}(\mathbf{r}_2, t)]\}. \quad (3)$$

Here σ_1 and σ_2 are the Pauli spin matrices for the two protons, and μ_p is the magnetic moment in nuclear magnetons (2.793). The A^2 terms have been omitted in accordance with the usual rules of quantum electrodynamics.¹¹

Although we shall treat the electromagnetic potential only to first order, we wish to treat the nuclear potential V_N , exactly. If T_N is defined to be the exact scattering matrix which would correspond to the Hamiltonian H_N , then the complete T matrix for the Hamiltonian H can be written, using the results of Lippmann,¹² in terms of T_N and V_{em} . The result to lowest order in V_{em} is

$$T = T_N + V_{em} + T_N G_0 V_{em} + V_{em} G_0 T_N + T_N G_0 V_{em} G_0 T_N. \quad (4)$$

G_0 is the free particle propagator, $(E + i\epsilon - H_0)^{-1}$. The term T_N on the right side of Eq. (4) represents proton-proton scattering without photon emission and is not of interest to us. The term V_{em} represents photon emission, by a free proton, which, of course, is kinematically impossible. The term $T_N G_0 V_{em}$ represents a photon emission followed by p - p scattering and the term $V_{em} G_0 T_N$ represents p - p scattering followed by photon emission. These are the main terms which we will calculate. The last term in Eq. (4) represents a process in which the nucleons interact both before and after the gamma ray is produced. This term is neglected in the present calculation, though an estimate of it will be given later. Thus, the T matrix of interest to us now, is

$$T' = T_N G_0 V_{em} + V_{em} G_0 T_N. \quad (5)$$

Since V_{em} as given by Eq. (3) has one term for each proton, T' consists of four terms altogether. These are calculated by inserting complete sets of intermediate states of H_0 between the operators in Eq. (5). The result, for protons of initial momenta \mathbf{p}_1 and \mathbf{p}_2 , and

¹⁰ We here use a system of units in which $\hbar = 1$.

¹¹ R. P. Feynman, *Quantum Electrodynamics* (W. A. Benjamin, Inc., New York, 1961).

¹² B. Lippmann, *Ann. Phys. (N. Y.)* **1**, 113 (1957).

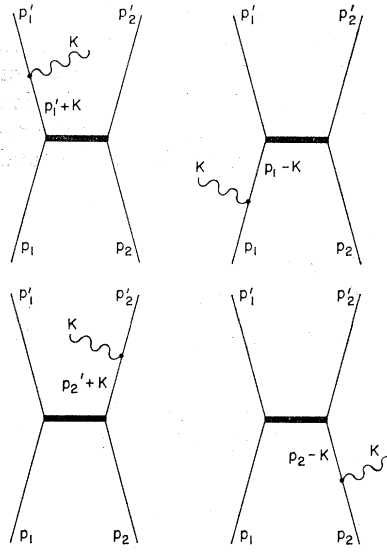


FIG. 1. Diagrams representing the four processes of photon emission considered. The heavy bars represent the matrix elements of T_N [see Eq. (6) of the text].

