Exchange-Current Contribution to the Deuteron Magnetic Moment*

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An estimate is made of the contribution to the deuteron's magnetic moment from an exchange-current diagram containing the $\rho\pi\gamma$ vertex. The result is sensitive to the form of the deuteron wave function at small distances: Using Hulthén and repulsive core wave functions, contributions of 0.11 and 0.01 nm, respectively, are obtained.

As is expected from the most naive considerations, the magnetic moment of the deuteron is very nearly equal to the sum of the neutron and proton moments.[1,2] There is, however, a discrepancy of about 2.6%, with a wide variety of possible causes. The best understood of these are the contributions from impurities in the spin states and from the orbital circulation of the proton's charge, which are proportional to the percentage of *d* wave in the predominantly *s*-wave deuteron wave function.[1,2] This percentage is, unfortunately, not known accurately, but present estimates give a correction to the deuteron magnetic moment larger than 2.6%,[3] indicating that additional corrections must be considered.

The remaining corrections arise because the deuteron cannot be accurately described by the nonrelativistic wave function for two rigid nucleons used in the calculations referred to above.[2,3] Clearly a relativistic equivalent of a wave function should be used, and the fact that nucleons have meson clouds, which will be modified in the bound state, should somehow be taken into account. In this note we present an attempt to estimate one of the simplest of these corrections: that in which the photon interacts with three pions, two from one nucleon's cloud and one from the other's, as pictured in Fig. 1. This correction, which we shall call an exchange-current (perhaps more properly a "proximity-induced" current[4]) correction, has been mentioned by Gourdin,[3] who points out that it is the simplest of its type, since the $\gamma\pi\pi$ vertex cannot contribute to the deuteron electromagnetic structure. We shall here simplify the calculation by treating the two pions from the same nucleon as a ρ meson. As will become clear below, our calculation contains many other approximations and uncertainties, but it may serve to illustrate some of the important features of exchange-current corrections and to give some idea about their size.

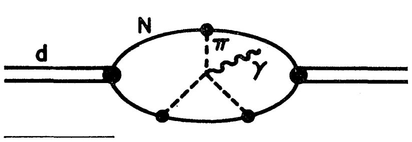

FIG. 1. The simplest exchange-current diagram.

We shall assume that the important contributions to the deuteron electromagnetic form factors can be included in the expression

$$
\begin{aligned}
&\langle d;\mathbf{K}',m'|j_\mu(0)|d;\mathbf{K},m\rangle \\
&\quad=\tfrac{1}{2}\sum_{\alpha_1',\alpha_2'}\sum_{\sigma_1',\sigma_2'}\int\frac{d^3q'}{(2\pi)^3} \\
&\quad\times\tfrac{1}{2}\sum_{\alpha_1,\alpha_2}\sum_{\sigma_1,\sigma_2}\int\frac{d^3q}{(2\pi)^3}\varphi_{\alpha_1'\alpha_2',\sigma_1'\sigma_2'}{}^{m'*}(\mathbf{q}') \\
&\quad\times\langle\tfrac{1}{2}\mathbf{K}'+\mathbf{q}',\alpha_1',\sigma_1';\tfrac{1}{2}\mathbf{K}'-\mathbf{q}';\alpha_2',\sigma_2'|j_\mu(0) \\
&\quad\times|\tfrac{1}{2}\mathbf{K}+\mathbf{q},\alpha_1,\sigma_1;\tfrac{1}{2}\mathbf{K}-\mathbf{q},\alpha_2,\sigma_2\rangle'\varphi_{\alpha_1\alpha_2,\sigma_1\sigma_2}{}^{m}(\mathbf{q}),
\end{aligned}
\tag{1}
$$

where $\varphi_{\alpha_1\alpha_2;\sigma_1\sigma_2}{}^m(\mathbf{q})$ is the nonrelativistic deuteron wave function and

$$\langle\mathbf{p}_1',\alpha_1'\sigma_1';\mathbf{p}_2',\alpha_2',\sigma_2'|j_\mu(0)|\mathbf{p}_1,\alpha_1,\sigma_1;\mathbf{p}_2,\alpha_2,\sigma_2\rangle'$$

is the matrix element of the electromagnetic current operator between two-nucleon states. The prime is to remind us that this matrix element is, in general, off the mass shell and, further, that it should not include contributions from those diagrams which have parts which could be included in deuteron wave-function diagrams. In (1) the σ's and α's are, respectively, spin and isospin indices.

