Production and Scattering in Simple Models*

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Some properties of the $V\theta$ scattering amplitude in the Lee model are discussed with reference to the influence of production ($N\theta\theta$) on scattering. As long as the coupling is not too weak, production has important effects on scattering even at low energies. For a wide class of well-behaved cutoff functions there can be no resonance in $V\theta$ scattering unless production is taken into account. Static model calculations for meson-baryon scattering neglecting production can therefore be quite misleading. There is in general no simple relation between the locations of resonances in $V\theta$ scattering and the production threshold, even in the strong coupling limit $Z\approx 0$. A modified version of the Lee model with odd-$NV\theta$ parity is discussed. Two different sets of integral equations for the $V\theta\rightarrow V\theta$ and $V\theta\rightarrow N\theta\theta$ amplitudes are derived, in which the influence of spin enters in a nontrivial way.

I. INTRODUCTION

THE Lee model[1] has proved useful in clarifying many dynamical questions[2–4] as well as questions of renormalization.[1] In particular, $V\theta$ scattering provides an example of scattering with a production ($N\theta\theta$) channel. In the first part (Sec. II) of this paper we derive and discuss some properties of the $V\theta$ scattering amplitude, with special attention given to the influence of production on scattering, and the possibility of a resonance resulting from the opening of the production channel as proposed by Ball and Frazer.[5] It will be seen that the existence of the production channel enhances elastic scattering even below the production threshold, and under certain circumstances a resonance below the production threshold is possible. The location of the resonance however depends on the form of the cutoff function and in general bears no simple relation to the production threshold. Except in the weak-coupling limit, the inclusion of production changes the high-energy behavior of the scattering amplitude and greatly modifies the low-energy behavior of the real part of the scattering amplitude. This throws some doubt on the adequacy of static model calculations which neglect production.

In the second part (Sec. III) of this paper we consider a modified version of the Lee model with a pseudoscalar θ. In this pseudoscalar model the V and $N\theta$ states are trivial modifications of the scalar model, essentially because $N\theta$ scattering takes place only through one ($p_{1/2}$) channel. For $V\theta$ scattering both $p_{1/2}$ and $p_{3/2}$ channels are open, resulting in coupled integral equations not soluble in closed form by standard methods.[6] In this model $V\theta$ scattering is analogous to low-energy $K^{+}p$ scattering (in view of the odd KY parity) neglecting pionic effects. The fact that this model gives only P-wave scattering in disagreement with experiment may be an indication of the importance of the $K-\pi$ interaction in $K-N$ scattering.

II. SCALAR CASE

The $V\theta$ scattering amplitude has been obtained by Amado[6] in closed form and may be written[7]

$$T(\omega)=\frac{g^2}{(\omega-\Delta)}\left[\frac{1-(\omega-\Delta)C(\omega)}{1+(\omega-\Delta)C(\omega)}+\beta(\omega)\right]^{-1}\equiv\frac{g^2/(\omega-\Delta)}{D(\omega)},\quad(1)$$

where g denotes the renormalized coupling constant and we have included the (renormalized) $V-N$ mass difference $m_V-m_N\equiv\Delta$. The functions β and C are given by[8]

$$\beta(\omega)=-\frac{g^2(\omega-\Delta)}{4\pi^2}\int_1^\infty\frac{(\omega_1^2-1)^{1/2}f^2(\omega_1)d\omega_1}{(\omega_1-\Delta)^2(\omega_1-\omega-i\epsilon)}\quad(2)$$

and

$$C(\omega)=\frac{g^2}{4\pi^2}\times\int_1^\infty\frac{(\omega_1^2-1)^{1/2}f^2(\omega_1)\beta(\omega+\Delta-\omega_1)d\omega_1}{(\omega_1-\Delta)^2|1-\beta(\omega_1)|^2(\omega_1-\omega)[1-\beta(\omega+\Delta-\omega_1)]},\quad(3)$$

the cutoff function $f^2(\omega)$ being such that all integrals converge and no ghost arises. This means that

$$1-\beta(\infty)=Z,\quad\text{where}\quad O<Z<1\quad(4)$$

and we shall require that

$$f^2(\omega)=O(1/\omega^\nu)\text{ for large }\omega,\text{ where }\nu>0.\quad(5)$$

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7 Throughout this paper we consider only the case when the V is stable: $m_V-m_N<m_\theta$. All the masses and coupling constants appearing in the equations are renormalized quantities.
8 The mass of the θ has been taken to be unity.

We begin by first neglecting production. The function $D(\omega)$ in this approximation is then simply $1+\beta(\omega)$.[6] For a resonance to occur, it is necessary that $\mathrm{Re}D(\omega)=0$. Since $\beta(1)<0$, the quantity $1+\beta(1)$ is less than unity and may be negative.[9] At the high-energy limit, we have $1+\beta(\infty)=2-Z>0$. Suppose $1+\beta(1)<0$. If the cutoff function $f^2(\omega)$ is sufficiently smooth (differentiability would suffice, but is not necessary) for $\omega\geq 1$, the real part of $\beta(\omega)$ is continuous in the physical region so the phase shift begins as π at zero energy and falls to zero at infinite energy, passing through $\frac{1}{2}\pi$ at some finite energy.[10] This behavior corresponds to the existence of a bound state in the $V\theta$ system according to Levinson's theorem.[11] It is well known, however, that such a case in general does not give a bump in the scattering cross section, as follows from a theorem of Wigner.[12]

Suppose then $1+\beta(1)>0$. We shall show that for a wide class of cutoff functions, $\mathrm{Re}[1+\beta(\omega)]\geq 1+\beta(1)$ for $\omega\geq 1$, and therefore no resonance is possible.[13] More precisely, if $f^2(\omega)(\omega+1)^{1/2}/(\omega-\Delta)$ is a nonincreasing function of ω for $\omega\geq 1$, then $\mathrm{Re}\beta(\omega)\geq\beta(1)$ for $\omega\geq 1$. To prove this, let

$$F(\omega)=(\omega-\Delta)P\int_1^\infty \frac{f^2(\omega_1)(\omega_1{}^2-1)^{1/2}d\omega_1}{(\omega_1-\Delta)^2(\omega_1-\omega)},$$

so that $\mathrm{Re}\beta(\omega)=-g^2F(\omega)/4\pi^2$. We wish to show that $F(\omega)\leq F(1)$ for $\omega\geq 1$. But for $\omega>1$ we have

$$\begin{aligned}F(\omega)-F(1)&\\ &=(\omega-1)P\int_1^\infty \frac{f^2(\omega_1)(\omega_1{}^2-1)^{1/2}d\omega_1}{(\omega_1-\Delta)(\omega_1-1)(\omega_1-\omega)}\\ &=(\omega-1)\left\{\frac{f^2(\omega)(\omega+1)^{1/2}}{(\omega-\Delta)}P\int_1^\infty\frac{d\omega_1}{(\omega_1-1)^{1/2}(\omega_1-\omega)}\right.\\ &\quad+\int_1^\infty\left[\frac{(\omega_1+1)^{1/2}}{(\omega_1-\Delta)}f^2(\omega_1)-\frac{(\omega+1)^{1/2}}{(\omega-\Delta)}f^2(\omega)\right]\\ &\quad\left.\times\frac{d\omega_1}{(\omega_1-1)^{1/2}(\omega_1-\omega)}\right\}.\qquad(6)\end{aligned}$$

