High-Energy Behavior of Total Cross Sections*

YOICHIRO NAMBU
Enrico Fermi Institute for Nuclear Studies and Department of Physics, University of Chicago, Chicago, Illinois

AND

MASAO SUGAWARA
Department of Physics, Purdue University, Lafayette, Indiana

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A proof of Pomeranchuk's theorem regarding the high-energy limits of the total cross sections is presented. The proof consists of assuming the usual analyticity for the forward elastic amplitudes and the assumption that these amplitudes become pure imaginary in the high-energy limit. This proof does not require that the total cross sections have finite limits. It is also shown that the total cross-section $\sigma(s)$ as a function of s, the total c.m. energy squared, behaves asymptotically as $\sigma(s)=\sigma(\infty)+\delta/s^{\frac{1}{2}}+O(1/s)$ when $\delta(\infty)$ is nonzero, where δ is some constant. This asymptotic form is based upon a more specific assumption that high-energy elastic scattering is described by an effective complex and energy-dependent potential which satisfies a dispersion relation in the energy variable. However, the above asymptotic form is valid independently of the dependence of this effective potential on the spacial coordinate. It is argued that the term $\delta/s^{\frac{1}{2}}$ in the above asymptotic form should be regarded as genuinely asymptotic, while the term of the order of $1/s$ is not. According to this criterion, the available high-energy $p-p$ data are not so close to the asymptotic region as the $\pi^{\pm}-p$ data in the same laboratory momentum range.

1. INTRODUCTION AND SUMMARY

WE denote by $\sigma(s)$ and $\bar{\sigma}(s)$ the total cross sections for a particle and its antiparticle, respectively, incident on the same target as functions of s, the total c.m. energy squared. Pomeranchuk's theorem[1] states that $\sigma(s)$ and $\bar{\sigma}(s)$ approach the same limit as $s\rightarrow\infty$ if they have finite limits. Several proofs[2] of this theorem were given which consist of assuming the forward dispersion relation together with some additional assumptions. The experimental check[3,4] on its validity is not yet conclusive, not only because of large experimental errors but also because nothing is known theoretically about the asymptotic forms of $\sigma(s)$ and $\bar{\sigma}(s)$.

The purpose of this paper is to present another proof of this theorem and also the theoretical asymptotic forms of $\sigma(s)$ and $\bar{\sigma}(s)$ which are given by (14) below. Besides its considerable generality and simplicity, our new proof does not require that the total cross sections have finite limits. We assume, as the alternative assumption, that the forward elastic amplitude becomes pure imaginary as $s\rightarrow\infty$. This is equivalent to assuming that high-energy elastic scattering becomes dominantly absorptive. We show that our assumptions imply that $\sigma(s)\simeq\bar{\sigma}(s)\propto s^0$, s^{-2}, etc., as $s\rightarrow\infty$. This result is even stronger than Pomeranchuk's theorem.

The asymptotic forms (14) are based upon a more specific assumption, that high-energy elastic scattering is described by an effective complex and energy-dependent potential $V(r,s)$ of the type proposed previously.[5] Since this model cannot[5] give rise to zero asymptotic limit of $\sigma(s)$, the asymptotic forms (14) apply to the case when $\sigma(\infty)$ and $\bar{\sigma}(\infty)$ are nonzero. However, these forms are valid independently of the dependence of $V(r,s)$ on the spatial coordinate r. The asymptotic forms (14) are due to the dispersion relation in s satisfied by $V(r,s)$, which is in turn due to the microscopic causality.[5] The comparison of (14) with experiments[3,4] and the relating discussion are given in Sec. 4.