Since our calculation will be, of necessity, crude, we shall not carry along the *d*-wave component of the deuteron wave function, but take simply

$$\varphi_{\alpha_1\alpha_2;\sigma_1\sigma_2}{}^m(\mathbf{q})=\chi_{0;\alpha_1\alpha_2}(I)\chi_{1;\sigma_1\sigma_2}{}^m(S)\varphi(q)\,,\tag{2}$$

where

$$
\begin{aligned}
\chi_{0;\alpha_1\alpha_2}(I)&=\delta_{\alpha_1+}\delta_{\alpha_2-}-\delta_{\alpha_1-}\delta_{\alpha_2+}\,,\\
\chi_{1;\sigma_1\sigma_2}{}^m(S)&=\delta_{\sigma_1+}\delta_{\sigma_2+}\,,\quad m=+1\\
&=(2)^{-\frac{1}{2}}(\delta_{\sigma_1+}\delta_{\sigma_2-}+\delta_{\sigma_1-}\delta_{\sigma_2+})\,,\quad m=0\\
&=\delta_{\sigma_1-}\delta_{\sigma_2-}\,,\quad m=-1\,,
\end{aligned}
\tag{3}
$$

and

$$\varphi(q)=\int d^3r\exp(i\mathbf{q}\cdot\mathbf{r})(4\pi)^{-\frac{1}{2}}R^{-1}u(r)\,.\tag{4}$$

In (4), $(4\pi)^{-\frac{1}{2}}R^{-1}u(r)$ is the (pure *s*-wave) deuteron wave function in configuration space.

If we take the contribution to the two-nucleon matrix

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[1] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952), Chap. II.

[2] L. Hulthén and M. Sugaware, *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. 39. Other references can be found here and in Ref. 3.

[3] M. Gourdin, Nuovo Cimento **28**, 533 (1963).

[4] R. K. Osborne and L. L. Foldy, Phys. Rev. **79**, 795 (1960).

element corresponding to the diagram of Fig. 2(a),

$$\langle 1';2'|\mathbf{j}(0)|1;2\rangle_a=(2\pi)^3\delta^3(p_2-p_2')\delta_{\alpha_2\alpha_2'}\delta_{\sigma_2\sigma_2'}$$
$$\times\langle 1'|j(0)|1\rangle+\text{three equivalent combinations,} \quad (5)$$

with[5]

$$\langle 1'|\mathbf{j}(0)|1\rangle=i\frac{e}{2m}[F_{\alpha_1'\alpha_1}{}^{\text{Mag}}(q^2)\boldsymbol{\sigma}_{\sigma_1'\sigma_1}$$
$$\times(\mathbf{p}_1'-\mathbf{p}_1)-F_{\alpha_1'\alpha_1}{}^{\text{Ch}}(q^2)\delta_{\sigma_1'\sigma_1}(\mathbf{p}_1'+\mathbf{p}_1)], \quad (6)$$

then, taking the limit $\mathbf{Q}\equiv\mathbf{K}'-\mathbf{K}\rightarrow 0$, we find the well-known results that, with our approximations, the deuteron magnetic moment is just the sum of the neutron and proton moments:

$$\langle d;K',m'|\mathbf{j}(0)|d;K,m\rangle_a$$
$$=\langle m'|\mathbf{S}|m\rangle\times\mathbf{Q}i\frac{e}{2m}(F_{++}{}^{\text{Mag}}(0)+F_{--}{}^{\text{Mag}}(0)), \quad (7)$$

where $\mathbf{S}$ is the total spin operator, or

$$\mu_a(d)=F_{++}{}^{\text{Mag}}(0)+F_{--}{}^{\text{Mag}}(0)=\mu(p)+\mu(n)\approx 0.88. \quad (8)$$

Among the many corrections to this result is the exchange current contribution from the diagram of Fig. 2(b). Taking the nonrelativistic limit, we find

$$\langle 1';2'|\mathbf{j}(0)|1;2\rangle_b=igg_\rho f_{\pi\rho\gamma}\boldsymbol{\tau}_{\alpha_1'\alpha_1}\cdot\boldsymbol{\tau}_{\alpha_2'\alpha_2}$$
$$\times\frac{\delta_{\sigma_2'\sigma_2}[\boldsymbol{\sigma}_{\sigma_1'\sigma_2}\cdot(\mathbf{p}_1'-\mathbf{p}_1)][(\mathbf{p}_1'-\mathbf{p}_1)\times(\mathbf{p}_2'-\mathbf{p}_2)]}{2m[m_\pi^2+(\mathbf{p}_1'-\mathbf{p}_1)^2][m_\rho^2+(\mathbf{p}_2'-\mathbf{p}_2)^2]}$$
$$+\text{three equivalent combinations,} \quad (9)$$