[9] That this is possible without violating the unitarity condition (4) may be seen by letting $\Delta=0$, $f^2(\omega)=1/(\omega+1)$. Then condition (4) gives $0<1-g^2[\frac{1}{2}\pi-1]/4\pi^2<1$, or $g^2/4\pi^2<2/(\pi-2)\approx 1.4$. Taking $g^2/4\pi^2=1.3$, for example, we have $1+\beta(1)=1-1.3<0$.

[10] It is, in principle, possible that the function $\mathrm{Re}[1+\beta(\omega)]$ may go through zero 3, 5, 7, etc. times before approaching the high-energy limit $2-Z>0$. For the 2nd, 4th, etc. zeros Wigner's theorem (Ref. 12) does not apply. However, it is easy to convince oneself that such behavior calls for quite unreasonable cutoff functions.

[11] N. Levinson, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **25**, No. 9 (1949).

[12] E. P. Wigner, Phys. Rev. **98**, 145 (1955).

[13] Our result below is useful also in discussing the $N\theta$ scattering amplitude, which has the form $g^2(\Delta-\omega)^{-1}[1-\beta(\omega)]^{-1}$, and shows that the real part of $1-\beta(\omega)$ *can* vanish for $\omega>1$. It is therefore possible to have a resonance in $N\theta$ scattering, in spite of the fact that the $N\theta$ interaction is repulsive. This conclusion has also been reached independently by Aitchison (private communication) and Fonda (to be published).

The principal value integral in the braces may be directly evaluated and has the value zero. So Eq. (6) reduces to

$$\begin{aligned}F(\omega)-F(1)&\\ &=(\omega-1)\int_1^\infty\left[\frac{(\omega_1+1)^{1/2}}{(\omega_1-\Delta)}f^2(\omega_1)-\frac{(\omega+1)^{1/2}}{(\omega-\Delta)}f^2(\omega)\right]\\ &\quad\times\frac{d\omega_1}{(\omega_1-1)^{1/2}(\omega_1-\omega)},\qquad(7)\end{aligned}$$

which is nonpositive if $[(\omega+1)^{1/2}/(\omega-\Delta)]f^2(\omega)$ is nonincreasing, for then the integrand is never positive. It is clear from (7) that our assumption is sufficient but by no means necessary. For example, if $f^2(\omega)=1/(\omega+\alpha)$, then $F(\omega)\leq F(1)$ for all $\alpha\geq 0$ and $\omega\geq 1$. Other examples may also be readily constructed, but there is not much point in pursuing this further, as the behavior of $\mathrm{Re}D(\omega)$ is quite different when production is taken into account. We merely conclude that for a wide class of well-behaved cutoff functions, there is no resonance in $V\theta$ scattering if production is neglected.

Next we discuss the full amplitude (1). It is convenient to list here some properties of the function $C(\omega)$. Properties (a) and (b) are obvious from the definitions while the others will be proved.

(a) $\mathrm{Im}C(\omega)=0$ for $\omega\leq 2-\Delta$
≥ 0 for $\omega>2-\Delta$.

It follows from this that

$$\mathrm{Im}\{[1-(\omega-\Delta)C(\omega)]/[1+(\omega-\Delta)C(\omega)]\}\leq 0$$

for all ω. Recalling $\mathrm{Im}\beta(\omega)\leq 0$, we see that production has the effect of enhancing the imaginary part of the scattering amplitude above the production threshold.

(b) $C(1)\geq 0$. This means that $D(1)\leq 1+\beta(1)<1$, and $D(1)$ can be negative for large values of g^2.

(c) For $1\leq\omega<2-\Delta$, $C(\omega)$ is a real, increasing function of ω. To prove this let $\Gamma=g^2/4\pi^2$ and

$$\varphi(z)=\int_1^\infty\frac{f^2(\omega)(\omega^2-1)^{1/2}d\omega}{(\omega-\Delta)^2(\omega-z)}.$$

Then (4) implies that $\varphi(\omega)\to-\lambda/\omega$ as $\omega\to\infty$, where $\lambda\Gamma<1$. One has

$$\Gamma^{-2}C(\omega)=\int_1^\infty\frac{(\omega_1{}^2-1)^{1/2}f^2(\omega_1)d\omega_1}{(\omega_1-\Delta)^2|1-\beta(\omega_1)|^2}\times\frac{\varphi(\omega+\Delta-\omega_1)}{1+\Gamma(\omega-\omega_1)\varphi(\omega+\Delta-\omega_1)}.\qquad(8)$$

For $\omega<2-\Delta$ the argument of $\varphi(\omega+\Delta-\omega_1)$ over the region of integration ranges from $-\infty$ to $\omega+\Delta-1<1$, where φ is analytic and may be differentiated under the integral sign. We shall prove property (c) by showing that under the integral sign

$$\frac{d}{d\omega}\left[\frac{\varphi(\omega+\Delta-\omega_1)}{1+\Gamma(\omega-\omega_1)\varphi(\omega+\Delta-\omega_1)}\right]=\frac{-\Gamma-(d/d\omega)(1/\varphi)}{[1+\Gamma(\omega-\omega_1)\varphi]^2(1/\varphi)^2}>0, \quad (9)$$

where we have suppressed the argument of $\varphi(\omega+\Delta-\omega_1)$. For this purpose define a function[14]

$$\Lambda(z)=(1/\lambda)+(1/z\varphi(z)), \quad (10)$$

which is clearly analytic in the upper half-plane with a cut from $z=1$ to $z=\infty$, and a pole at $z=0$ with residue $r=1/\varphi(0)>0$. The function $\Lambda(z)$ vanishes at infinity, and therefore has an integral representation of the form

$$\Lambda(z)=\frac{r}{z}-\int_1^\infty \rho(E)dE/(E-z). \quad (11)$$

Since $(1/2i)[\varphi(E+i\epsilon)-\varphi(E-i\epsilon)]\geq 0$ over the cut, it follows that

$$\frac{1}{2i}[\Lambda(E+i\epsilon)-\Lambda(E-i\epsilon)]=\frac{1}{2i}\left[\frac{1}{\varphi(E+i\epsilon)}-\frac{1}{\varphi(E-i\epsilon)}\right]\leq 0,$$