2. ASYMPTOTIC LIMITS OF TOTAL CROSS SECTIONS

Let $A(s)$ and $\bar{A}(s)$ be the forward elastic amplitudes for the particle and its antiparticle, respectively. We normalize them such that $A(s)\simeq s\sigma(s)$ and $\bar{A}(s)\simeq s\bar{\sigma}(s)$ as $s\rightarrow\infty$. The crossing relation which connects these amplitudes is written as

$$A(u)=\bar{A}(s)\,, \qquad (1)$$

if we introduce u, the covariant total energy squared in the crossed channel. One knows that $s+u=2(m^2+m'^2)$, where m and m' are the masses of the colliding particles. The usual analyticity assumption implies that $A(s)$ and $\bar{A}(s)$ are analytic in s except for finite numbers of poles and cuts which are given by $\infty>s\geq s_0$ and $\infty>u\geq u_0$ for $A(s)$ and by $\infty>s\geq u_0$ and $\infty>u\geq s_0$ for $\bar{A}(s)$, respectively, where s_0 and u_0 are some constants. If $\delta(s)$ is the phase of $A(s)$ along the cut, $\infty>s\geq s_0$, and $\bar{\delta}(s)$ is that of $\bar{A}(s)$ along the cut, $\infty>s\geq u_0$, the phase representations[6] for $A(s)$ and $\bar{A}(s)$ are written

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as

$$A(s)=\frac{P_1(s)}{P_2(s)}\exp\left\{\frac{s}{\pi}\int_{s_0}^{\infty}\frac{\delta(s')ds'}{s'(s'-s)}+\frac{u}{\pi}\int_{u_0}^{\infty}\frac{\bar{\delta}(s')ds'}{s'(s'-u)}\right\},$$
$$\bar{A}(s)=\frac{\bar{P}_1(s)}{\bar{P}_2(s)}\exp\left\{\frac{s}{\pi}\int_{u_0}^{\infty}\frac{\bar{\delta}(s')ds'}{s'(s'-s)}+\frac{u}{\pi}\int_{s_0}^{\infty}\frac{\delta(s')ds'}{s'(s'-u)}\right\},\quad(2)$$

where $P_i(s)$'s and $\bar{P}_i(s)$'s are finite real polynomials. The crossing relation (1) then implies that these polynomials satisfy

$$P_1(u)/P_2(u)=\bar{P}_1(s)/\bar{P}_2(s).\quad(3)$$

Therefore, the asymptotic limits[7] of $A(s)$ and $\bar{A}(s)$ are given by

$$A(s)\to cs^n\left[\frac{s_0}{s}\right]^{\delta(\infty)/\pi}\left[\frac{u_0}{s}\right]^{\bar{\delta}(\infty)/\pi}\exp[i\delta(\infty)],$$
$$\bar{A}(s)\to c(-s)^n\left[\frac{s_0}{s}\right]^{\delta(\infty)/\pi}\left[\frac{u_0}{s}\right]^{\bar{\delta}(\infty)/\pi}\exp[i\bar{\delta}(\infty)],\quad(4)$$

where n is an integer due to the polynomials in (2) and c is some real constant. It is assumed in deriving (4) that $\delta(s)$ and $\bar{\delta}(s)$ approach their limits not too slowly. This is equivalent to assuming that the amplitudes $A(s)$ and $\bar{A}(s)$ exhibit the power behavior in s as $s\to\infty$. It is clearly seen in (4) that $\sigma(\infty)$ and $\bar{\sigma}(\infty)$ do not have to be the same as long as $\delta(\infty)$ and $\bar{\delta}(\infty)$ remain arbitrary.