final momenta \mathbf{p}_1' and \mathbf{p}_2' , is

$$\begin{aligned}
 & \langle \mathbf{p}_1', \mathbf{p}_2' | T' | \mathbf{p}_1, \mathbf{p}_2 \rangle \\
 &= -\frac{ae}{Mc} e^{-i\omega t} \left[\frac{\mathbf{p}_1 \cdot \mathbf{e} + \frac{1}{2} i \mu_p \sigma_1 \cdot (\mathbf{K} \times \mathbf{e})}{E(\mathbf{p}_1) + E(\mathbf{p}_2) - E(\mathbf{p}_1' + \mathbf{K}) - E(\mathbf{p}_2')} \right. \\
 & \quad \times \langle \mathbf{p}_1' + \mathbf{K}, \mathbf{p}_2' | T_N | \mathbf{p}_1, \mathbf{p}_2 \rangle + \langle \mathbf{p}_1', \mathbf{p}_2' | T_N | \mathbf{p}_1 - \mathbf{K}, \mathbf{p}_2 \rangle \\
 & \quad \times \frac{\mathbf{p}_1 \cdot \mathbf{e} + \frac{1}{2} i \mu_p \sigma_1 \cdot (\mathbf{K} \times \mathbf{e})}{E(\mathbf{p}_1) + E(\mathbf{p}_2) - E(\mathbf{p}_1 - \mathbf{K}) - E(\mathbf{p}_2) - \omega} \\
 & \quad + \frac{\mathbf{p}_2' \cdot \mathbf{e} + \frac{1}{2} i \mu_p \sigma_2 \cdot (\mathbf{K} \times \mathbf{e})}{E(\mathbf{p}_1) + E(\mathbf{p}_2) - E(\mathbf{p}_1') - E(\mathbf{p}_2' + \mathbf{K})} \\
 & \quad \times \langle \mathbf{p}_1', \mathbf{p}_2' + \mathbf{K} | T_N | \mathbf{p}_1, \mathbf{p}_2 \rangle + \langle \mathbf{p}_1', \mathbf{p}_2' | T_N | \mathbf{p}_1, \mathbf{p}_2 - \mathbf{K} \rangle \\
 & \quad \left. \times \frac{\mathbf{p}_2 \cdot \mathbf{e} + \frac{1}{2} i \mu_p \sigma_2 \cdot (\mathbf{K} \times \mathbf{e})}{E(\mathbf{p}_1) + E(\mathbf{p}_2) - E(\mathbf{p}_1) - E(\mathbf{p}_2 - \mathbf{K}) - \omega} \right], \quad (6)
 \end{aligned}$$

where $E(\mathbf{p}) = p^2/2M$. These four terms are represented in Fig. 1. The matrix elements of T_N in Eq. (6) all conserve momentum, but they do not conserve energy. For example, in the first term, $E(\mathbf{p}_1' + \mathbf{K}) + E(\mathbf{p}_2') \neq E(\mathbf{p}_1) + E(\mathbf{p}_2)$, the matrix elements of T' must be antisymmetrized with respect to the two protons. This is identical to antisymmetrizing the matrix elements of T_N .

III. CALCULATIONS

A. Kinematics

In the final state there are three particles with nine degrees of freedom. These are reduced to five by the conservation equations. In the planned experiment, the direction and energy of both protons will be detected,

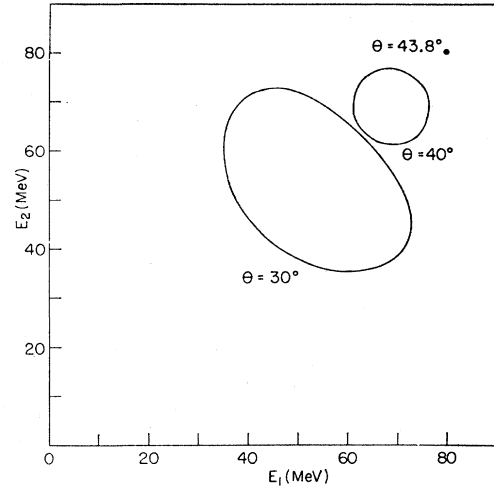


FIG. 2. Kinematics of p - p bremsstrahlung for 160-MeV incident protons. E_1 and E_2 are final energies of protons scattered through angle θ to left and right, respectively. The point at $\theta = 43.8^\circ$ corresponds to elastic scattering.

so we take as the five independent parameters the directions of both protons and the energy of one. The kinematic equations then give two solutions for the energy of the second proton. These solutions have been calculated by Gottshalk⁹ and the results are shown in Fig. 2 for an incident energy 160 MeV, where the final energy of proton 2 is plotted against that of proton 1 (all in the laboratory system).