for $E\geq 1$. Thus, $\rho(E)$ is positive definite. We have

$$\frac{1}{\varphi(z)}=z\Lambda(z)-\frac{z}{\lambda}=r-\frac{z}{\lambda}-z\int_1^\infty\frac{\rho(E)dE}{E-z} \quad (12)$$

and for $z=\omega+\Delta-\omega_1<1$ we may differentiate under the integral sign, giving

$$\frac{d}{d\omega}\left(\frac{1}{\varphi}\right)=\frac{1}{\lambda}-\int_1^\infty\frac{\rho(E)dE}{E-z}-z\int_1^\infty\frac{\rho(E)dE}{(E-z)^2}=\frac{1}{\lambda}-\int_1^\infty\frac{E\rho(E)dE}{(E-z)^2}, \quad (13)$$

so that

$$-\Gamma-\frac{d}{d\omega}\left(\frac{1}{\varphi}\right)=-\Gamma+\frac{1}{\lambda}+\int_1^\infty\frac{E\rho(E)dE}{(E-z)^2}>-\Gamma+\frac{1}{\lambda}>0, \quad (14)$$

since $\lambda\Gamma<1$. This shows that $d/d\omega C(\omega)>0$ for $\omega<2-\Delta$ and completes the proof.

(d) The function $(\omega-\Delta)C(\omega)$ approaches the negative real value $-(-1+1/Z)^2$ as $\omega\rightarrow\infty$.

[14] This trick is due to Schwinger [Ann. Phys. (N. Y.) **9**, 169 (1960)].

To prove this, recall that $f^2(\omega)=O(1/\omega^\nu)$, $\nu>0$, for large ω. By definition

$$(\omega-\Delta)C(\omega)=\Gamma\int_1^\infty\frac{(\omega_1^2-1)^{1/2}f^2(\omega_1)d\omega_1}{(\omega_1-\Delta)^2|1-\beta(\omega_1)|^2}\frac{\omega-\Delta}{\omega_1-\omega}\times\frac{\beta(\omega+\Delta-\omega_1)}{1-\beta(\omega+\Delta-\omega_1)}. \quad (15)$$

Since $|1-\beta(\omega_1)|^{-2}$ is bounded, there exists for any $\epsilon>0$ an energy E such that

$$\frac{(\omega_1^2-1)^{1/2}}{(\omega_1-\Delta)^{1+\nu/2}|1-\beta(\omega_1)|^2}<\epsilon,$$

for $\omega_1>E$. Furthermore, for any $\delta>0$ choose ω to be so large that $\omega\gg E$ and

$$\left|\frac{1-Z}{Z}-\frac{\beta(\omega+\Delta-E)}{1-\beta(\omega+\Delta-E)}\right|<\delta.$$

Then we have

$$(\omega-\Delta)C(\omega)=\Gamma\left[\int_1^E+\int_E^\infty\right]\rightarrow\Gamma\left[\int_1^E\frac{(\omega_1^2-1)^{1/2}f^2(\omega_1)d\omega_1}{(\omega_1-\Delta)^2|1-\beta(\omega_1)|^2}(-1)\left(\frac{1-Z}{Z}\right)+\int_E^\infty\frac{f^2(\omega_1)d\omega_1}{(\omega_1-\Delta)^{1-\nu/2}}\frac{\beta(\omega+\Delta-\omega_1)}{1-\beta(\omega+\Delta-\omega_1)}\frac{\omega-\Delta}{\omega_1-\omega}\times\frac{(\omega_1^2-1)^{1/2}}{(\omega_1-\Delta)^{1+\nu/2}|1-\beta(\omega_1)|^2}\right]. \quad (16)$$

The last integral has an absolute value less than

$$\epsilon\int_E^\infty\frac{f^2(\omega_1)d\omega_1}{(\omega_1-\Delta)^{1-\nu/2}}\frac{\beta(\omega+\Delta-\omega_1)}{1-\beta(\omega+\Delta-\omega_1)}\frac{\omega-\Delta}{\omega_1-\omega},$$

which can be made arbitrarily small since the integral converges.[15] Finally, we have, letting $E\rightarrow\infty$ and $\omega\rightarrow\infty$

$$(\omega-\Delta)C(\omega)\rightarrow-\frac{1-Z}{Z}\Gamma\int_1^\infty\frac{(\omega_1^2-1)^{1/2}f^2(\omega_1)d\omega_1}{(\omega_1-\Delta)^2|1-\beta(\omega_1)|^2} \quad (17)$$

$$=-\left(\frac{1-Z}{Z}\right)^2,$$

[15] The integral is bounded by a constant times

$$(\omega-\Delta)P\int_E^\infty d\omega_1/\omega_1^{1+\nu/2}(\omega_1-\omega),$$

which approaches a negative constant for large ω. [See R. Omnes, Nuovo Cimento **8**, 316 (1958).]

where the integral has been evaluated by integrating the function $(1-\beta(\omega))^{-1}$ from $\infty-i\epsilon$ to $1-i\epsilon$ and then from $1+i\epsilon$ to $\infty+i\epsilon$, closing the contour around a circle at infinity and applying Cauchy's theorem.

It follows from property (c) that for $1\leq\omega<2-\Delta$ the function $[1-(\omega-\Delta)C)(\omega)]/[1+(\omega-\Delta)C(\omega)]<1$ is real and decreasing. This means that even below the production threshold, the presence of the production channel is felt through a decrease in the real part of $D(\omega)$, thus enhancing the elastic scattering when $1+\beta(\omega)$ is positive, and inhibiting elastic scattering when $1+\beta(\omega)$ is negative. The function $\mathrm{Re}D(\omega)$ may be rewritten, for $\omega\leq 2-\Delta$

$$\mathrm{Re}D(\omega) =1-[2(\omega-\Delta)C(\omega)/(1+(\omega-\Delta)C(\omega))]+\mathrm{Re}\beta(\omega)\,, \quad (18)$$

which shows that below production threshold $\mathrm{Re}D(\omega)$ consists of a *decreasing* part, roughly of order g^4, and the term $\mathrm{Re}\beta(\omega)$ which is of order g^2 and which, according to the discussion following Eq. (7), in general, *increases* with ω at low energies. For large values of g^2 the production term may dominate over $\mathrm{Re}\beta(\omega)$ and produce a zero of $\mathrm{Re}D(\omega)$ below the production, which is, in general, impossible if production is neglected.