Our additional assumption, that the forward elastic amplitude becomes pure imaginary as $s\to\infty$, implies that $\delta(\infty)$ and $\bar{\delta}(\infty)$ can only be $\pm\frac{1}{2}\pi$ except for some integer multiples of π. It then follows that the limits in (4) are both proportional to s^m, m being an integer. One can argue that this integer m must be unity or less. Let $A(s,t)$ be the scattering amplitude as a function of s and t, the covariant momentum-transfer squared, normalized such that $A(s,t=0)=A(s)$. In the case when $A(s,t)\propto s^m\beta(t)$ as $s\to\infty$, where $\beta(t)$ is sufficiently peaked in t around $t=0$, one estimates the total elastic cross section, $\sigma_{\rm el}(s)\propto\int^0|A(s,t)|^2dt/s^2$, as proportional to s^{2m-2}. Since $\sigma_{\rm el}(s)\leq\sigma(s)$, one finds $2m-2\leq m-1$, that is $m\leq1$. One can argue similarly and conclude the same, also, in the cases when $A(s,t)\propto s^m\beta(t)\exp(\alpha t\ln s)$ or even $s^m(\ln s)^n\beta(t)\exp(\alpha t\ln s)$ as $s\to\infty$, where α is a positive constant and n is an integer.

One then proves Pomeranchuk's theorem by direct computations. First, the case of $m=1$ can be attained in (4) by various combinations of $\delta(\infty)$ and $\bar{\delta}(\infty)$. However, one finds always that the limits in (4) are the same. For example, when $\delta(\infty)=\pi/2$ and $\bar{\delta}(\infty)=-\pi/2$, one finds that $n=1$ and the limits in (4) are both $ics(s_0/u_0)^{1/2}$, implying that $\sigma(\infty)=\bar{\sigma}(\infty)=c(s_0/u_0)^{1/2}$. In the case of $m=0$, however, one finds always that the limits in (4) have different signs. Since this contradicts the optical theorem, the case of $m=0$ is not permissible. One finds similarly that the case of $m=-1$ is permissible and that $\sigma(s)\simeq\bar{\sigma}(s)\propto s^{-2}$ as $s\to\infty$ in this case. This way one proves that the total cross sections behave asymptotically only as s^0, s^{-2}, etc., and always $\sigma(s)\simeq\bar{\sigma}(s)$ as $s\to\infty$.

It is likely that the forward elastic amplitudes for strongly interacting particles become pure imaginary in the high-energy limit. If this is actually the case, our proof indicates that Pomeranchuk's theorem is valid in all pairs of the total cross sections which involve strongly interacting particles.

3. ASYMPTOTIC FORMS OF TOTAL CROSS SECTIONS

We assume in this section that high-energy elastic scattering is described by an effective complex and energy-dependent potential $V(r,s)$ of the type proposed previously.[5] According to our previous work,[5] the requirements that $V(r,s)$ does not vanish at $s=\infty$ and becomes pure imaginary as $s\to\infty$ and that $V(r,s)$ is analytic in s, imply that $\mathrm{Im}V(r,s)$ diverges as $s\to\infty$ as $s^{1/2}$, $s^{3/2}$, etc. We assume in this paper that $\mathrm{Im}V(r,s)$ diverges as $(s)^{1/2}$, because this is probably the only physically plausible behavior. Then, $V(r,s)$ satisfies a dispersion relation

$$V(r,s)=V(r,0)+\frac{s}{\pi}\int_a^{\infty}\frac{\mathrm{Im}V(r,s')ds'}{s'(s'-s)}+\text{poles}.\quad(5)$$

Suppose that $\mathrm{Im}V(r,s)\simeq-(s)^{1/2}V_I(r)$ for $s\geq b$, where $V_I(r)$ is a real positive function of r, and b is some large number. Then, the principal value integral in (5) approaches a finite limit as $s\to\infty$, since

$$\frac{s}{\pi}P\int_a^{\infty}\frac{\mathrm{Im}V(r,s')ds'}{s'(s'-s)}=\frac{s}{\pi}\int_a^{b}\frac{\mathrm{Im}V(r,s')ds'}{s'(s'-s)}-\frac{s}{\pi}P\int_b^{\infty}\frac{(s')^{1/2}V_I(r)ds'}{s'(s'-s)}\xrightarrow[s\to\infty]{}-(1/\pi)\times\int_a^b\mathrm{Im}V(r,s')ds'/s'-(2(b)^{1/2}/\pi)V_I(r).\quad(6)$$

Therefore, $V(r,s)$ behaves as $s\to\infty$ as

$$V(r,s)\to V_R(r)-i(s)^{1/2}V_I(r),\quad(7)$$

where $V_R(r)$ is some real, finite function of r. However, the sign of $V_R(r)$ cannot be determined by that of $V_I(r)$ alone.

