Analytic Structure of Partial-Wave Amplitudes for Production and Decay Processes*

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We consider a model theory for studying overlapping final-state interaction effects in 3-body production and decay amplitudes. The model is given in terms of dispersion relations similar to those given by Khuri and Treiman for the process $K \rightarrow 3\pi$. We extend the partial-wave projections into the complex plane, and determine their analytic properties, giving explicitly a set of cuts and discontinuities. These consist of the usual right-hand cut with normal discontinuity, together with a "left-hand" cut for which the discontinuity is expressed as an integral over the projections. The right-hand cut can be factored out in the usual way, and thus one can hope to obtain the solution by iteration for the left-hand cut contribution.

I. INTRODUCTION

THERE has been a great deal of interest recently in the problem of overlapping final-state interactions in production and decay processes. One particular viewpoint is that of Peierls and Tarski[1] in which one uses a model theory. This model is defined by a dispersion relation of the type first proposed by Khuri and Treiman,[2] for the process $K \rightarrow 3\pi$.

In this paper we study the S-wave projection of such an amplitude. We determine its complete analytic structure, and obtain a possible set of cuts and associated discontinuities. We hence obtain a single-variable integral equation for the partial amplitude. This equation is somewhat similar to that found by MacDowell[3] for the partial amplitudes in $\pi+N \rightarrow \pi+N$; there being a right-hand and a left-hand cut. The discontinuity across the right-hand cut is given directly by unitarity, while that for the left-hand cut is given in terms of an integral over the partial amplitude. It is straightforward to factorize out the right-hand cut, and one may therefore hope to obtain a complete solution by iteration.

The organization of this paper is as follows. In Sec. 2 we define the basic model, being rather careful about the definition of "S-wave projection." In Sec. 3 we extend the definition into the whole complex plane, proceeding in several stages. Thus, we study a crucial mapping transformation, make an initial foray into the complex plane in Sec. 3.2, digress briefly into the second sheet in Sec. 3.3, and present the final analytic structure in Sec. 3.4, obtaining a possible set of cuts and discontinuities. Finally, in Sec. 4, we consider the integral equation satisfied by the S-wave projection, and outline in iterative method of solution, based on a factorization of the problem into a right- and a left-hand cut, the right-hand cut having the usual Omnés type solution.

There is also a brief Appendix in which we present an alternative method of expressing our conclusions. (I am very grateful to Tran Truong for suggesting this method.)

2. BASIC FORMULATION

We consider processes of the type $A \rightarrow a+b+c$ or $A+B \rightarrow a+b+c$ in which all the structure of the matrix element M is due to the final-state interactions (fsi) taking place between the final-state particles (which, for simplicity, are assumed to be neutral and have spin 0). Again, for simplicity, we consider only *two*-body S-wave interactions, but assume that these occur between more than one pair.

Let the 4 momenta of the final-state particles be k_1, k_2, k_3, with $K=k_1+k_2+k_3$. Define $s_1=(k_2+k_3)^2=(K-k_1)^2$, etc., with $s_1+s_2+s_3\equiv 3s_0=K^2+m_a^2+m_b^2+m_c^2$, where $k_1^2=m_a^2$, etc. Our assumption that all the structure is due to final-state interactions can now be more precisely stated in the form that, for a *fixed* $K^2=m^2$, the amplitude M depends only on s_1, s_2, s_3. We wish to determine the form of this dependence.

The dispersion relation which defines the model is

$$M(s_1,s_2,s_3)=D+A(s_1)+B(s_2)+C(s_3)\,, \qquad (2.1)$$

with

$$A(s)=\frac{(s-s_0)}{\pi}\int_{(m_b+m_c)^2}^{\infty}\frac{ds'\alpha(s')}{(s'-s-i\epsilon)(s'-s_0-i\epsilon)};\ \text{etc.} \qquad (2.2)$$

Here and throughout "etc." will denote cyclic permutations on (A,B,C), (α,β,γ), (a,b,c), and $(1,2,3)$. The spectral functions are given by

$$\alpha(s)=f_1^*(s)M_1(s)\,,\ \text{etc.}\,, \qquad (2.3)$$

$$f_1^*(s)=\exp[-i\delta_1(s)]\sin\delta_1(s)\,,\ \text{etc.}\,, \qquad (2.4)$$

and

$$M_1(s)=D+A_1(s)+B_1(s)+C_1(s)\,,\ \text{etc.} \qquad (2.5)$$

Here $\delta_1(s)$ is the S-wave phase shift in the two-particle scattering channel $b+c \rightarrow b+c$.[4]

* Work supported, in part, by the U. S. Atomic Energy Commission. An initial account of this work was presented by Tran Truong and the author in Bull. Am. Phys. Soc. 8, 300 (1963).

[1] R. F. Peierls and Jan Tarski, Phys. Rev. **129**, 981 (1963). This paper gives an up to date review of the recent literature on this subject.

[2] N. N. Khuri and S. B. Treiman, Phys. Rev. **119**, 1115 (1960).

[3] S. W. MacDowell, Phys. Rev. **116**, 774 (1959).

[4] At certain points in the following we will have to assume all the δ_i are real; that is, we ignore the competing inelastic channels in f_i. This is reasonable since they are ignored in the model dispersion relation for M.

Finally, we must define the quantities in (2.5). We go to the "1" reference frame in which $\mathbf{K}-\mathbf{k}_1=\mathbf{k}_2+\mathbf{k}_3=0$, and define the angle between $\mathbf{k}_2-\mathbf{k}_3$ and $\mathbf{k}_1$ as θ_{23}. We express s_2 and s_3 in terms of s_1 and $z_1=\cos\theta_{23}$. Then

$$M_1(s)=\tfrac{1}{2}\oint_{-1}^{1} dz_1 M(s_1,z_1) \tag{2.6}$$

and similarly, for $A_1(s_1)$, $B_1(s_1)$, $C_1(s_1)$ [of course $A_1(s_1)=A(s_1)$]. (The circle across the integral sign should be ignored at present, though it will be very important later on.) Thus, M_1, A_1, B_1, C_1 may be though of as being the S-wave projections of M, A, B, C in the reference frame "1." Similarly, M_2, A_2, B_2, C_2 are the S-wave projections taken in the "2" reference frame $\mathbf{K}-\mathbf{k}_2=\mathbf{k}_3+\mathbf{k}_1=0$, and similarly for 3.[5] The definition (2.6) is only meaningful for $(m_b+m_c)^2\leq s_1 \leq (m-m_a)^2$, and has to be given an appropriate meaning by analytic continuation for $s_1>(m-m_a)^2$.

The structure of the model should now be evident, and can be symbolized in the set of dispersion relation diagrams of Fig. 1, in which the double line represents the incoming particle or particles, the broken line is the absorptive part intermediate state, all permutations of a, b, c are to be taken, and also all higher iterations.

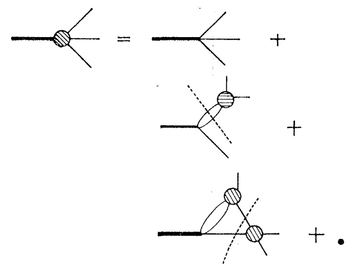

FIG. 1. The basic dispersion theoretic diagrams defining the model.

It is apparent that the restriction to S-wave scattering is not essential.[6] Further, the basic production or decay process can also be made more general, if one replaces D by a suitable polynomial in the s_i, and then makes further subtractions. In most applications only a few partial waves will be necessary, and the generalization of our method should be straightforward. From now on, we formally ignore even the first subtraction, though it will be resuscitated in the final equations.