The high-energy limit of $D(\omega)$ follows directly from property (d). It is real and equal to $Z/(2Z-1)$. If g^2 is small, $Z\approx 1$ and $Z/(2Z-1)$ is positive, as is $D(1)$. Therefore, when the interaction is sufficiently weak, the real part of $D(\omega)$ remains positive for all $\omega\geq 1$ and no resonance occurs. In the strong-coupling limit $Z\approx 0$, $Z/(2Z-1)$ is negative. A resonance is then possible when $D(1)>0$, for if $\mathrm{Re}D(\omega)$ is continuous it must pass through zero at least once. This need not be the case if either the cutoff function is singular such that $\mathrm{Re}\beta(\omega)$ and $C(\omega)$ are discontinuous, or the factor $1+(\omega-\Delta)C(\omega)$ vanishes for some $\omega>1$. Let us examine the latter possibility a little more closely. By property (b), the quantity $1+(1-\Delta)C(1)$ is positive, and by property (c) the function $(\omega-\Delta)C(\omega)$ is real and increasing for $1\leq\omega<2-\Delta$. Therefore, $1+(\omega-\Delta)C(\omega)$ cannot vanish for $1\leq\omega<2-\Delta$. Above the production threshold $\omega\geq 2-\Delta$, the imaginary part of $C(\omega)$ is given by

$$\left(\frac{g^2}{4\pi^2}\right)^2 \mathrm{Im}\int_1^{\omega+\Delta-1}\frac{(\omega_1^2-1)^{1/2}f^2(\omega_1)d\omega_1}{(\omega_1-\Delta)^2|1-\beta(\omega_1)|^2}\frac{\varphi(\omega+\Delta-\omega_1)}{1-\beta(\omega+\Delta-\omega_1)}$$
$$=\left(\frac{g^2}{4\pi^2}\right)^2\int_1^{\omega+\Delta-1}\frac{(\omega_1^2-1)^{1/2}f^2(\omega_1)d\omega_1}{(\omega_1-\Delta)^2|1-\beta(\omega_1)|^2}\times\frac{\mathrm{Im}\,\varphi(\omega+\Delta-\omega_1)}{|1-\beta(\omega+\Delta-\omega_1)|^2},$$

where

$$\varphi(z)=\int_1^\infty\frac{(\omega_1^2-1)^{1/2}f^2(\omega_1)d\omega_1}{(\omega_1-\Delta)^2(\omega_1-z)},$$

and

$$\mathrm{Im}\,\varphi(z)=\pi(z^2-1)^{1/2}f^2(z)\theta(z-1)/(z-\Delta)^2.$$

Since the integrand is non-negative, $\mathrm{Im}C(\omega)$ can vanish only if $f^2(\xi)f^2(\omega+\Delta-\xi)=0$ for $1\leq\xi\leq\omega+\Delta-1$. Hence, the function $1+(\omega-\Delta)C(\omega)$ has no zeros for $\omega\geq 1$ if the product $f^2(\omega_1)f^2(\omega+\Delta-\omega_1)$ is different from zero over a finite interval in the range $1\leq\omega_1\leq\omega-1+\Delta$, provided $f^2(\omega)$ is not so singular that $C(\omega)$ is discontinuous at $\omega=2-\Delta$.

For large values of g^2 it may also happen that $D(1)<0$ and $\mathrm{Re}D(\omega)$ remain negative in the physical region, so that there is again no resonance.

Summarizing, we give a set of sufficient conditions for a resonance (defined as a situation where $\mathrm{Re}D(\omega)$ decreases through zero at some finite energy) to occur in $V\theta$ scattering:

(i) $D(1)>0$ and $Z<\frac{1}{2}$.
(ii) The cutoff function $f^2(\omega)$ is sufficiently smooth such that both $\beta(\omega)$ and $C(\omega)$ are continuous for $\omega\geq 1$.
(iii) $\int_1^{\omega+\Delta-1}f^2(\omega_1)f^2(\omega+\Delta-\omega_1)d\omega_1>0$ for $2-\Delta<\omega<\infty$.

The location and width of the resonance clearly depend on the product $g^2f^2(\omega)$ in an essential way and the resonance in general needs not lie above or close to the production threshold.

III. PSEUDOSCALAR CASE

We now turn to the case when the $NV\theta$ relative parity is odd. As is well known, the spins of the heavy particles are irrelevant in $N\theta$ and $V\theta$ scattering in the standard Lee model, and the same scattering amplitudes are obtained if the V and the N are scalar particles instead of fermions. This "degeneracy" is removed when the θ is pseudoscalar and the interaction is of the type $\boldsymbol{\sigma}\cdot\nabla$ which can flip the spin.

We begin with the Hamiltonian

$$H=H_0+H_I$$
$$H_0=Zm_V\psi_V^\dagger\psi_V+m_N\psi_N^\dagger\psi_N+\Sigma\,\omega_{\mathbf{K}}a_{\mathbf{K}}^\dagger a_{\mathbf{K}}$$
$$H_I=g\sum_{\mathbf{K}}\frac{f(\omega)}{(2\omega V)^{1/2}}[\psi_V^\dagger\boldsymbol{\sigma}\cdot\mathbf{K}\psi_N a_{\mathbf{K}}+\psi_N^\dagger\boldsymbol{\sigma}\cdot\mathbf{K}\psi_V a_K^\dagger]+Z\delta m_V\psi_V^\dagger\psi_V\,, \quad (19)$$

where

$$\psi_V=\sum_\alpha b_V{}^\alpha v_\alpha\,,\quad \psi_N=\sum_\alpha b_N{}^\alpha n_\alpha\,,$$

v_α and n_α being Pauli spinors of the V and N fields. Otherwise the notation is standard,[2] with the commutation relations

$$\{b_V{}^{\alpha\dagger},b_V{}^\beta\}=\frac{1}{Z}\delta_{\alpha\beta}\,,\quad \{b_N{}^{\alpha\dagger},b_N{}^\beta\}=\delta_{\alpha\beta}\,,$$
$$[a_{\mathbf{K}'}{}^\dagger,a_{\mathbf{K}}]=\delta_{\mathbf{KK}'},\ \text{etc.} \quad (20)$$

The V and $N\theta$ states are trivial generalizations of those in the scalar case. We shall merely list the results here as they will be needed later.