The asymptotic form (7) of $V(r,s)$ then determines the asymptotic form of the phase of the forward ampli-

[7] The asymptotic limits of the exponential factors in (2) are derived in Ref. 6.

tude $A(s)$ by means of the well-known formula[8]

$$A(s)=4\pi is\int_0^\infty\left[1-\exp\left\{-\frac{i}{2(s)^{1/2}}\int_{-\infty}^{+\infty}V(r,s)dz\right\}\right]bdb,\tag{8}$$

where $V(r,s)$ is given by (7). One easily finds from (8) that the phase $\delta(s)$ of $A(s)$ behaves as $s\to\infty$ as

$$\delta(s)-\delta(\infty)\simeq\delta/(s)^{1/2},\tag{9}$$

where δ is a real constant.

The asymptotic form (9) of the phase should be valid for both $\delta(s)$ and $\bar{\delta}(s)$ in (2) because both $A(s)$ and $\bar{A}(s)$ are the forward elastic amplitudes. In order to compute the asymptotic forms of $A(s)$ and $\bar{A}(s)$ by means of (2) and (9), one rewrites[9] (2) as

$$A(s)=\frac{P_1(s)}{P_2(s)}Q\left[\frac{s_0}{s_0-s}\right]^{\delta(\infty)/\pi}\left[\frac{u_0}{u_0-u}\right]^{\bar{\delta}(\infty)/\pi}\gamma(s),\tag{10}$$

where Q is a positive real constant and

$$\gamma(s)\equiv\exp\left\{\frac{1}{\pi}\int_{s_0}^\infty\frac{\delta(s')-\delta(\infty)}{s'-s}ds'+\frac{1}{\pi}\int_{u_0}^\infty\frac{\bar{\delta}(s')-\bar{\delta}(\infty)}{s'-u}ds'\right\},\tag{11}$$

and a similar expression for $\bar{A}(s)$. In terms of (9) for $s\geq s_1$ and $\bar{\delta}(s)-\bar{\delta}(\infty)\simeq\bar{\delta}/(s)^{1/2}$ for $s\geq u_1$, where s_1 and u_1 are some large, but finite numbers, one finds after simple computation that

$$\gamma(s)\to\left\{1+\frac{\bar{\delta}}{(s)^{1/2}}+O\left(\frac{1}{s}\right)\right\}+i\frac{\delta}{(s)^{1/2}}\tag{12}$$

as $s\to\infty$. One thus obtains our final results,

$$\begin{aligned}A(s)&\to is\sigma(\infty)\left[\left\{1+\frac{\bar{\delta}}{(s)^{1/2}}+O\left(\frac{1}{s}\right)\right\}+i\frac{\delta}{(s)^{1/2}}\right],\\ \bar{A}(s)&\to is\sigma(\infty)\left[\left\{1+\frac{\delta}{(s)^{1/2}}+O\left(\frac{1}{s}\right)\right\}+i\frac{\bar{\delta}}{(s)^{1/2}}\right],\end{aligned}\tag{13}$$

and, therefore,

$$\begin{aligned}\sigma(s)&\to\sigma(\infty)\left[1+\frac{\bar{\delta}}{(s)^{1/2}}+O\left(\frac{1}{s}\right)\right],\\ \bar{\sigma}(s)&\to\sigma(\infty)\left[1+\frac{\delta}{(s)^{1/2}}+O\left(\frac{1}{s}\right)\right],\end{aligned}\tag{14}$$

as $s\to\infty$.

[8] The expression (8) is the same as the equation (7) of Ref. 5.