We wish to determine M from a complete knowledge of the δ_i. M is a function of three variables, and therefore a direct solution will be rather involved (except in certain kinematic limits, such as the static limit[1]). We prefer to convert the problem to a determination of the various S-wave projections. Once these are known, it is straightforward to obtain M. The great advantage of the projections M_i is that they depend on only a single variable.

One can obtain integral equations for the M_i by taking projections of (2.2), at least in the physical region. These equations can then be extended for all *real* s, and so one is led to integral equations in the *real* variable s for the A_i. These can then be solved by iteration.[7]

In this paper we instead extend the definitions of the $M_i(s)$ to all *complex* s, on one Reimann sheet. This sheet is defined with certain cuts, and the discontinuities across these cuts are obtained, leading to a rather different type of integral equation.

In order to simplify the presentation, we now assume that all the final-state masses are equal. Thus, keeping $K^2=m^2$ we set $k_1^2=k_2^2=k_3^2=1$ (this is at least appropriate for three pion final states from K or η decay, or even e^+e^- annihilation[8]).

We consider M_1, the projection in the reference frame 1. One has

$$\begin{aligned} s_2&=G(s_1)+F(s_1)z_1\,,\\ s_3&=G(s_1)-F(s_1)z_1\,, \end{aligned} \tag{2.7}$$

with

$$G(s)=\tfrac{1}{2}(\Sigma-s)\,;\quad \Sigma=m^2+3=3s_0\,, \tag{2.8}$$

and

$$F(s)=\{(s-4)[s-(m-1)^2][s-(m+1)^2]/4s\}^{1/2}. \tag{2.9}$$

Define $s_\pm=G\pm F$, then

$$M_1(s_1)=\frac{1}{2F(s_1)}\oint_{s_-(s_1)}^{s_+(s_1)} ds_2 M(s_1,\,s_2,\,\Sigma-s_1-s_2)\,. \tag{2.10}$$

The limits $s_\pm(s_1)$ are the two s_2 roots of the physical region boundary curve[9]

$$\begin{aligned}\Gamma\equiv s_1s_2s_3-(m^2-1)^2\\ =s_1s_2(m^2+3-s_1-s_2)-(m^2-1)^2=0\,,\end{aligned} \tag{2.11}$$

which is plotted in Fig. 2. The "decay" region is indicated by D, and has $4\leq s_i\leq (m-1)^2$; while the other regions are the related "scattering" regions, e.g., I: $K+k_1\rightarrow k_2+k_3$, $s_1\geq(m+1)^2$, $s_2\leq 0$, $s_3\leq 0$, etc.

Since we certainly require the projections for all $s_1\geq 4$, and in fact will also consider all complex s_1, we define $F(s)$ in the whole complex plane by the cuts and limits shown in Fig. 3. Then $s_+(s)\geq s_-(s)$ for real s in the regions $s\leq 0$ and $4\leq s\leq (m-1)^2$ (the decay region) but $s_+\leq s_-$ for $s\geq (m+1)^2$ (region I). We call the cuts L: $0\leq s\leq 4$, and R: $(m-1)^2\leq s\leq (m+1)^2$, with suffices $\pm$ denoting the upper and lower edges.

[5] The method underlying our notation is that A and α refer to quantities expressed naturally in the s_1 variable appropriate to the two-body $(b+c)$ channel; while A_i is the projection of A relevant to the ith channel. Since f_i and δ_i will only appear in their natural variables the notation for these is straightforward.

[6] Actually the methods of this paper as well as the model can be generalized immediately to deal with higher partial waves. The partial projections for different angular momenta will be coupled, but the basic nature of the equations will not change. In particular, the partial projections will be defined by integrals over paths precisely identical to those we find explicitly for the S-wave projections, the only difference being that the integrand will contain Legendre *polynomials*.

[7] J. Bronzan, thesis, Princeton University, 1963 (to be published).

[8] D. R. Harrington, Phys. Rev. **130**, 2502 (1963).

[9] T. W. B. Kibble, Phys. Rev. **117**, 1159 (1960).

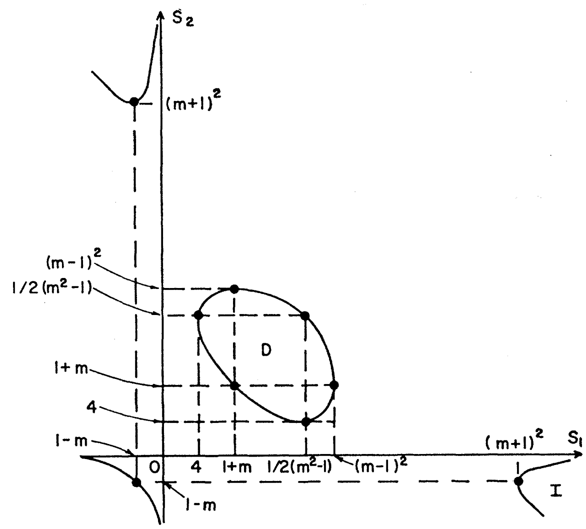

FIG. 2. The physical regions; i.e., the curve $\Gamma(s_1,s_2)=0$.

Now from (2.5) we see that $B_1(s)$ is a typical example of a nontrivial projection, and once $B_1(s)$ is fully understood, the full consideration of the M_i is straightforward. For most of the time we therefore will consider specifically the projection B_1 and only at the end do we turn to M_1. From (2.10) and (2.2) we have (ignoring subtractions)

$$B_1(s_1)=\frac{1}{2F(s_1)\pi}\oint_{s_-(s_1)}^{s_+(s_1)} ds_2\int_4^\infty \frac{ds'\beta(s')}{s'-s_2-i\epsilon}. \quad (2.12)$$

Since the spectral functions β usually correspond to sums over intermediate states and, since B_1 is one such term (cf. 2.3), the natural definition of the projection operation would seem to be a normal average over $z_1=\cos\theta_{23}$, or equivalently over s_2, at least in the decay region itself. Thus, one would expect that an interchange in the order of integration in (2.12) would be permissible, at least in the decay region, i.e.,

$$B_1(s_1)=\frac{1}{2F(s_1)\pi}\int_4^\infty ds'\beta(s')\ln\left[\frac{s'-s_-(s_1)-i\epsilon}{s'-s_+(s_1)-i\epsilon}\right] \quad (2.13)$$

where the log is on its principal sheet, for $4\leq s\leq (m-1)^2$. However, such a definition disagrees with perturbation theory. This question has recently been investigated by Bronzan and the present author.[10] They find, for all real $s_1\geq 4$, that perturbation theory leads to the prescription

$$B_1(s_1)=\frac{1}{2F(s_1)\pi}\int_4^\infty ds'\beta(s')\oint_{s_-(s_1)}^{s_+(s_1)} \frac{ds_2}{s'-s_2}, \quad (2.14)$$

where the integral with circle denotes a contour integral along any path between s_- and s_+ which avoids the real s' axis for $4\leq s'<\infty$. Further, where necessary, $s_\pm(s_1)$ are taken infinitesimally above or below the real axis according to the prescription obtained by replacing $m^2\to m^2+i\delta$, $\delta\to 0+$, with s_1 *real*, in the defining equations (2.8)–(2.11).