Let

$$f_\beta(t)=\left(-i\frac{d}{dt}+m_V\right)b_V{}^\beta(t)\,,\quad f_\beta=f_\beta(0)\,,$$

$$j(t)=\frac{(2\omega V)^{1/2}}{f(\omega)}\left(-i\frac{d}{dt}+\omega\right)a_{\mathbf{K}}(t)\,,\quad j=j(0)\,.$$

Then we have

$$f_\beta(t)=-\delta m_V b_V{}^\beta(t)-\frac{g}{Z}\sum_{\mathbf{K}}\frac{f(\omega)}{(2\omega V)^{1/2}}v_\beta{}^\dagger\boldsymbol{\sigma}\cdot\mathbf{K}\psi_N a_{\mathbf{K}}\,,\quad(21)$$

$$j(t)=-g\psi_N{}^\dagger\boldsymbol{\sigma}\cdot\mathbf{K}\psi_V\,,\quad(22)$$

$$\langle N_\beta|j|V_\alpha\rangle=-gn_\beta{}^\dagger\boldsymbol{\sigma}\cdot\mathbf{K}\,v_\alpha\,,\quad(23)$$

$$\langle 0|f_\beta|N_\alpha\mathbf{K},\text{in}\rangle=-\frac{gf(\omega)}{(2\omega V)^{1/2}[1-\tilde{\beta}(\omega)]}v_\beta{}^\dagger\boldsymbol{\sigma}\cdot\mathbf{K}n_\alpha\,,\quad(24)$$

$$\langle N_\beta\mathbf{K}',\text{out}|N_\alpha\mathbf{K},\text{in}\rangle=\delta_{\alpha\beta}\delta_{\mathbf{KK}'}+2\pi i\delta(\omega-\omega')\frac{f(\omega')}{(2\omega'V)^{1/2}}\times\langle N_\beta|j'|N_\alpha\mathbf{K},\text{in}\rangle\,,\quad(25)$$

$$\langle N_\beta|j'|N_\alpha\mathbf{K},\text{in}\rangle=\frac{g^2f(\omega)n_\beta{}^\dagger\boldsymbol{\sigma}\cdot\mathbf{K}'\boldsymbol{\sigma}\cdot\mathbf{K}n_\alpha}{(2\omega V)^{1/2}(\Delta-\omega)[1-\tilde{\beta}(\omega)]}\,,\quad(26)$$

$$\tilde{\beta}(\omega)=-\frac{g^2(\omega-\Delta)}{4\pi^2}\int_1^\infty\frac{(\omega_1{}^2-1)^{3/2}f^2(\omega_1)d\omega_1}{(\omega_1-\Delta)^2(\omega_1-\omega-i\epsilon)}\,.\quad(27)$$

The $N\theta$ scattering phase shift δ_N is given by

$$e^{i\delta_N}\sin\delta_N=g^2f^2(\omega)(\omega^2-1)^{3/2}/4\pi(\Delta-\omega)[1-\tilde{\beta}(\omega)]\,.\quad(28)$$

Equations (25) and (26) show that $N\theta$ scattering takes place only through the $p_{1/2}$ state. This can also be seen by inspection of the relevant Feynman diagrams, and is the main reason of the close analogy to the scalar case. The analogy is largely lost for $V\theta$ scattering, which has two amplitudes instead of one.

Next consider the amplitudes for $V\theta\to V\theta$ and $V\theta\to N\theta\theta$. Again the calculations are quite similar to the scalar case,[6] and we sketch only those steps where our equations look appreciably different from the scalar case. The relevant amplitudes are

$$\langle V_\beta\mathbf{K}',\text{out}|V_\alpha\mathbf{K},\text{in}\rangle=\delta_{\alpha\beta}\delta_{\mathbf{KK}'}+2\pi i\delta(\omega-\omega')\frac{f(\omega)f(\omega')}{2V(\omega\omega')^{1/2}}T_{\beta\alpha}(\mathbf{K}',\mathbf{K})\,,$$

where

$$T_{\beta\alpha}(\mathbf{K}',\mathbf{K})=\frac{(2\omega V)^{1/2}}{f(\omega)}\langle V_\beta|j'|V_\alpha\mathbf{K},\text{in}\rangle\equiv v_\beta{}^\dagger[\mathbf{K}\cdot\mathbf{K}'\alpha(\omega)+i\boldsymbol{\sigma}\cdot(\mathbf{K}\times\mathbf{K}')\gamma(\omega)]v_\alpha\,,\quad(29)$$

and

$$\langle N_\beta\mathbf{K}',\mathbf{K},\text{out}|V_\alpha\mathbf{K}_0,\text{in}\rangle=2\pi i\delta(\omega_0-\omega-\omega')\frac{1}{\sqrt{2}}\frac{f(\omega)}{(2\omega V)^{1/2}}\times\langle N_\beta\mathbf{K}',\text{out}|j|V_\alpha\mathbf{K}_0,\text{in}\rangle\,.\quad(30)$$

Instead of (30) it is more convenient to work with the amplitude[6]

$$\langle N_\beta\mathbf{K}',\text{in}|j|V_\alpha\mathbf{K}_0,\text{in}\rangle=e^{-2i\delta_N}\langle N_\beta\mathbf{K}',\text{out}|j|V_\alpha\mathbf{K}_0,\text{in}\rangle\,.$$

Thus let

$$\frac{2V(\omega_0\omega')^{1/2}}{f(\omega')f(\omega_0)}\langle N_\beta\mathbf{K}',\text{in}|j|V_\alpha\mathbf{K}_0,\text{in}\rangle\equiv F_{\beta\alpha}(\mathbf{K}',\mathbf{K},\mathbf{K}_0)$$
$$=n_\beta{}^\dagger[\text{Born term}+\mathbf{K}_0\cdot\mathbf{K}\boldsymbol{\sigma}\cdot\mathbf{K}'F(\omega',\omega_0)+\mathbf{K}_0\cdot\mathbf{K}'\boldsymbol{\sigma}\cdot\mathbf{K}G(\omega',\omega_0)+\mathbf{K}\cdot\mathbf{K}'\boldsymbol{\sigma}\cdot\mathbf{K}_0H(\omega',\omega_0)+i\mathbf{K}_0\cdot(\mathbf{K}\times\mathbf{K}')I(\omega',\omega_0)]v_\alpha\,.\quad(31)$$

Proceedings in the standard way,[6] we find

$$F_{\beta\alpha}(\mathbf{K}',\mathbf{K},\mathbf{K}_0)=\frac{2Vg\omega_0}{f^2(\omega_0)}n_\beta{}^\dagger\boldsymbol{\sigma}\cdot\mathbf{K}\,v_\alpha\delta_{\mathbf{K}_0\mathbf{K}'}-\frac{(2\omega_0V)^{1/2}}{f(\omega_0)}\times\sum_n\left[\frac{\langle N_\beta|j'|n\rangle\langle n|j|V_\alpha\mathbf{K}_0,\text{in}\rangle}{m_N+\omega'-E_n-i\epsilon}-\frac{\langle N_\beta|j|n\rangle\langle n|j'|V_\alpha\mathbf{K}_0,\text{in}\rangle}{\omega'+E_n-m_V-\omega_0-i\epsilon}\right]\,,\quad(32)$$

where only $|n\rangle=|V\rangle$ and $|n\rangle=|N\theta\rangle$ contribute. The sum over the intermediate states is therefore equal to[16]