where g, g_ρ, and $f_{\pi\rho\gamma}$ are the πNN, ρNN, and $\pi\rho\gamma$ coupling constants, respectively (we have ignored a possible magnetic moment type ρNN coupling). Inserting this in (1), and performing the isospin and spin sums, we have

$$\langle d;\mathbf{K}',m'|\mathbf{j}(0)|d;\mathbf{K},m\rangle_b$$
$$=-i6gg_\rho\frac{f_{\pi\rho\gamma}}{2m}\int\frac{d^3q'}{(2\pi)^3}\varphi^*(q')\int\frac{d^3q}{(2\pi)^3}\varphi(q)$$
$$\times\frac{[(\tfrac{1}{2}\mathbf{Q}+\mathbf{q}'-\mathbf{q})\cdot\langle m'|\mathbf{S}|m\rangle][(\mathbf{q}'-\mathbf{q})\times\mathbf{Q}]}{[m_\pi^2+(\tfrac{1}{2}\mathbf{Q}+\mathbf{q}'-\mathbf{q})^2][m_\rho^2+(\tfrac{1}{2}\mathbf{Q}-\mathbf{q}'+\mathbf{q})^2]}. \quad (10)$$

In the limit $Q\rightarrow 0$ this reduces to

$$\langle d;\mathbf{K}',m'|\mathbf{j}(0)|d;\mathbf{K},m\rangle=-i2gg_\rho(2m)^{-1}f_{\pi\rho\gamma}$$
$$\times\langle m'|\mathbf{S}|m\rangle\times\mathbf{Q}\int\frac{d^3q'}{(2\pi)^3}\varphi^*(q')\int\frac{d^3q}{(2\pi)^3}\varphi(q)$$
$$\times\frac{(\mathbf{q}'-\mathbf{q})^2}{[m_\pi^2+(\mathbf{q}'-\mathbf{q})^2][m_\rho^2+(\mathbf{q}'-\mathbf{q})^2]}. \quad (11)$$

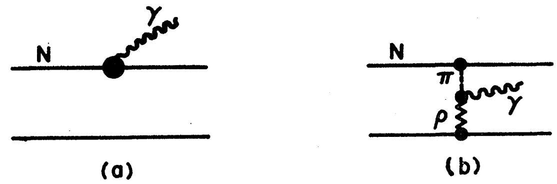

FIG. 2. (a) The diagram giving the main contribution to the electromagnetic structure of the deuteron. (b) The exchange-current correction considered in this paper.

Going over to configuration space, we find that the contribution to the deuteron magnetic moment from the diagram of Fig. 2(b), in nuclear magnetons, is

$$\mu_b(d)=-2\frac{gg_\rho}{4\pi}\frac{m_\pi f_{\pi\rho\gamma}}{e}\frac{1}{m_\pi(m_\rho^2-m_\pi^2)}$$
$$\times[m_\rho^2I(m_\rho)-m_\pi^2I(m_\pi)], \quad (12)$$

with

$$I(m)=\int_0^\infty dru^2(r)\frac{e^{-mr}}{r}. \quad (13)$$

To obtain a numerical result we need to know the deuteron wave function and, because the mass of the ρ meson is large, we need to know it accurately at small distances ($m_\rho^{-1}\approx\frac{1}{4}$F). This is particularly important because the $I(m_\rho)$ and $I(m_\pi)$ terms tend to cancel; for a deuteron with a dense core the $I(m_\rho)$ term will dominate, for a deuteron with a hollow core the $I(m_\pi)$ term will dominate, and there will be complete cancellation for some intermediate case. Again, because the mass of the ρ is so large, we can see from Eq. (11) that we cannot hope our nonrelativistic calculation to be accurate unless $\varphi(q)$ cuts off rapidly at large q. This, too, depends upon the deuteron wave function at small distances.