These paths are shown in Fig. 4, for the ranges (i) $s_1\geq(m+1)^2$, (ii) $(m+1)^2\geq s_1\geq(m-1)^2$, (iii) $(m-1)^2\geq s_1 \geq\frac{1}{2}(m^2-1)$, and (iv) $\frac{1}{2}(m^2-1)\geq s_1\geq 4$. At $s_1=\frac{1}{2}(m^2-1)$, $s_-(s_1)=4$, so that the transition from (iii) to (iv) takes place in a continuous fashion.[11]

Both (iii) and (iv) belong to the physical-decay region, yet only for (iv) are Eqs. (2.6) (2.10), (2.12), and (2.13) actually correct (the significance of the integral sign with circle is to indicate that the integration must be performed in a specified and nonstraightforward manner). It will turn out that the definition of

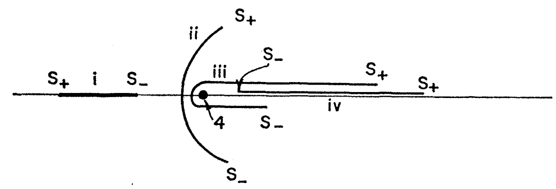

FIG. 4. The paths appropriate to (2.14); the different cases are: (i) $s_1>(m+1)^2$, (ii) $(m+1)^2>s_1>(m-1)^2$, (iii) $(m-1)^2>s_1>\frac{1}{2}(m^2-1)$, and (iv) $\frac{1}{2}(m^2-1)>s_1>4$.

the channel-1 S-wave projection appropriate for the absorptive part is obtained by suitable *analytic* continuation in s_1, for fixed unstable m^2, of the straightforward definition applicable to the scattering process I into the decay region. This prescription therefore has some plausibility, even if it does not agree with the straightforward definition as given in the decay region. Since our dispersion relation is only a model theory, (and no such relation has ever been proved) it might be argued that we are at liberty to define the projection operation in the straightforward way in the decay region. We disagree, and feel that one should *always* follow the dictates of pertubation theory provided they do not lead to meaningless conclusions.[12]

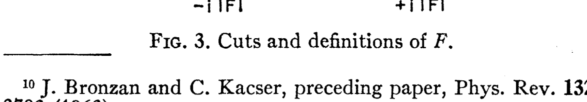

FIG. 3. Cuts and definitions of F.

[10] J. Bronzan and C. Kacser, preceding paper, Phys. Rev. **132**, 2703 (1963).

[11] After the completion of the work described here and in Ref. 10, we were informed by Professor V. V. Anisovich of a paper by himself, A. A. Ansel'm, and V. N. Gribov, Zh. Eksperim. i Teor. Fiz. 42, 224 (1962) [translation: Soviet Phys.—JETP **15**, 159 (1962)] which gives a prescription identical with that given by our Fig. 4, though only for the lowest order triangle graph.

[12] In fact it is possible to proceed from the "straightforward" definition by following the same methods as presented here. One finds a rather more involved analytic structure, in which the "left-hand cut" divides the complex plane into two completely separate regions, there being a cut along the positive real axis to infinity.

Equation (2.14) forms the basis for a set of *real* variable integral equations for the M_i, which can be solved by iteration. In this paper we extend (2.14) for all complex s_1, in the next section.

3. EXTENSION INTO THE COMPLEX PLANE

Equation (2.14) seems eminently suitable for analytic continuation into the s_1 complex plane, since $F(s_1)$ and $s_\pm(s_1)$ are known analytic functions of s_1, and the s' and s_2 integration paths are stipulated to be nonintersecting. However, we have already remarked that for real s_1 in the decay region the location of $s_\pm(s_1)$ relative to the real axis is to be obtained by the prescription $m^2 \to m^2+i\delta$, rather than by either $s_1 \to s_1+i\epsilon$ or $s_1 \to s_1-i\epsilon$. In fact, for $4 \leq s_1 \leq m+1$, neither of the latter two prescriptions agree with the former [and if we were foolhardy enough to follow the straightforward prescription, we would encounter difficulty also for

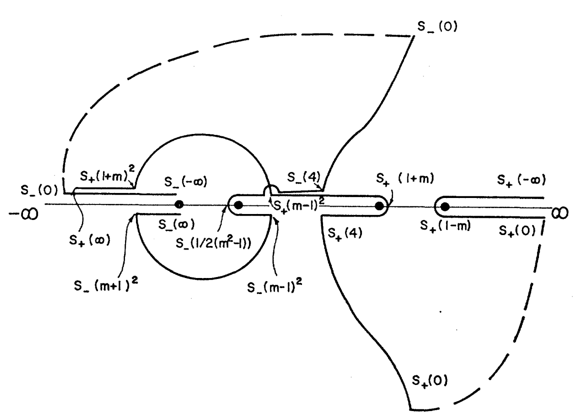

FIG. 5. The section of the mappings $s_\pm(s_1)$ for s_1 moving along the real axis, *below* the F cuts. The dots are at 0, 4, $(m-1)^2$ and $(m+1)^2$.

$\frac{1}{2}(m^2-1) \leq s_1 \leq (m-1^2)$]. It is this which makes the problem more than a trivial generalization of that for scattering amplitudes.

It is clear that the mapping $s_1 \to s_\pm(s_1)$ is fundamental to our problem, so we digress somewhat to present its more important features.

3.1. The $s_\pm(s_1)$ Mappings

The mapping $s_1 \to s_\pm(s_1)$ is given by the two s_2 roots of $\Gamma(s_1,s_2)=0$ [cf. (2.11)]. This is actually symmetric under $s_1 \leftrightarrows s_2$ for the equal mass case. The mapping from the entire real s_1 axis is straightforward, in part being given by Fig. 2. The ranges $0 \leq s_1 \leq 4$ and $(m-1)^2 \leq s_1 \leq (m+1)^2$ are also straightforward, but one must observe the F cuts of Fig. 3. We show the results in Fig. 5, for $s_1 - i\epsilon$.

One next asks for the locus of all *complex* $s_1 = x+iy$ such that *one* of $s_+(s_1)$ or $s_-(s_1)$ is real. Since $\mathrm{Im}G = -\frac{1}{2}y = \mp \mathrm{Im}F$ is needed, therefore $F = \mathrm{Re}F \pm \frac{1}{2}iy$,

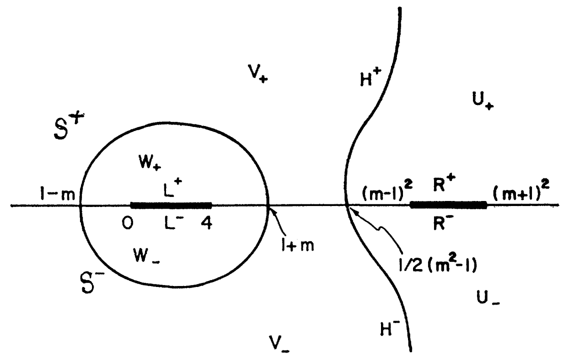

FIG. 6. The curves S and H.

and, hence, $F^2 = [(\mathrm{Re}F)^2 - y^2/4] \pm iy\,\mathrm{Re}F$. Therefore,

$$\mathrm{Re}F^2 = [(\mathrm{Im}F^2)/y]^2 - y^2/4.$$

Since F^2 is a rational algebraic function of $s_1 = x+iy$, we can always find $\mathrm{Re}F^2$ and $\mathrm{Im}F^2$ explicitly, and substitute into the above. One obtains, for the equal mass case,

$$y^2 = -\frac{[x-\frac{1}{2}(m^2-1)][x-(1+m)][x-(1-m)]}{[x-\frac{1}{2}(m^2+3)]} \tag{3.1}$$

implies one of $s_\pm(x+iy)$ real. There are two branches of (3.1) which we call S and H, respectively. Each of these has parts S^+ and S^- (H^+ and H^-) depending on whether $y \gtrless 0$. We show S and H in Fig. 6, where we also introduce names for certain domains. We remark that because of the s_1, s_2 symmetry in the equal-mass case, the complex curves in Figs. 5 and 6 are identical.