$$-g\left\{\frac{n_\beta{}^\dagger\boldsymbol{\sigma}\cdot\mathbf{K}'v_\rho}{\omega'-\Delta}v_\rho{}^\dagger[\mathbf{K}_0\cdot\mathbf{K}\alpha(\omega_0)+i\boldsymbol{\sigma}\cdot(\mathbf{K}_0\times\mathbf{K})\gamma(\omega_0)]v_\alpha-\frac{n_\beta{}^\dagger\boldsymbol{\sigma}\cdot\mathbf{K}\,v_\rho}{\omega'-\omega_0-i\epsilon}v_\rho{}^\dagger[\mathbf{K}_0\cdot\mathbf{K}'\alpha(\omega_0)+i\boldsymbol{\sigma}\cdot(\mathbf{K}_0\times\mathbf{K}')\gamma(\omega_0)]v_\alpha\right\}$$
$$-\frac{(2\omega_0V)^{1/2}}{f(\omega_0)}\sum_{\mathbf{K}_1}\left[\frac{\langle N_\beta|j'|N_\rho\mathbf{K}_1,\text{in}\rangle\langle N_\rho\mathbf{K}_1,\text{in}|j|V_\alpha\mathbf{K}_0,\text{in}\rangle}{\omega'-\omega_1-i\epsilon}-\frac{\langle N_\beta|j|N_\rho\mathbf{K}_1,\text{in}\rangle\langle N_\rho\mathbf{K}_1,\text{in}|j'|V_\alpha\mathbf{K}_0,\text{in}\rangle}{\omega'+\omega_1-\omega_0-\Delta-i\epsilon}\right]\,.\quad(33)$$

[16] In the following, repeated indices are to be summed over.

Substituting (26) and (31) into (33) and carrying out the angular integrations, we find for the sum over $\mathbf{K}_1$

$$-\frac{1}{12\pi^2}n_\beta^\dagger\Bigg\{\int_1^\infty\frac{f^2(\omega_1)(\omega_1^2-1)^{3/2}A(\omega_1)d\omega_1}{\omega'-\omega_1-i\epsilon}[3\mathbf{K}_0\cdot\mathbf{K}\boldsymbol{\sigma}\cdot\mathbf{K}'F(\omega_1,\omega_0)+\boldsymbol{\sigma}\cdot\mathbf{K}'\boldsymbol{\sigma}\cdot\mathbf{K}_0\boldsymbol{\sigma}\cdot\mathbf{K}G(\omega_1,\omega_0)+\boldsymbol{\sigma}\cdot\mathbf{K}'\boldsymbol{\sigma}\cdot\mathbf{K}\boldsymbol{\sigma}\cdot\mathbf{K}_0H(\omega_1,\omega_0)$$
$$+i\boldsymbol{\sigma}\cdot\mathbf{K}'\boldsymbol{\sigma}\cdot(\mathbf{K}_0\times\mathbf{K})I(\omega_1,\omega_0)]-\int_1^\infty\frac{f^2(\omega_1)(\omega_1^2-1)^{3/2}A(\omega_1)d\omega_1}{\omega_1+\omega'-\omega_0-\Delta-i\epsilon}[\mathbf{K}\leftrightarrow\mathbf{K}']\Bigg\}v_\alpha, \quad (34)$$

where

$$A(\omega)=g^2/(\Delta-\omega)[1-\tilde{\beta}(\omega)] \quad (35)$$

is related to the $N\theta$ scattering phase shift by

$$e^{i\delta_N}\sin\delta_N=f^2(\omega)A(\omega)(\omega^2-1)^{3/2}/4\pi.$$

Returning to (32) and equating coefficients of various spin invariants, one obtains the following integral equations:

$$\begin{aligned}
F(\omega',\omega_0)=&-g\left[\frac{\alpha(\omega_0)}{\omega'-\Delta}-\frac{\gamma(\omega_0)}{\omega'-\omega_0-i\epsilon}\right]-gA(\omega_0)\left(\frac{1}{\omega'-\omega_0-i\epsilon}-\frac{1}{\omega'-\Delta}\right)\\
&+\frac{1}{3\pi}\int_1^\infty K(\omega_1)d\omega_1\left[\frac{3F(\omega_1,\omega_0)+G(\omega_1,\omega_0)+H(\omega_1,\omega_0)}{\omega_1-\omega'+i\epsilon}+\frac{G(\omega_1,\omega_0)-H(\omega_1,\omega_0)+I(\omega_1,\omega_0)}{\omega_1+\omega'-\omega_0-\Delta-i\epsilon}\right],\\
G(\omega',\omega_0)=&-g\left[\frac{\gamma(\omega_0)}{\omega'-\Delta}-\frac{\alpha(\omega_0)}{\omega'-\omega_0-i\epsilon}\right]-gA(\omega_0)\left(\frac{1}{\omega'-\omega_0-i\epsilon}-\frac{1}{\omega'-\Delta}\right)\\
&+\frac{1}{3\pi}\int_1^\infty K(\omega_1)d\omega_1\left[\frac{G(\omega_1,\omega_0)-H(\omega_1,\omega_0)+I(\omega_1,\omega_0)}{\omega_1-\omega'+i\epsilon}-\frac{3F(\omega_1,\omega_0)+G(\omega_1,\omega_0)+H(\omega_1,\omega_0)}{\omega_1+\omega'-\omega_0-\Delta-i\epsilon}\right],
\end{aligned} \quad (36)$$

$$\begin{aligned}
H(\omega',\omega_0)=&g[\gamma(\omega_0)-A(\omega_0)]\left(\frac{1}{\omega'-\Delta}-\frac{1}{\omega'-\omega_0-i\epsilon}\right)\\
&-\frac{1}{3\pi}\int_1^\infty K(\omega_1)d\omega_1[G(\omega_1,\omega_0)-H(\omega_1,\omega_0)+I(\omega_1,\omega_0)]\left(\frac{1}{\omega_1-\omega'+i\epsilon}+\frac{1}{\omega_1+\omega'-\omega_0-\Delta-i\epsilon}\right),
\end{aligned} \quad (37)$$

$$\begin{aligned}
I(\omega',\omega_0)=&g[A(\omega_0)-\gamma(\omega_0)]\left(\frac{1}{\omega'-\Delta}+\frac{1}{\omega'-\omega_0-i\epsilon}\right)\\
&+\frac{1}{3\pi}\int_1^\infty K(\omega_1)d\omega_1[G(\omega_1,\omega_0)-H(\omega_1,\omega_0)+I(\omega_1,\omega_0)]\left(\frac{1}{\omega_1-\omega'+i\epsilon}-\frac{1}{\omega_1+\omega'-\omega_0-\Delta-i\epsilon}\right),
\end{aligned} \quad (38)$$

where $K(\omega)=e^{i\delta_N}\sin\delta_N$.

The Equations (36)–(38) have the following crossing properties:

$$F\leftrightarrow G,\quad H\leftrightarrow -H,\quad I\leftrightarrow +I, \quad (39)$$

under interchange of the two outgoing θ particles: $\omega'\leftrightarrow\omega_0-\omega'+\Delta$.