[9] This is explained in full detail in Ref. 6.

We remark that the $1/(s)^{1/2}$ terms in (13) and (14) depend only on the asymptotic phase (9) and, therefore, are entirely due to the asymptotic form (7) of the effective potential. However, the $1/s$ terms in (13) and (14) depend also on the low-energy aspects, such as the branch points s_0 and u_0. Therefore, it seems reasonable to regard those energy regions as asymptotic in which the $1/s$ terms in the total cross sections become insignificant.

We cannot estimate even the signs of δ and $\bar{\delta}$ in (13) and (14) because they depend on both $V_R(r)$ and $V_I(r)$ in (7). However, we can argue that $\delta=\bar{\delta}$ at least in the case of $\pi^\pm-p$ scsttering. If $\delta\neq\bar{\delta}$, then $\sigma(s)-\bar{\sigma}(s)$ approaches zero as $(s)^{-1/2}$. This means that $A(s)-\bar{A}(s)$ diverges as $(s)^{1/2}$. However, $A(s)-\bar{A}(s)$ is known empirically to satisfy a no-subtraction dispersion relation in the case of $\pi^\pm-p$ scattering. Therefore, one must have $\delta=\bar{\delta}$. In fact, the asymptotic forms (14) become consistent with this empirical fact if $\delta=\bar{\delta}$.

4. COMPARISON WITH EXPERIMENTS

We now compare with (14) the experimental cross sections for $\pi^\pm-p$[3], $p-p$[4], and $\bar{p}-p$[4] scattering in the lab momentum range 10 to 20 BeV/c [in s, 20 to 40 (BeV)^2]. All these data clearly do not satisfy (14) without the $1/s$ terms. This is because $\sigma_{\pi^-p}(s)-\sigma_{\pi^+p}(s)$ is still appreciable and $\sigma_{\bar{p}p}(s)-\sigma_{pp}(s)$ decreases too rapidly to fit a $(s)^{-1/2}$ dependence in spite of large experimental errors. Therefore, we have done the following analysis. We put, for simplicity, $\delta=\bar{\delta}$ in (14) also for $p-p$ and $\bar{p}-p$ scattering. Thus, we fit $\sigma(s)-\bar{\sigma}(s)$ by a pure $1/s$ term. We then fit $\sigma(s)+\bar{\sigma}(s)$ by a constant plus a $(s)^{-1/2}$ term alone, also for simplicity. The results in mb and (BeV)^2 units are

$$\begin{aligned}\sigma_{\pi^\pm p}(s)&=23.2\left[1+\frac{0.48}{(s)^{1/2}}\mp\frac{0.71}{s}\right],\\ \sigma_{pp}(s),\ \sigma_{\bar{p}p}(s)&=35\left[1+\frac{1.6}{(s)^{1/2}}\mp\frac{5.6}{s}\right].\end{aligned}\tag{15}$$

These formulas fit the data very well, though they should not be taken too seriously because the experimental errors are large and also the arbitrary choices are made for the parameters. It is, however, interesting to note that $\sigma_{pp}(s)$ in (15) varies from 38 mb very slowly to 39 mb over the momentum range concerned.

The major point of our asymptotic forms (14) is that both $\sigma(s)$ and $\bar{\sigma}(s)$ have relatively slow approaches to their limits at $s=\infty$. Furthermore, it is the asymptotic phase of the amplitude for its antiparticle (not the particle itself) that determines the asymptotic form of the total cross section. According to (14), therefore, one should judge whether the energy is high enough to be asymptotic, not merely by a nearly constant behavior of $\sigma(s)$, but rather by observing both $\sigma(s)$ and $\bar{\sigma}(s)$ to see if they behave similarly in the sense of (14). For this reason and also based upon the fit (15), we are inclined to conclude that available $p-p$ data are not so close to the asymptotic region as the $\pi^{\pm}-p$ data for the same available momentum range. We recall that the model underlying the asymptotic forms (14) predicts[5] no shrinkage in the forward peak of high-energy elastic scattering. Therefore, we understand at least qualitatively the reason why the recent experimental data[10] indicate no shrinkage in $\pi^{\pm}-p$ scattering, but appreciable shrinkage in $p-p$ scattering.