Figures 2, 5, and 6 together with some algebra contain all the information we need. Since G and F are both real algebraic functions of s_1, we have the mirror property

$$s_+(s_1^*) = [s_+(s_1)]^*, \quad s_-(s_1^*) = [s_-(s_1)]^*. \tag{3.2}$$

We find that the various domains $u_\pm$, $v_\pm$, $w_\pm$ map into each other, i.e.,

$$\begin{aligned} s_+:\quad & u_\pm \to v_\mp & s_-:\quad & u_\pm \to w_\pm \\ & v_\pm \to u_\mp & & v_\pm \to w_\mp \\ & w_\pm \to u_\pm & & w_\pm \to v_\mp. \end{aligned} \tag{3.3}$$

Further, we find that certain arcs map into each other, i.e.,

$$\begin{aligned} s_+:\quad & L_\pm \to H_\pm & s_-:\quad & L_\pm \to H_\mp \\ & R_\pm \to S_\mp & & R_\pm \to S_\pm \\ & H_\pm \to H_\mp & & H_\pm \to L \\ & S_\pm \to R & & S_\pm \to S_\mp. \end{aligned} \tag{3.4}$$

Equations (3.3) and (3.4) are the most important features of the mapping; other details can be read off the figures.

While the equal-mass case mapping has the great simplification of (s_1,s_2) symmetry the more important

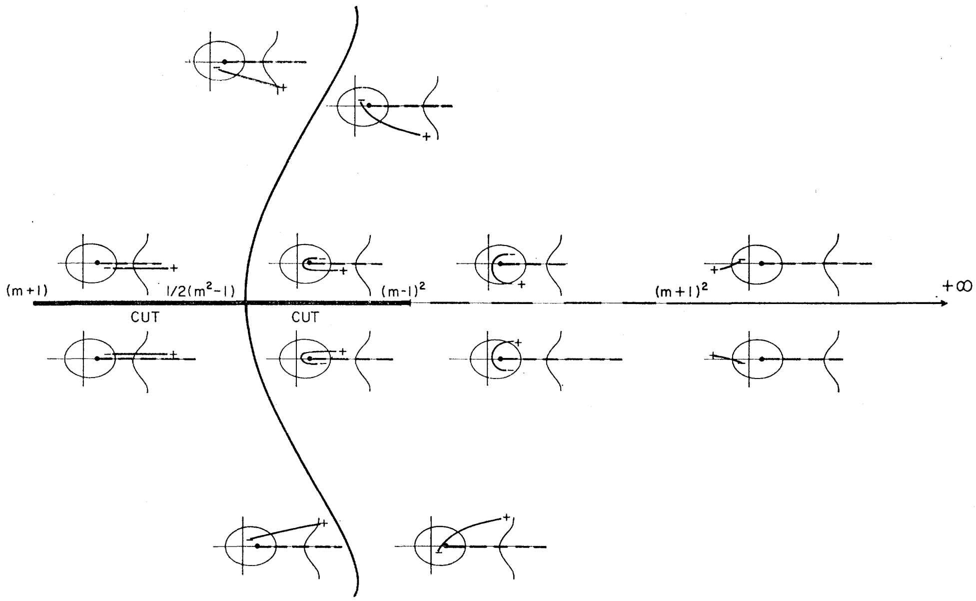

FIG. 7. The function $\hat{B}_1(s_1)\equiv B_1(s_1)$ for the right half s_1 plane. The dashed curve represents the s' path P, and the solid curve is the s_2 path C.

features of the mappings will remain the same for arbitrary masses.

3.2. Initial Extension into the Complex Plane

The definitions of $B_1(s_1)$ given by Fig. 4 for real s_1 may be called the *physical* limit $B_{1\text{phys}}$, which is therefore defined for all real $s_1\geq 4$. We define, for all complex s_1, the function obtained by *straightforward* extension of (2.14), i.e.,

$$\hat{B}_1(s_1)=\frac{1}{2F(s_1)\pi}\int_4^\infty ds'\beta(s')\int_C \frac{ds_2}{s'-s_2}, \qquad (3.5)$$

where C denotes a contour from $s_-(s_1)$ to $s_+(s_1)$ which does not cross the real s_2 axis above $s_2=4$.

We see immediately that

$$s_1\geq (m+1)^2:\ \hat{B}_1(s_1+i\epsilon)=\hat{B}_1(s_1-i\epsilon)=B_{1\text{phys}}(s_1). \quad (3.6)$$

Thus, we can analytically continue B_1 from the line $s_1\geq (m+1)^2$ into both the upper and lower half planes, until some difficulty arises. These continuations are shown in Fig. 7, for the right-hand half of the s_1 plane. The inset diagrams show the location of $s_\pm(s_1)$ for s_1 at that point, and the path C.

Since $F(s)$ changes sign on crossing the real axis between $(m-1)^2\leq s\leq (m+1)^2$ at the same time as s_+ and s_- go into each other, we see that this continuation satisfies

$$(m-1)^2\leq s_1\leq (m+1)^2:$$
$$\hat{B}_1(s_1+i\epsilon)=\hat{B}_1(s_1-i\epsilon)=B_{1\text{phys}}(s_1), \quad (3.7)$$

that is, the right-hand F cut is not a cut of $\hat{B}_1$. We find that $s=(m-1)^2$ is a branch point of $\hat{B}_1$, since for real s_1 in $4\leq s_1\leq (m-1)^2$ both $s_\pm$ are real ≥ 4. From Figs. 2, 4, and 5, we see that

$$m+1\leq s_1\leq (m-1)^2:\ \hat{B}_1(s_1-i\epsilon)=B_{1\text{phys}}(s_1) \neq\hat{B}_1(s_1+i\epsilon). \quad (3.8)$$

Thus, $\hat{B}_1$ is cut along $(m+1)\leq s_1\leq (m-1)^2$, and the physical limit of B_1 is from below the cut. Nonetheless $\hat{B}_1$ is still a suitable definition of $\hat{B}_1$ both in the upper and lower s_1 half planes, starting from the real axis, for *all* $s_1\geq m+1$. (We deliberately do not specify how far the continuation may proceed into the half planes.)

So far everything is trivial, and it may be wondered why we are proceeding so cautiously. The answer comes when we consider the last physical region $4\leq s_1\leq (m+1)$. The prescription of Fig. 4 has small positive imaginary parts for both $s_\pm$, yet this cannot be achieved by either $s_1\pm i\epsilon$ (cf. Fig. 5). In fact, as we move slightly below the real s_1 axis from the range $\frac{1}{2}(m^2-1)\geq s_1\geq m+1$ to $(m+1)\geq s_1\geq 4$, we see from Fig. 5 that $s_+(s_1)$ attempts to push through the s' integration path from above, at $s_1=m+1$, $s_+=(m-1)^2$.

The perturbation theory analysis[10] has no singularity in the physical limit at this point; hence, the projection must be analytic at this point. This implies that the motion of s_+ pushes ahead of itself the s' integration path. As long as the (negative) imaginary part of s_1 is infinitesimal, the necessary distortion is also infinitesimal; nonetheless it *is* necessary. Once this has been realized, we see that this generalizes to finite (negative) imaginary part to s_1, for s_1 in w_- (cf. 3.3). This is actually the key remark of the present analysis.

For the moment let us assume that this distortion of the s' path into the lower half plane (actually u_-)can be achieved without encountering any singularities of $\beta(s')$ (we discuss this in Sec. 3.3). Then we see that (3.5) must be generalized to

$$B_1(s_1)=\frac{1}{2F(s_1)\pi}\int_P ds'\beta(s')\int_C \frac{ds_2}{s'-s_2}, \qquad (3.9)$$

where P is a suitably distorted contour from 4 to ∞, obtained by following the motion of $s_+(s_1)$.