In addition to (36)–(38), two more equations are obtained on further reducing the amplitude $T_{\beta\alpha}(\mathbf{K},\mathbf{K}_0)$:

$$T_{\beta\alpha}(\mathbf{K}',\mathbf{K}_0)=\frac{(2\omega_0V)^{1/2}}{f(\omega_0)}\sum_{\mathbf{K}_1}\frac{\langle 0|f_\beta|N_\sigma\mathbf{K}_1,\text{ in}\rangle\langle N_\sigma\mathbf{K}_1,\text{ in}|j|V_\alpha\mathbf{K}_0,\text{ in}\rangle}{\omega_1-\Delta}, \quad (40)$$

which gives after some straightforward algebra

$$\alpha(\omega_0)=A(\omega_0)+\frac{1}{3\pi g}\int_1^\infty K(\omega_1)d\omega_1[3F(\omega_1,\omega_0)+G(\omega_1,\omega_0)+H(\omega_1,\omega_0)], \quad (41)$$

$$\gamma(\omega_0)=A(\omega_0)+\frac{1}{3\pi g}\int_1^\infty K(\omega_1)d\omega_1[G(\omega_1,\omega_0)-H(\omega_1,\omega_0)+I(\omega_1,\omega_0)]. \quad (42)$$

The equations (36)–(38) may be decoupled by introducing the amplitudes $P=3F+2G+I$ and $Q=3F-G+3H-2I$. The decoupled equations take the following form:

$$P(\omega',\omega_0)=g[\alpha(\omega_0)+\gamma(\omega_0)-2A(\omega_0)]\left(\frac{2}{\omega'-\omega_0-i\epsilon}-\frac{3}{\omega'-\Delta}\right)+\frac{1}{\pi}\int_1^\infty K(\omega_1)P(\omega_1,\omega_0)\left(\frac{1}{\omega_1-\omega'+i\epsilon}+\frac{\frac{2}{3}}{\omega_1+\omega'-\omega_0-\Delta-i\epsilon}\right)d\omega_1, \quad (43)$$

$$Q(\omega',\omega_0)=g[2\gamma(\omega_0)-\alpha(\omega_0)-A(\omega_0)]\left(\frac{1}{\omega'-\omega_0-i\epsilon}+\frac{3}{\omega'-\Delta}\right)+\frac{1}{\pi}\int_1^\infty K(\omega_1)Q(\omega_1,\omega_0)\left(\frac{1}{\omega_1-\omega'+i\epsilon}-\frac{\frac{1}{3}}{\omega_1+\omega'-\omega_0-\Delta-i\epsilon}\right)d\omega_1. \quad (44)$$

All six amplitudes F, G, H, I, and α, γ can be obtained from P and Q by quadratures since $3F+G+H=\frac{1}{3}(2P+Q)$ and $G-H+I=\frac{1}{3}(P-Q)$. The decoupled Eqs. (43) and (44) differ from the corresponding integral equation in the scalar model[6] in the unequal coefficients for the "direct" and "crossed" terms under the integral sign. This lack of symmetry under the transformation $\omega'\leftrightarrow\omega_0-\omega'+\Delta$ leaves an arbitrary function in the standard solution of equations of this type discussed by Blankenbecler and Gartenhaus,[17] rendering the solution ambiguous. For this reason the solution of (43) and (44) in closed form does not seem easy. Numerical integration is of course possible, but it will be necessary to specify the cutoff function. As remarked before at the end of Sec. I, this model is probably not sufficiently realistic to warrant extensive numerical work.

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APPENDIX

An alternative set of integral equations for $V\theta$ scattering and $N\theta\theta$ production in the pseudoscalar model may be derived by solving the Schrödinger equation. Again for completeness we record the results for the V and $N\theta$ states as well.

We shall denote the physical and bare V states by $|V\rangle$ and $|v\rangle$, respectively. The former is given by[16]

$$|V_\alpha\rangle=Z^{1/2}|v_\alpha\rangle+\sum_{\mathbf{K}}\phi_{\alpha\beta}(\mathbf{K})|n_\beta\mathbf{K}\rangle \quad (A1)$$

with

$$Z=1-\frac{g^2}{2V}\sum_{\mathbf{K}}\frac{K^2f^2(\omega)}{\omega(\omega-\Delta)^2}, \quad (A2)$$

$$\phi_{\alpha\beta}(\mathbf{K})=\frac{gf(\omega)}{(\Delta-\omega)(2\omega V)^{1/2}}n_\beta^\dagger\boldsymbol{\sigma}\cdot\mathbf{K}\,v_\alpha. \quad (A3)$$

For $N\theta$ scattering we need an $N\theta$ state with an incident θ of momentum $\mathbf{K}_0$ with unit amplitude,

$$|n_\alpha\theta\rangle=|n_\alpha\mathbf{K}_0\rangle+\sum_{\mathbf{K}\neq\mathbf{K}_0}\chi_{\alpha\beta}(\mathbf{K})|n_\beta\mathbf{K}\rangle+Z^{1/2}\lambda_{\alpha\beta}|v_\beta\rangle. \quad (A4)$$

Then it is easily verified that

$$\lambda_{\alpha\beta}=gf(\omega_0)\,v_\beta^\dagger\boldsymbol{\sigma}\cdot\mathbf{K}_0n_\alpha/(2\omega_0V)^{1/2}(\omega_0-\Delta)[1-\tilde{\beta}(\omega_0)], \quad (A5)$$

and

$$\chi_{\alpha\beta}(\mathbf{K})=\frac{f(\omega_0)f(\omega)A(\omega_0)}{2V(\omega_0\omega)^{1/2}(\omega-\omega_0-i\epsilon)}n_\beta^\dagger\boldsymbol{\sigma}\cdot\mathbf{K}\boldsymbol{\sigma}\cdot\mathbf{K}_0n_\alpha, \quad (A6)$$

where $\tilde{\beta}(\omega_0)$ and $A(\omega_0)$ are given by (27) and (35), respectively. The scattering amplitude is given up to trivial factors by the part of $\chi_{\alpha\beta}(\mathbf{K})$ which contains the delta function $\delta(\omega-\omega_0)$.