The calculation was done for the special case where the incident momentum and the final momenta of both protons are all coplanar. (This restriction is not necessary, but was chosen for experimental convenience.) Proton 1 is scattered to the left through angle θ and proton 2 to the right through the same angle θ . Figure 2 shows the results for $\theta = 30$ and 40° . The angle corresponding to elastic scattering, with no gamma rays, is $\theta = 43.8^\circ$. Each point on a closed curve in Fig. 2 corresponds to a definite gamma-ray energy and direction. Gamma-ray energies range from 34 to 74 MeV for $\theta = 30^\circ$ and from 12 to 32 MeV for $\theta = 40^\circ$.

B. Calculation of T_N Matrix Elements

The matrix elements of T_N are most simply calculated in the center-of-mass system. The T_N matrix is obtained from the scattering matrix M by multiplying the latter by simple kinematic factors, including a momentum conserving delta function; and the M matrix is given by

$$M = \langle \mathbf{k}_f | V_N | \psi_{\mathbf{k}_i} \rangle. \quad (7)$$

Here $\langle \mathbf{k}_f |$ is a plane-wave state with the c.m. momentum \mathbf{k}_f and $|\psi_{\mathbf{k}_i}\rangle$ is an exact scattering state, a solution of the Schrödinger equation with incident momentum \mathbf{k}_i . For ordinary elastic scattering, $|\mathbf{k}_i| = |\mathbf{k}_f|$

and M can be computed in terms of phase shifts¹³ without doing the integral over V_N implied in Eq. (7). Invariance requirements restrict M to the form

$$M = A + B(\boldsymbol{\sigma}_1 \cdot \hat{n})(\boldsymbol{\sigma}_2 \cdot \hat{n}) + C(\boldsymbol{\sigma}_1 \cdot \hat{n} + \boldsymbol{\sigma}_2 \cdot \hat{n}) \\ + E(\boldsymbol{\sigma}_1 \cdot \hat{p})(\boldsymbol{\sigma}_2 \cdot \hat{p}) + F(\boldsymbol{\sigma}_1 \cdot \hat{q})(\boldsymbol{\sigma}_2 \cdot \hat{q}). \quad (8)$$

Here \hat{n} , \hat{p} , \hat{q} , are unit vectors in the directions $\mathbf{k}_i \times \mathbf{k}_f$, $\mathbf{k}_i + \mathbf{k}_f$, and $\mathbf{k}_i - \mathbf{k}_f$, respectively.¹⁴ The coefficients A , B , C , E , and F are functions of the only two scalars which can be formed from \mathbf{k}_i and \mathbf{k}_f , namely k_i^2 and $\mathbf{k}_i \cdot \mathbf{k}_f$.

We are interested in the matrix elements of M between states for which $|\mathbf{k}_i| \neq |\mathbf{k}_f|$. For such off-energy-shell matrix elements the phase shift series no longer applies. Instead M must be calculated by solving the Schrödinger equation numerically for $\psi_{\mathbf{k}_i}$ and doing the integral over V_N numerically. The result will still have the form of Eq. (8), provided \hat{p} and \hat{q} are redefined as follows: \hat{p} is a unit vector along the line which bisects the vectors \mathbf{k}_i and \mathbf{k}_f , and \hat{q} is a unit vector perpendicular to \hat{p} , in the scattering plane. The coefficients A , B , etc., are now functions of the three scalars k_i^2 , k_f^2 and $\mathbf{k}_i \cdot \mathbf{k}_f$.

The basic equations for calculating M , with V_N containing central, spin-orbit, and tensor terms, and a hard core, have been worked out by Everett.³ His work has been followed with certain modifications.¹⁵ A program was written for the IBM 7090 computer at the MIT Computation Center, to calculate the matrix element M . All states of orbital angular momentum $l \leq 3$ were included in this calculation. Extensive checks were carried out on this program. The numerical integration of the Schrödinger equation, i.e., the calculation of $\psi_{\mathbf{k}_i}$, was checked by comparing the phase shifts obtained using the Gammel-Thaler potential with the published values.¹⁶ The phase shifts all agreed within 1½% except for the singlet- S phase shift which differed by 6%.