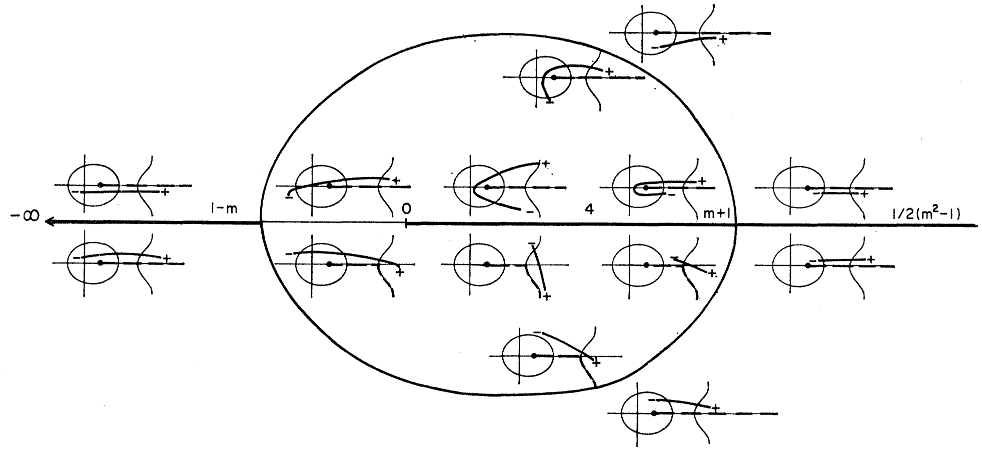

FIG. 8. The function $B_1(s_1)$ for the left half s_1 plane (cf. Fig. 7).

From (3.3) we see that for s_1 in the lower half plane the distortion of P away from P_0 (the undistorted path) is only necessary in w_-; and that the distortion is always within u_-. Further, as s_- crosses S_- from w_- to v_-, the distortion becomes unnecessary, so that (3.9) links up analytically with $\hat{B}_1(s_1)$. Formally, throughout u_- we take P as maximally distorted, going along the real axis from 4 to $\frac{1}{2}(m^2-1)$ and then along H_- to ∞; however, this distortion of P can always be reduced towards P_0 until it encounters $s_+(s_1)$.

In u_- and v_- we can take $P\rightarrow P_0$, i.e., the undistorted definition (cf. $\hat{B}_1$), and we have therefore obtained a definition $B_1(s_1)$ for all complex s_1 in the lower half plane which properly approaches the physical limit. In the upper half plane we do not have the added restriction that B_1 must tend to a specified limit. We therefore take $P\rightarrow P_0$, i.e., $B_1=\hat{B}_1$. The choice in the upper half plane is somewhat arbitrary, but we emphasize that the choice in the lower half plane is really forced upon us by the physical limit definitions obtained from perturbation theory.

We present our choice for P and C in Fig. 8, for the left half s_1 plane, stressing that the distortion of P can always be undone until it encounters C. However, we must now turn to the problem of the analyticity of $\beta(s')$ in the region through which the distortion $P_0\rightarrow P$ is actually performed, i.e., u_0. Now

$$\beta(s)=f_2^*(s)[A_2(s)+B_2(s)+C_2(s)]. \qquad (3.10)$$

A_2 and C_2 have similar properties to B_1, while B_2 has a cut for real $s\geq 4$, and f_2^* has cuts for real $s\leq 0$ and $s\geq 4$. Thus, we have a self-consistency problem, in which we must show that our choice of distorted contours P does not lead to cuts of $\beta(s)$ which prevent the distortion. We hence turn to this question.

3.3. Digression onto the Second Sheet

In order to distort the s' contour P in the integral (3.9), we must ensure that $\beta(s')$ in (3.10) is analytic between P_0 and P; and, further, we must explicitly analytically continue β in this region.

We consider the various factors in (3.10) in turn. $f_2^*(s)=\exp[-i\delta_2(s)]\sin\delta_2(s)$ has cuts along the real s axis for $-\infty<s\leq 0$, and $4\leq s<\infty$ (note that we ignore any inelastic threshold branchpoints). It also has the possibility of *first* sheet poles, arising from resonances in $f_2(s)=\exp[+i\delta_2(s)]\sin\delta_2(s)$ due to second sheet poles of $f_2(s)$. (It is straightforward to verify this interchange of second and first sheet properties of f and f^* in a Breit-Wigner relativistic resonance formula, but the result is general, as we shall see immediately.) These possible first sheet singularities of f^* do *not* cause any difficulty, because in the integrand of (3.9), δ is to be taken in its physical limit, which is from *above* the right-hand cut. Thus, when we continue β downwards from P_0 to P, we must continue f^* onto its *second* sheet reached by crossing the right-hand cut from above.

Let δ_+ and δ_- be the physical-sheet limits of δ_2 just above and below the right-hand cut, and similarly for other quantities. Then applying elastic unitarity to $\mathfrak{F}=\omega f/k$, i.e., $\mathrm{Im}\mathfrak{F}_+=(\mathfrak{F}_+-\mathfrak{F}_-)/2i=k|\mathfrak{F}_+|^2/\omega$ and noting that $k_+=-k_-$, we have immediately (cf. Ref. 13)

$$\delta_+=-\delta_-. \qquad (3.11)$$

Hence,

$$(f^*)_+=\exp(-i\delta_+)\sin\delta_+=-\exp(+i\delta_-)\sin\delta_- =(-f)_-. \qquad (3.12)$$

That is, $(-f)$ has, as its boundary value on the lower edge of its right-hand cut, the value which f^* has on the upper edge of that cut. Thus, $(-f)$ provides the necessary second-sheet continuation of f^*; formally we write

$$f^*_{\mathrm{II}}\equiv -f_{\mathrm{I}}=-e^{i\delta}\sin\delta. \qquad (3.13)$$

This general result proves our assertion that first (second) sheet singularities of f^* correspond precisely to second (first) sheet singularities of f. We see that the continued integrand in (3.9) contains $-f$, and so has *no* singularities between $P_0(s-i\epsilon)$ and P.

We next turn to $M_2=A_2+B_2+C_2$. Consider first B_2.

[13] R. Oehme, Phys. Rev. **121**, 1840 (1961); R. Blankenbecler, M. L. Goldberger, S. W. MacDowell, and S. B. Treiman, Phys. Rev. **123**, 692 (1961).

Now

$$B_2(s)=B(s)=\frac{1}{\pi}\int_4^{\infty}\frac{ds'\beta(s')}{s'-s-i\epsilon}. \tag{3.14}$$

Here again B comes naturally as the limit from above the right-hand cut, and must be converted to its second sheet. We define

$$B_{\mathrm{II}}(s)=B(s)+2i\beta(s)\,, \tag{3.15}$$

where $\beta(s)$ is the *analytic* continuation of β into the region u_- that we are seeking, with the required property that the limit onto the real axis $s\geq 4$ from *below* reproduces β, the spectral function in (3.14). From (3.14) and (3.15) we see that $B_{\mathrm{II}-}=B_+=B_{\mathrm{phys}}$.

In principle, B_{II} can introduce new singularities into our problem. While B (3.14) possesses only a right-hand cut, B_{II} also has the singularities of β. Since β contains $f_2{}^*$ as a factor, β has first-sheet resonance poles if f_2 has second sheet poles. For our present purpose such poles cause difficulty only if they are located in $u_\pm$. Throughout the rest of this work we assume that the various f_i do *not* contain second-sheet poles in the domains $u_\pm$, associated with resonances (but see note added in proof). With this assumption B causes no difficulty in the distortion $P_0\rightarrow P$.