Similarly the $V\theta\to V\theta$ and $V\theta\to N\theta\theta$ amplitudes are contained in the state vector

$$|V_\alpha\mathbf{K}_0\rangle+\sum_{\mathbf{K}\neq\mathbf{K}_0}M_{\alpha\beta}(\mathbf{K})|V_\beta\mathbf{K}\rangle+\sum_{\mathbf{K}_1}\sum_{\mathbf{K}_2}N_{\alpha\beta}(\mathbf{K}_1,\mathbf{K}_2)|n_\beta\mathbf{K}_1,\mathbf{K}_2\rangle$$
$$\equiv Z^{1/2}|v_\alpha\mathbf{K}_0\rangle+\sum_{\mathbf{K}\neq\mathbf{K}_0}A_{\alpha\beta}(\mathbf{K})|v_\beta\mathbf{K}\rangle+\sum_{\mathbf{K}_1}\sum_{\mathbf{K}_2}B_{\alpha\beta}(\mathbf{K}_1,\mathbf{K}_2)|n_\beta\mathbf{K}_1,\mathbf{K}_2\rangle, \quad (A7)$$

where

$$A_{\alpha\beta}(\mathbf{K})=Z^{1/2}M_{\alpha\beta}(\mathbf{K}) \quad (A8)$$

and

$$B_{\alpha\beta}(\mathbf{K}_1,\mathbf{K}_2)=\tfrac{1}{2}[M_{\alpha\rho}(\mathbf{K}_1)\phi_{\rho\beta}(\mathbf{K}_2)+M_{\alpha\rho}(\mathbf{K}_2)\phi_{\rho\beta}(\mathbf{K}_1)]+N_{\alpha\beta}(\mathbf{K}_1,\mathbf{K}_2). \quad (A9)$$

[17] R. Blankenbecler and S. Gartenhaus, Phys. Rev. **116**, 1297 (1959). The arbitrary function corresponds to the function g in their Eq. (10).

The Schrödinger equation then gives the following relations:

$$Z(\omega-\omega_0+\delta m_V)A_{\alpha\beta}(\mathbf{K})=-g\sum_{\mathbf{K}_1}\frac{f(\omega_1)}{(2\omega_1 V)^{1/2}}[B_{\alpha\rho}(\mathbf{K}_1,\mathbf{K})+B_{\alpha\rho}(\mathbf{K},\mathbf{K}_1)]\,v_\beta{}^\dagger\boldsymbol{\sigma}\cdot\mathbf{K}n_\rho\,, \tag{A10}$$

$$B_{\alpha\rho}(\mathbf{K}_1,\mathbf{K}_2)+B_{\alpha\rho}(\mathbf{K}_2,\mathbf{K}_1)=-g(\omega_1+\omega_2-\omega_0-\Delta)^{-1}$$
$$\times\left[\frac{f(\omega_2)}{(2\omega_2 V)^{1/2}}n_\rho{}^\dagger\boldsymbol{\sigma}\cdot\mathbf{K}_2\,v_\alpha\delta_{\mathbf{K}_0\mathbf{K}_2}+\frac{f(\omega_2)}{(2\omega_2 V)^{1/2}}M_{\alpha\mu}(\mathbf{K}_1)n_\rho{}^\dagger\boldsymbol{\sigma}\cdot\mathbf{K}_2\,v_\mu+1\leftrightarrow 2\right]. \tag{A11}$$

Eliminating $B_{\alpha\rho}$ and carrying out the renormalization gives for $\mathbf{K}\neq\mathbf{K}_0$

$$(\omega-\omega_0)[1-\tilde{\beta}(\omega_0+\Delta-\omega)]A_{\alpha\beta}(\mathbf{K})=\frac{g^2 f(\omega_0)f(\omega)}{2V(\omega_0\omega)^{1/2}(\omega-\Delta)}v_\beta{}^\dagger\boldsymbol{\sigma}\cdot\mathbf{K}_0\boldsymbol{\sigma}\cdot\mathbf{K}\,v_\alpha$$
$$+\frac{g^2 f(\omega)}{(2\omega V)^{1/2}}\sum_{\mathbf{K}_1}\frac{f(\omega_1)}{(2\omega_1 V)^{1/2}}\frac{v_\beta{}^\dagger\boldsymbol{\sigma}\cdot\mathbf{K}_1\boldsymbol{\sigma}\cdot\mathbf{K}\,v_\rho}{\omega_1+\omega-\omega_0-\Delta}A_{\alpha\rho}(\mathbf{K}_1). \tag{A12}$$

Let

$$A_{\alpha\beta}(\mathbf{K})=\frac{f(\omega_0)f(\omega)}{2V(\omega_0\omega)^{1/2}}v_\beta{}^\dagger[\mathbf{K}_0\cdot\mathbf{K}a(\omega,\omega_0)+i\boldsymbol{\sigma}\cdot(\mathbf{K}_0\times\mathbf{K})b(\omega,\omega_0)]\,v_\alpha\,, \tag{A13}$$

then (A12) gives the following integral equations

$$h(\omega,\omega_0)a(\omega,\omega_0)=\frac{g^2}{\omega-\Delta}+\frac{g^2}{12\pi^2}\int_1^\infty\frac{f^2(\omega_1)(\omega_1^2-1)^{3/2}[a(\omega_1,\omega_0)+2b(\omega_1,\omega_0)]d\omega_1}{\omega_1+\omega-\omega_0-\Delta}, \tag{A14}$$

$$h(\omega,\omega_0)b(\omega,\omega_0)=\frac{g^2}{\omega-\Delta}+\frac{g^2}{12\pi^2}\int_1^\infty\frac{f^2(\omega_1)(\omega_1^2-1)^{3/2}a(\omega_1,\omega_0)d\omega_1}{\omega_1+\omega-\omega_0-\Delta}, \tag{A15}$$

where

$$h(\omega_1\omega_0)=(\omega-\omega_0)[1-\beta(\omega_0+\Delta-\omega)].$$

The equations (A14) and (A15) may also be decoupled by introducing the linear combinations $c=\frac{1}{2}(a+b)$ and $d=2b-a$. The result is

$$h(\omega,\omega_0)c(\omega,\omega_0)=\frac{g^2}{\omega-\Delta}+\frac{g^2}{12\pi^2}\int_1^\infty\frac{f^2(\omega_1)(\omega_1^2-1)^{3/2}c(\omega_1,\omega_0)d\omega_1}{\omega_1+\omega-\omega_0-\Delta}, \tag{A16}$$

$$h(\omega,\omega_0)d(\omega,\omega_0)=\frac{g^2}{\omega-\Delta}-\frac{g^2}{12\pi^2}\int_1^\infty\frac{f^2(\omega_1)(\omega_1^2-1)^{3/2}d(\omega_1,\omega_0)d\omega_1}{\omega_1+\omega-\omega_0-\Delta}. \tag{A17}$$

These equations determine $A_{\alpha\beta}(\mathbf{k})$ and therefore the $V-\theta$ scattering amplitude $M_{\alpha\beta}(\mathbf{k})$ by (A8) together with (A3). The amplitude $B_{\alpha\beta}(\mathbf{k}_1,\mathbf{k}_2)$ is then given by (A11), from which the $N-\theta-\theta$ production amplitude $N_{\alpha\beta}(\mathbf{k}_1,\mathbf{k}_2)$ may be obtained with the aid of (A9).