It is unfortunately true that our knowledge of the inner regions of the deuteron is extremely limited, and is based upon the somewhat circular process of using our knowledge of the isoscalar electromagnetic properties of the nucleon to determine the deuteron wave function, and conversely (aided, of course, by the latest two-nucleon potential). An extensive discussion of the deuteron wave function has been given by Durand[6] in connection with inelastic electron-deuteron scattering. For purposes of illustration we shall follow him in using the analytically simple Hulthén (H) and repulsive core (R) wave functions to evaluate our integrals:

$$u_H=Ne^{-\alpha r}(1-e^{-\gamma r}), \quad (14)$$
$$u_R=Ne^{-\alpha r}(1-e^{-\mu r})^3, \quad (15)$$

with

$$N^2=0.766\ \text{F}^{-1},$$
$$\alpha=0.232\ \text{F}^{-1},$$
$$\gamma=1.202\ \text{F}^{-1}, \quad (16)$$

and

$$\mu=2.13\ \text{F}^{-1}.$$

[5] F. J. Ernst, R. G. Sachs, and K. C. Wali, Phys. Rev. **119**, 1105 (1960).

[6] L. Durand, Phys. Rev. **123**, 1393 (1961).

These parameters have been chosen to agree with the deuteron binding energy, the triplet effective range, and, in the case of the repulsive-core wave function, the low momentum transfer elastic electron-deuteron scattering.

With these simple wave functions the function $I(m)$, defined by Eq. (13), can be evaluated analytically. Then, substituting the values for the parameters given in (16), we obtain, with $m_\rho^2=29m_\pi^2$,

$$m_\rho^2 I_H(m_\rho)=1.44m_\pi^3\,,\quad m_\pi^2 I_H(m_\pi)=0.30m_\pi^3\,, \tag{17}$$

$$m_\rho^2 I_R(m_\rho)=0.34m_\pi^3\,,\quad m_\pi^2 I_R(m_\pi)=0.23m_\pi^3\,. \tag{18}$$

For the coupling constants, we take[7–10]

$$\begin{aligned}(4\pi)^{-1}g^2&=15\,,\\ (4\pi)^{-1}g_\rho^2&\approx 1\,,\\ (m_\pi f_{\pi\rho\gamma})^2&\approx 0.1e^2.\end{aligned} \tag{19}$$

Assembling our numbers, we obtain, with the Hulthén wave function,

$$|\mu_{b,H}(d)|=0.11 \tag{20}$$

and, with the repulsive-core wave function,

$$|\mu_{b,R}(d)|=0.01\,. \tag{21}$$

The correction to the deuteron magnetic moment obtained using the Hulthén wave function seems excessively large. If correct, it would require either a drastic revision of our present estimates for the amount of d wave in the deuteron, or large corrections of opposite sign from other diagrams to cancel it. The size of the correction obtained using the repulsive-core wave function, on the other hand, is quite reasonable, but sensitive to the parameters and shape used.

The calculation above could be easily extended to estimate corrections to the deuteron electromagnetic form factors for arbitrary momentum transfers. In view of its crudity, however, it is not clear that this would be worth the additional effort. A really satisfactory calculation will have to await the development of an accurate relativistic deuteron wave function and a method for including off-mass-shell effects.

After the body of this paper had been written, it was pointed out to the author that a similar calculation had previously been made by Fujii and Kawaguchi.[11] These authors follow a somewhat different path, beginning with a $\gamma 3\pi$ rather than a $\gamma\rho\pi$ vertex, but, after making a one-pole approximation to the two-pion contribution to the nucleon form factors, arrive at a formula which is, except for the coupling constants multiplying the integrals, identical to Eq. (12) above. Furthermore, if we insert the actual numerical values which are used for the coupling constants in the two calculations, we find that the numerical factors agree to within about 10%. This agreement, while quite possibly fortuitous, is especially striking considering that Fujii and Kawaguchi[11] use a value for the $\gamma 3\pi$ coupling constant calculated from a nucleon loop diagram, while the value for the $\gamma\rho\pi$ coupling constant used above was taken from an analysis of double-pion photoproduction experiments.[7] With a formula which is thus essentially identical to (12), these authors, using a hard-core wave function and a somewhat lower value for the mass of the ρ resonance than is used above, obtain a contribution to the deuteron magnetic moment of about 0.03 nm, providing another example of the sensitivity of the result to the detailed form of the deuteron wave function.

ACKNOWLEDGMENTS

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[7] W. Alles and D. Boccaletti, Nuovo Cimento **27**, 306 (1963).
[8] J. Bowcock, N. W. Cottingham, and D. Lurie, Nuovo Cimento **16**, 918 (1960).
[9] J. P. Burq and J. K. Walker, Phys. Rev. (to be published).
[10] J. S. Ball and D. Y. Wong (to be published).

[11] Y. Fujii and M. Kawaguchi, Progr. Theoret. Phys. (Kyoto) **26**, 519 (1961). See also M. Kawaguchi and H. Yokomi, Progr. Theoret. Phys. Suppl. (Kyoto) **21**, 71 (1962).