Finally, we consider A_2 and C_2. These have properties very similar to B_1. Thus, they have "left-hand" cuts, with, in particular, a cut along $4\leq s\leq (m-1)^2$; but no cuts or singularities in the lower half plane. The physical limit appropriate for β is reached from *below* this cut, so that there is no difficulty in the continuation from P_0 to P.[14]

We have now expressed each factor in β as the limit from below of some function, if necessary on the second sheet. Thus, from 3.15, using (3.13) and (3.10), we define M_2 as a function in the lower half plane by the condition that the limit from below reproduces the factor in β; and similarly for B_{II}. Hence,

$$B_{\mathrm{II}}=B+2i\beta=B+2i(-f)[B_{\mathrm{II}}+A_2+C_2].$$

Therefore,

$$B_{\mathrm{II}}=\exp(-2i\delta_2)[B-2if(A_2+C_2)]\,,\ \text{etc.}, \tag{3.16}$$

where each factor is defined as a function of a *complex* variable;

$$\begin{aligned}M_2&=A_2+C_2+B_{\mathrm{II}}\\&=\exp(-2i\delta_2)(B+A_2+C_2)\,,\ \text{etc.},\end{aligned} \tag{3.17}$$

and

$$\beta=-fM_2,\ \text{etc.} \tag{3.18}$$

[14] We emphasize that the presence of branch points of A_2 and C_2 at $(m-1)^2$, which is *on* the undistorted contour P_0, does not affect the distortion of the path of integration to P, since the integrand is the *same* analytic function along all of P_0. This is not the case if we were to include inelastic contributions, in the form of other *distinct* spectral functions β_{in}, with integrals running along undistorted $P_{\mathrm{in};0}$: $s_{\mathrm{in}}\leq s\leq\infty$. In that case we would be misled when writing a *single* total spectral function β_{tot}, and path P_0, and we would not be able to distort P_0 to P away from the inelastic threshold s_{in}. We have neglected inelastic contributions throughout this work. (I am grateful to Professor S. B. Treiman for raising this question.)

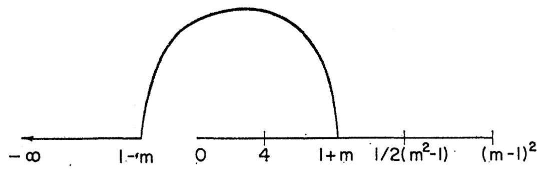

FIG. 9. The cuts of $B_1(s_1)$.

Equations (3.16–3.18) enable us to perform the distortion $P_0\rightarrow P$ which we introduced in Sec. 3.2.

3.4. The Cuts and Discontinuities of B_1

In Figs. 7 and 8 we have given a prescription defining the function $B_1(s_1)$ of a complex variable s_1, for each of the six regions $u_\pm$, $v_\pm$, and $w_\pm$. The cuts of B_1 occur at the boundaries between two such regions, for which the prescription is discontinuous. By inspection we see that the only possible cuts are the real axis for $s_1\leq(m\pm1)^2$, S_+, and S_-.

Actually S_- is not a cut of B_1, since along S_- $s_-(s_1)$ is on S_+, while $s_+(s_1)$ is on R: $(m-1)^2\leq s\leq(m+1)^2$. Thus, the distortion $P_0\rightarrow P$ can be undone, leaving no discontinuity. [It is an essential feature of this that the ds' integration contour is not tied down to $(m-1)^2$.] Similar arguments show that $1-m\leq s_1\leq 0$ is not a cut of $B_1(s_1)$. We have already shown that $(m-1)^2\leq s_1\leq(m+1)^2$ is not a cut, and, in fact, the cuts are S_+, $-\infty\leq s_1\leq 1-m$, and $0\leq s\leq(m-1)^2$, and are shown in Fig. 9.

The discontinuities of B_1 across these cuts can be obtained straightforwardly. We consider the cut $-\infty\leq s_1\leq 1-m$ as a typical example. On this cut $s_+>4$, $s_-<4$ and $\mathrm{Im}s_+/\mathrm{Im}s_1<0$. Hence,

$$B_1(s_1\pm i\epsilon)=\frac{1}{2F\pi}\int_4^{\infty}ds'\left\{\int_{s_-(s_1)}^{4}\frac{ds_2}{s'-s_2}+\int_4^{s_+(s_1)}\frac{ds_2}{s'-s_2\pm i\epsilon}\right\}\beta(s').$$

Therefore,

$$\begin{aligned}\mathrm{disc}B_1(s_1)&\equiv B_1(s_1+i\epsilon)-B_1(s_1-i\epsilon)\\&=-\frac{1}{2F\pi}\int_4^{s_+}ds_2\oint_{\mathrm{R}}\frac{ds'\beta(s')}{s'-s_2}\\&=-\frac{i}{F(s_1)}\int_4^{s_+(s_1)}ds_2\beta(s_2)\,,\end{aligned}$$

where R denotes an anticlockwise contour encircling the real axis $4\leq s\leq\infty$, of infinitesimal width. (This contour is to go below any singularities of β on the real axis, so that we only get the pole contribution.) In this way we find the following discontinuities (always taken from the side with greater positive imaginary part).

$-\infty<s_1\leq 1-m$; $s_+\geq(m+1)^2$:

$$\mathrm{disc}B_1(s_1)=-\frac{i}{F(s_1)}\int_4^{s_+(s_1)}\beta(s_2)ds_2\,, \tag{3.19a}$$

$s_1 \in S_+$; $(m+1)^2 \geq s_+ \geq (m-1)^2$:

$$\mathrm{disc}B_1(s_1) = -\frac{i}{F(s_1)} \int_4^{s+(s_1)} \beta(s_2) ds_2, \quad (3.19\text{b})$$

$(m+1) \leq s_1 \leq \frac{1}{2}(m^2-1)$; $(m-1)^2 \geq s_+ \geq \frac{1}{2}(m^2-1)$; $(m+1) \geq s_- \geq 4$:

$$\mathrm{disc}B_1(s_1) = -\frac{i}{F(s_1)} \int_{s-(s_1)}^{s+(s_1)} \beta(s_2) ds_2, \quad (3.19\text{c})$$

$\frac{1}{2}(m^2-1) \leq s_1 \leq (m-1)^2$; $\frac{1}{2}(m^2-1) \geq s_+ \geq m+1 \geq s_- \geq 4$:

$$\mathrm{disc}B_1(s_1) = -\frac{i}{F(s_1)} \left[\int_4^{s+(s_1)} + \int_4^{s-(s_1)} \right] \beta(s_2) ds_2, \quad (3.19\text{d})$$

$4 \leq s_1 \leq m+1$; $\frac{1}{2}(m^2-1) \geq s_- \geq m+1$:

$$\mathrm{disc}B_1(s_1) = +\frac{i}{F(s_1)} \int_4^{s-(s_1)} \beta(s_2) ds_2, \quad (3.19\text{e})$$

$0 \leq s_1 \leq 4$; $s_+ \in H_-$:

$$\mathrm{disc}B_1(s_1) = \frac{-2i}{s_+(s_1) - s_-(s_1)} \int_4^{s+(s_1)} \beta(s_2) ds_2. \quad (3.19\text{f})$$

The only discontinuity which may cause difficulty is (3.19f). This is resolved by specifying that $s_\pm$ are evaluated *below* the F cut $0 \leq s \leq 4$ for both limits of $\beta_1(s_1)$.