The integration over V_N , to find M , was checked by doing it for the special case $|\mathbf{k}_i| = |\mathbf{k}_f|$. In this case the M matrix is given both by Eq. (7) and by the ordinary scattering formulas in terms of the phase shifts.¹³ The latter calculations were done by hand and there was complete agreement between these and the results of integrating over V_N .

The extent to which a matrix element is off the energy shell is measured by the parameter $|\mathbf{k}_i|/|\mathbf{k}_f|$. Typical values of this parameter ranged from 1.6 to 1.9 for $\theta = 30^\circ$, and from 1.2 to 1.4 for $\theta = 40^\circ$.

¹³ H. Stapp, T. Ypsilantis, and N. Metropolis, Phys. Rev. **105**, 302 (1957).

¹⁴ L. Wolfenstein, Ann. Rev. Nucl. Sci. **6**, 43 (1956).

¹⁵ For a hard core of radius r_c there is a finite contribution to the integral in Eq. (7) from the region $0 < r < r_c$. This can be calculated by a simple limiting process.

¹⁶ J. L. Gammel and R. M. Thaler, Phys. Rev. **107**, 291 (1957).

C. Calculation of the Cross Section

The cross section is given in terms of the matrix

$$\mathfrak{T} = \langle p_1', p_2' | T' | p_1, p_2 \rangle \text{ by}$$

$$d\sigma = 2\pi^{\frac{1}{4}} \text{Tr}(\mathfrak{T}^\dagger \mathfrak{T}) \frac{d^3 p_1'}{(2\pi)^3} \frac{d^3 p_2'}{(2\pi)^3} \frac{d^3 K}{(2\pi)^3} \\ \times \delta[E(\mathbf{p}_1) + E(\mathbf{p}_2) - E(\mathbf{p}_1') - E(\mathbf{p}_2') - \omega]. \quad (9)$$

Each of the four terms in \mathfrak{T} can be expressed in terms of the sixteen matrices, 1 , $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$, and $\boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2$. These matrices form an orthonormal basis in the sense that $\text{Tr}(0_j^\dagger 0_i) = 4\delta_{ij}$, where 0_i is any one of these sixteen matrices. Thus $\text{Tr}(\mathfrak{T}^\dagger \mathfrak{T})$ can be expressed as the sum of the absolute squares of sixteen terms, each term being a combination of the amplitudes A , B , C , E , and F (divided by the appropriate energy denominators) for the four kinematically distinct terms in Eq. (6). In choosing the basis matrices, one must take account of the fact that the unit vectors \hat{p} and \hat{q} are different for each of the four terms in Eq. (6).

In the terms arising only from the electric interaction ($\mathbf{p} \cdot \mathbf{A}$) there is a large amount of cancellation: 90% or more for $\theta = 30^\circ$, and 95% or more for $\theta = 40^\circ$. This is to be expected on the basis of the argument of Ashkin and Marshak⁸ described in the Introduction. The magnetic moment interaction ($\boldsymbol{\sigma} \cdot (\mathbf{K} \times \mathbf{A})$) does not have such a cancellation. It will give rise to a zero p - p bremsstrahlung cross section only in the case of a nuclear potential which depends upon only total spin and parity.⁸ Since the actual potentials used have large tensor and spin-orbit parts, there is no tendency for the magnetic moment term to be small.

The magnetic moment interaction is weaker than the electric interaction by a factor $\frac{1}{2}\mu_p K/p$ where p is a typical proton momentum. But the cancellation described above and kinematic factors arising from the average over photon polarization tend to make the magnetic moment contribution larger than the electric contribution to p - p bremsstrahlung. Typically, for $\theta = 30^\circ$, the electric interaction contributes 30%, the magnetic interaction 60%, and the cross term between them 10% to the cross section. For $\theta = 40^\circ$ these numbers are roughly 66, 29, and 10%, respectively.