If one combines the fit (15) with (13), one can estimate a deviation from the optical point as

$$|\text{Re}A(s)/\text{Im}A(s)|^2 \simeq \delta^2/s \simeq 1\% \qquad (15)$$

at the lab momentum 10 BeV/c for $\pi^{\pm}-p$ scattering. This figure violently disagrees with 23±10%, a figure suspected in a recent report.[11] The same estimate gives a deviation of 13% for $p-p$ scattering at the same lab momentum.

We remark finally that all our arguments are valid also when the particles have spins. Our arguments then apply individually to the amplitudes with the spin directions specified and the corresponding total cross sections. Therefore, our arguments apply also to the spin-averaged ones.

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Two-Pion-Exchange Contribution to the Three-Body Λ-Nucleon Interaction*

J. D. Chalk, III,† and B. W. Downs‡
University of Colorado, Boulder, Colorado

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The two-pion-exchange contribution to the three-body Λ-nucleon interaction is derived from a static model and also from covariant perturbation theory. It is found that the local part of the potential calculated by the latter method is similar to that part of the static-model potential which corresponds to the formation of lambda-antisigma pairs in intermediate states. This potential is noncentral and has the form $(\boldsymbol{\tau}^1\cdot\boldsymbol{\tau}^2)(\boldsymbol{\sigma}^1\cdot\mathbf{r}_1)(\boldsymbol{\sigma}^2\cdot\mathbf{r}_2)f(r_1,r_2)$, where $\boldsymbol{\sigma}^i$ and $\boldsymbol{\tau}^i$ are the spin and isotopic-spin operators for the two nucleons, and $\mathbf{r}_1$ and $\mathbf{r}_2$ are the Λ-nucleon separation vectors. An estimate is made of the importance of this potential in the binding of the hypertriton by calculating its expectation value with respect to hypertriton wave functions corresponding to two-body interactions with hard cores. In these calculations, the three-body potential is found to contribute less than 5% of the expectation value of the total Λ-nucleon interaction.

I. INTRODUCTION

Analyses of the binding-energy data for the hypernuclei with $A \geqslant 3$ have been made to determine characteristics of the Λ-nucleon interaction.[1–4] Uncertainties in these analyses have precluded the deduction of a complete set of parameters characterizing these interactions; in particular, it has not been possible to establish the presence of Λ-nucleon-nucleon three-body interactions. When three-body interactions have been neglected, these analyses have led to the specification of parameters characterizing central two-body S-wave potentials which include the effect of possible tensor components.[1–3] The resulting two-body potentials are strong and highly spin-dependent. It has been noted that the deduced spin dependence depends critically upon the assumption that the effect of three-body interactions is negligible in the binding of hypernuclei.[1,4,5] Bodmer and Sampanthar[4] have recently made a quantitative connection between the assumed strength of three-body potentials of the form

$$(\boldsymbol{\tau}^1\cdot\boldsymbol{\tau}^2)(\boldsymbol{\sigma}^1\cdot\boldsymbol{\sigma}^2)V(\mathbf{R}_1,\mathbf{R}_2,\mathbf{R}_\Lambda), \qquad (1)$$

and the spin dependence of the corresponding two-body interactions required to account for the binding energies of the lightest hypernuclei. [In (1), 1, 2 and Λ denote the coordinates of the two nucleons and the Λ particle, respectively.] Previously, Weitzner[5] had similarly determined the required strength of a potential of the form

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† Present address: The Department of Physics, Rice University, Houston 1, Texas.

‡ Address during academic year 1963–64: Department of Theoretical Physics, The University of Oxford, England.

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