We make some comments on (3.19). The discontinuity joins continuously from each range to the next except at $s_1 = 4$. This is evident everywhere except at $s_1 = \frac{1}{2}(m^2-1)$, while at this point $s_-(s_1) = 4$, so the transition is smooth. The discontinuity has inverse square-root singularities at $s_1 = 4$ and at $s_1 = (m-1)^2$, arising from the vanishing of F. Near $s_1 = 0$, $F \to \infty$, so that the discontinuity goes as $(s_1)^{1/2}$, provided β has suitable asymptotic behavior (see below).

As further remarks we state again that B_1 has no singularities in the lower half s_1 plane, so that the distortion $P_0 \to P$ encounters no difficulty. The choice of prescriptions for B_1 in the upper half plane s, and, hence, of the cuts is not unique, but our choice seems the most natural one. An alternative choice in w_+ would continue the distortion $P_0 \to P \to P_-$ where P_- is just the negative real axis $-\infty < s' \leq 4$; pushing the $0 \leq s_1 \leq 4$ cut up through w_+ to S_+. Whatever choice is made, it seems likely that at least one *discontinuity* will involve an integration over β taken to an unphysical s_+ or s_- (in our case for $0 \leq s_1 \leq 4$, cf. 3.19f).

Finally, we remark that the branch points we have found for B_1 are consistent with the singularities one would find by investigating all possible pinch and end point singularities of (3.9). However, such an analysis does not determine which singularities are on the physical sheet. (The arbitrariness in the choice of prescriptions is of course just an arbitrariness in the definition of the physical sheet. One has *one* single analytic but many sheeted function.)

4. THE INTEGRAL EQUATIONS

In the previous section we have obtained a possible set of cuts and discontinuities for B_1. Exactly similar results will hold for A_2 and C_2. Hence, we know the complete analytic structure of M_2,

$$M_2 = \exp(-2i\delta_2)(B + A_2 + C_2), \text{ etc.} \quad (3.17)$$

There are two sources of singularities for M_2. The cut structure is independent of the detailed dynamics, and is already implicit in the above. However, the factor $e^{-2i\delta_2} = f_2^*/f_2$ may possess (first sheet) poles arising from resonances in f_2, since these are then present in f_2^*. In such a case M_2 possesses these poles with certain residues. These poles then lead to an inhomogeneous term in the integral equation for M_2, the homogeneous terms arising from the cuts of M_2.

The discontinuities across the various cuts of M_2 can all be expressed in terms of the α, β, γ, and so we get a set of coupled integral equations for the M_i (in the case of identical particles these uncouple). Thus, recalling that δ_2, B, A_2 and C_2 may all have discontinuities at the same cut $[4 \leq s < (m-1)^2]$, we have most generally

$$\begin{aligned} M_{2+} - M_{2-} &= \exp(-2i\delta_{2+})(B_+ + A_{2+} + C_{2+}) \\ &\qquad - \exp(-2i\delta_{2-})(B_- + A_{2-} + C_{2-}) \\ &= [\exp(-2i\delta_{2+}) - \exp(-2i\delta_{2-})](B_- + A_{2-} + C_{2-}) \\ &\qquad + \exp(-2i\delta_{2+})[(B_+ - B_-) + (A_{2+} - A_{2-}) \\ &\qquad\qquad + (C_{2+} - C_{2-})] \\ &= \exp(-2i\delta_{2+}) - \exp(-2i\delta_{2-})] \exp(+2i\delta_{2-}) M_{2-} \\ &\qquad + \exp(-2i\delta_{2+})[2i\beta + \mathrm{disc}A_2 + \mathrm{disc}C_2], \text{ etc.} \end{aligned} \quad (4.1)$$

By integrating a Cauchy denominator around all the cuts of M_i, we hence can obtain integral equations for the M_i.

In fact a much more straightforward approach is to keep the cuts of B, and A_2 and C_2, separate in (3.17) and never go into the second sheet as regards the integral equation. Thus, from the original equations $M_2(s) = B + (A_2 + C_2)$ where B has a right-hand cut $4 \leq s < \infty$, and A_2 and C_2 have "left-hand" cuts as in Fig. 9. The physical limit is then obtained from above the right-hand cut, and below the left-hand cut, so that we have a situation as in Fig. 10. The discontinuity across the right-hand cut is that of B, given by unitarity in terms of $M_{2\,\mathrm{phys}}$; while we have already expressed the left-hand cut discontinuities in terms of α and γ. The only place where the second-sheet continuation is needed is for the discontinuity across $0 \leq s \leq 4$, and this can be dealt with straightforwardly (cf. next paragraph).

The great advantage of this method is that one can *immediately* factor out the right-hand cut. That is, one writes $M_2 = R_2 L_2$, where R_i and L_i have only the right- and left-hand cuts, respectively. Then R_i can be written down immediately in the standard Omnés form,[15] leaving the left-hand functions L_i to be ob-

[15] R. Omnés, Nuovo Cimento 8, 316 (1958); N. I. Mushkhelishvili, *Singular Integral Equations* (P. Noordhoff Ltd., Groningen, The Netherlands, 1953); also G. F. Chew and S. Mandelstam, Phys. Rev. 119, 467 (1960).

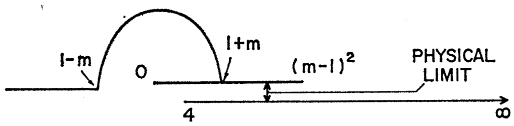

FIG. 10. The cuts of M_2 on the first sheet.

tained by iteration. (It is because R_i can be obtained explicitly that the downward distortion of β presents no difficulties in this approach.) Notice that since we deal with M_2 on its first sheet rather than its second-sheet continuation M_{II}, M_2 has no poles arising from resonances. (We still must require that there be no second-sheet poles of f_i in $u_\pm$.)

4.1 Asymptotic Behavior

In the preceding we have ignored the question of subtractions and behavior at infinity. Let us assume that all unsubtracted integrals converge in the original Khuri-Treiman equations (2.1) and (2.2). When $s\to\infty$, $G\to -\frac{1}{2}s$, $F\to -\frac{1}{2}s$, so $s_+\to -s$, $s_-\to 0$. Hence,

$$B_1(s_1)\to\frac{1}{-s}\int_0^{-s}ds_2\frac{1}{\pi}\int_4^\infty ds'\frac{\beta(s')}{s'-s_2}$$

over *undistorted* paths (cf. Figs. 6 and 7). Thus,

$$B_1(s)\to\frac{1}{s\pi}\int_4^\infty ds'\beta(s')\ln\frac{s'+s}{s'}\to 0\left(\frac{\ln s}{s}\right)\to 0 \quad (4.2)$$

and

$$B(s)\to\frac{1}{s}\to 0.$$

However, the equations we have will contain at least one subtraction, corresponding to the equal time or Born term. Let us denote "subtracted" quantities with a tilde ($\sim$). Then, cf. (2.1–2.5)

$$M=D+\tilde{A}(s_1)+\tilde{B}(s_2)+\tilde{C}(s_3)\,, \quad (4.3)$$

$$\tilde{A}(s)=\frac{(s-s_0)}{\pi}\int_4^\infty\frac{ds'\alpha(s')}{(s'-s-i\epsilon)(s'-s_0-i\epsilon)}\,,\ \text{etc.} \quad (4.4)$$

We define the partial projections by

$$\tilde{B}_1(s)=\frac{1}{2F(s)}\int_{s-(s)}^{s+(s)}ds_2\tilde{B}(s_2)$$
$$=\frac{1}{2F(s)}\int_{s_-}^{s_+}ds_2B(s_2)-b\,, \quad (4.5)$$

where

$$b=\frac{1}{\pi}\int_4^\infty\frac{\beta(s')ds'}{s'-s_0}.$$

Notice $\tilde{B}_1(s_0)\neq 0$. Then

$$M_1(s_1)=D+\tilde{A}(s_1)+\tilde{B}_1(s_1)+\tilde{C}_1(s_1)\,,\ \text{etc.} \quad (4.6)$$

The analytic structure of $\tilde{B}_1(s_1)$ is the same as that of $B_1(s_1)$ previously studied, with the *same* discontinuities as before, but different asymptotic behavior, owing to the presence of b.