In the corresponding case of n - p bremsstrahlung there will be no cancellation of the electric terms, while the magnetic moment contribution will be of the same order of magnitude as in the p - p case. For $\theta = 30^\circ$, corresponding to photon energies of order 50 MeV, we would expect the n - p bremsstrahlung cross section to be about four times the p - p cross section. Thus p - p bremsstrahlung may not be negligible compared with n - p as is often assumed. For $\theta = 40^\circ$ and smaller photon energies, the n - p cross section is expected to be more than twenty times the p - p .

IV. RESULTS AND DISCUSSION

The cross section $d\sigma/(d\Omega_1 d\Omega_2 dE_1)$ was calculated, where $d\Omega_1$ and $d\Omega_2$ are the solid angles into which proton 1 and 2, respectively, are scattered; E_1 is the laboratory energy of proton 1. We have used the Yale potential,⁶ and the BGT potential described by Brueckner and Gammel.⁷ The results are shown in Fig. 3 where the cross section is plotted against E_1 .

The shape of these curves including the characteristic divergence at both ends of the energy spectrum, is due primarily to kinematic factors. For a given potential and angle θ , the matrix elements of T_N vary very little.

The result for the Yale potential is consistently smaller than for BGT potential, by a factor ~ 2.5 in the case of $\theta = 30^\circ$, and 1.7 at $\theta = 40^\circ$. This is consistent with the nuclear matter calculations of Brueckner and Masterson² where the Yale potential is found to give significantly weaker binding than the BGT potential.

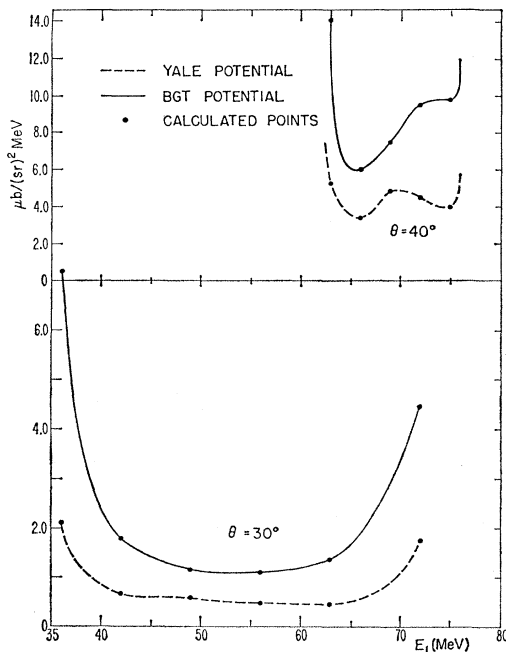


FIG. 3. The p - p bremsstrahlung cross section for 160 MeV incident protons. The curves give results obtained using the Breit potential (Ref. 6) and the BGT potential (Ref. 7).

The last term in Eq. (4), which represents a double-scattering process, has been neglected in this work. Rough preliminary estimates indicate that its square contributes an additional 1.2% to the cross section. Thus, cross terms between it and the main term may introduce corrections of up to 11%. Further work on this point is in progress.

Finally we note that in introducing the vector potential \mathbf{A} in Eq. (2), we neglected possible momentum dependence of the nuclear potential. There is in fact a spin-orbit term $\mathbf{S} \cdot (\mathbf{r} \times \mathbf{p})$ in the potentials used, and by the usual gauge invariance arguments this should be replaced by $\mathbf{S} \cdot \mathbf{r} \times (\mathbf{p} - (e/c)\mathbf{A})$. Since this nonlocality of the potential represents the meson origin of the nucleon-nucleon interaction, the perturbation $-(e/c)\mathbf{S} \cdot (\mathbf{r} \times \mathbf{A})$ represents photons emitted by the intermediate mesons.¹⁷ Preliminary estimates of this term in the T matrix indicate that its square contributes about 0.2% to the cross section.

In conclusion, it appears that in spite of the correction terms that may have to be taken into account in the cross section, a p - p bremsstrahlung experiment should be able to distinguish between the two potentials considered.

ACKNOWLEDGMENTS

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¹⁷ We are indebted to Professor M. Friedman for suggesting this point to us.