We then perform a Cauchy integral for M_1, i.e.,

$$\frac{(z-s_0)}{2\pi i}\oint\frac{M_1(z')dz'}{(z'-z)(z'-s_0)}$$
$$=M_1(z)-M_1(s_0)=\frac{z-s_0}{\pi}\int_{\text{RH cut}}\frac{\alpha(s')ds'}{(s'-z)(s'-s_0-i\epsilon)}$$
$$+\frac{(z-s_0)}{\pi}\int_{\text{LH cut}}\frac{\text{disc}M_1(z')dz'}{(z'-z)(z'-s_0+i\epsilon)} \quad (4.7)$$

and take this as our starting equation for solution, solving the left-hand cut by successive iteration.

ACKNOWLEDGMENTS

It is a pleasure to acknowledge the many stimulating discussions I have had on this subject with Tran Truong; also the helpful comments of Ian Aitchison, John Bronzan, and S. B. Treiman which have so greatly contributed to the final form of this work.

I am also grateful to Dr. G. C. Wick for extending the hospitality of Brookhaven National Laboratory to me during the summer of 1963, during which this work was put in final form.

Note added in proof. In a recent paper,[16] the late Georges Bonnevay has considered a model which is essentially the same as that presented here. He uses similar methods in extending the definition of the partial-wave projections from the physical region into the complex plane, with some interesting differences as follows:

1. Rather than factoring out the right-hand cut by the standard Omnès[15] function e^u (which has only the right-hand cut), he observes that $f_i^*M_i/k$ has no right-hand cut [cf. (3.11)], and considers an integral representation for this function. Since f^* has the *two*-body normal left-hand cut $-\infty<s\leq 0$, it is not clear which of these factorizations will lead to an easier final solution of the integral equation. The question rests on the magnitude of the contributions to *physical M* which arise from integrations over δ near the left-hand cut. This leads naturally to point 2.

2. In the present work we have attempted to distort the integration over x as little as possible, and this was our criterion for the upper half plane definitions. In this way the left-hand cut overlaps the physical region $4\leq s\leq(m-1)^2$; but the discontinuities involve a knowledge of δ only near the physical region (specifically all real $s\geq 4$, and also along H_- which is not far from the physical region, and is far from the two-body left-hand cut $-\infty<s\leq 0$). Hence, effective range or other *physical region* approximations to δ can be used with reasonable confidence. The price paid is that the integral equations for the M_i are then singular; that is, the integration path along the cuts goes along the physical region. Hence, in computing we are faced with principal part integrations. While these are unaesthetic, they do not cause difficulties of principle, only of practice.

[16] G. Bonnevay, Nuovo Cimento (to be published).

On the other hand, Bonnevay prefers to distort the left-hand cut away from the physical region. This he does by defining M_2 in the upper half plane by analytic continuation from below the cut $0\leq s\leq (m-1)^2$. He therefore ends up with a *left*-hand cut consisting of the two parts $-\infty\leq s\leq 0$ and $(m-1)^2\leq s\leq +\infty$. In this way he avoids troubles arising from a singular kernel, but requires a knowledge of the phase shifts δ_i on the left-hand cut. Of course an *exact* knowledge of the physical region $(4\leq s<\infty)$ phase shifts gives complete information everywhere; the problem is one of minimizing computational inaccuracies arising from two possible sources.

In our present paper we have specifically excluded the case in which f has second-sheet resonance poles in $u_\pm$, since these lead to extra (logarithmic) singularities in W_-. This case is the one studied by Bonnevay. He deals with the function $f_i{}^*M_i/k$, which has poles arising directly from f^*, and also the extra singularity in W_-. After performing the continuation described in paragraph 2, these "resonance" singularities are the singularities which lie closest to the physical sheet. Bonnevay proposes an iterative solution of the resultant integral equation in which the resonance contributions are treated as the inhomogeneous term (the residue at the pole being treated as an unknown parameter). The inhomogeneous term is then to be used as the first approximation in the iteration of the homogeneous terms.

It is a sad privilege to acknowledge that a study of Bonnevay's paper has enabled me to remove some initial errors in the present work, by restricting its applicability to cases with no resonance in the domains $u_\pm$. The case with such a resonance is the one explicitly treated by Bonnevay. The two papers therefore complement each other.

As a final remark, an integral equation *somewhat* similar to the Khuri-Treiman equation has been transformed into a *soluble* integral equation by V. V. Anisovich, Zh. Eksperim. i Teor. Fiz. **44**, 1593 (1963) [translation: Soviet Phys.—JETP **17**, 1072 (1963)]. I am grateful to Professor Anisovich for sending me a reprint of the original article.

APPENDIX: ALTERNATIVE FORMULATION

In Eq. (3.9) we have

$$B_1(s_1)=\frac{1}{2F(s_1)\pi}\int_P ds'\beta(s')\int_C \frac{ds_2}{s'-s_2}, \tag{A1}$$

where P and C are *non*intersecting contours given in Figs. 6 and 7. It is therefore permissible to perform the s_2 integration first, i.e.,

$$B_1(s_1)=\frac{1}{2F(s_1)\pi}\int_P ds'\beta(s')\ln\left\{\frac{s'-s_-(s_1)}{s'-s_+(s_1)}\right\}_C, \tag{A2}$$

where the suffix C on the ln specifies how the imaginary part of the ln is to be evaluated. One can next perform an integration by parts. Thus, define

$$b(s)=\int_4^s ds'\beta(s'). \tag{A3}$$

Then

$$\begin{aligned}B_1(s_1)&=\frac{1}{2F(s_1)\pi}\int_P ds'b(s')\left\{\frac{1}{s'-s_+}-\frac{1}{s'-s_-}\right\}_C\\&=\frac{1}{\pi}\int_P\frac{ds'b(s')}{\{[s'-G(s_1)]^2-F(s_1)^2\}_C}.\end{aligned} \tag{A4}$$

Form (A2) is appropriate when treating B_1 as a function of a *real* variable, for then $P\rightarrow P_0$. For complex s_1, the distortion of P is necessary, and hence one *cannot* give a unique prescription for the logarithmic kernel for all s_1, with s' restricted to lie on the real range $4\leq s'\leq\infty$. One can, of course, investigate the function $\hat{B}$ defined in (3.5), but this does *not* have the correct physical limit for $4\leq s_1\leq m+1$.

Form (A4) is most appropriate to our problem, and it is easy to see that it leads to the same cuts and discontinuities as presented above. The cuts occur when one or both of s_+ or s_- crosses P (not P_0!), and the discontinuity is then simply the residue at the pole or poles which crossed, viz. $(i/F)b(s_+)$, etc., precisely as found in (3.19).

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K^--p Total Cross Section between 2.7 and 5.2 BeV/c*

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The K^--p total cross section has been measured between 2.7 and 5.2 BeV/c, by means of a transmission experiment. Points with about 3% statistical errors have been obtained at momenta approximately 200 MeV/c apart.

Previous measurements of the K^--p total cross section at momenta of about[1-3] 4 BeV/c are widely spaced but collectively they are not consistent with a smooth variation of the cross section with momentum. In order to investigate this region more thoroughly, a transmission experiment was undertaken, the results of which are reported.

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