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Composite Particle Model for the Nucleon and the (3,3) Resonance*

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Partial-wave dispersion relations are used with interaction terms arising from the exchange of a nucleon, the N^* and the ρ meson to produce integral equations for the partial-wave amplitudes for pion-nucleon scattering. The solutions of these equations depend on a single arbitrary parameter, the energy at which the dispersion integrals are "cutoff." It is shown that when the value of the cutoff is adjusted to produce the N^* at the correct energy, a bound state, the nucleon, appears in the p wave, $I=J=\frac{1}{2}$ amplitude. Thus, the nucleon mass, the pion-nucleon coupling constant, the width of the N^*, and mass of the N^* are all determined by the cutoff parameter. It is shown that when the N^* mass has the experimental value the other quantities are in reasonable agreement with experiment. The effect of variations in the coupling constants controlling the interaction terms is also studied. In each case all of the $J\leq\frac{3}{2}$ partial-wave amplitudes are calculated and compared with experiment.

I. INTRODUCTION

IN a recent paper Chew[1] showed that the exchange of the N^* [the $(3-3)$ resonance] in pion-nucleon scattering can give rise to a sufficiently strong attractive force to produce the nucleon as a pion-nucleon bound state. The residue of this pole is proportional to the strength of the exchange force which is simply the width of the N^*. Since the residue of the nucleon pole is, by definition, the pion-nucleon coupling constant, one finds a relation between the pion-nucleon coupling constant and the width of the N^*. In fact, Chew showed that the relation obtained in the static approximation is almost identical to the well-known result of the Chew-Low theory where the nucleon pole provides the exchange force and the N^* is produced as a resonance. Thus, it appears that the nucleon pole and the $3-3$ resonance should be treated on an equal footing in the calculation of pion-nucleon scattering amplitudes. Since these are the most prominent features of pion-nucleon scattering, a dynamical theory of the pion-nucleon interaction should produce both the nucleon and the N^* as composite particle states.

In the static theory the masses of the nucleon and the N^* are each separately controlled by a cutoff parameter. This type of calculation is therefore limited to providing relations between the pion-nucleon coupling constant and the N^* width. The same relation is also found to hold in the static-field theory as discussed by Low.[2] On the other hand, it is not clear, *a priori*, whether the cutoff parameter needed to produce the nucleon pole is at all close to the cutoff parameter necessary to produce the N^*. The purpose of the present paper is to perform a calculation of pion-nucleon scattering using the fully relativistic dispersion relations and including as interaction the forces arising from the exchange of a single nucleon, the N^*, and the ρ meson. A single cutoff parameter is introduced. It is shown that when this parameter is adjusted to give the N^* at the observed energy the nucleon pole appears at approximately the correct energy. This result is quite remarkable in view of the fact that the positions of both the nucleon pole and the N^* are very sensitive to the cutoff parameter. Hence, the relation between N and N^* is much more intimate than one would have expected from Chew's results. Once the cutoff has been fixed, we calculate all of the $J\leq\frac{3}{2}$ partial-wave amplitudes. While the resulting bound state in the (1,1) amplitude and the width of the N^* are of primary interest, the other phase shifts are also compared with experiment.

Sections II, III, and IV contain the kinematical preliminaries, the definitions of the invariant amplitudes, and the partial-wave expansions. In Secs. V and VI we formulate the partial-wave dispersion relations with a set of parameters for the exchange of the nucleon, the (3.3) resonance, and the ρ resonance. All of these parameters are taken from experimental values except for the ρ-nucleon coupling parameter which is obtained through the isovector form factors. Numerical results are given in Sec. VII.

II. KINEMATICS

Scattering amplitudes for the process $p_1+q_1\rightarrow p_2+q_2$ shown in Fig. 1 are considered as functions of the

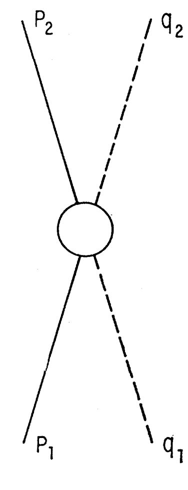

FIG. 1. Diagram representing the $\pi-N$ interaction.

* This work supported by the U. S. Atomic Energy Commission.
[1] G. F. Chew, Phys. Rev. Letters 9, 233 (1962).
[2] F. E. Low, Phys. Rev. Letters 9, 279 (1962).

familiar scalar variables[3]

$$s=-(p_1+q_1)^2=m^2+\mu^2+2k_s^2 +2[(k_s^2+m^2)(k_s^2+\mu^2)]^{1/2}, \quad (2.1)$$

$$t=-(p_1-p_2)^2=-2k_s^2(1-z_s), \quad (2.2)$$

$$u=-(p_1-q_2)^2=2m^2+2\mu^2-s-t, \quad (2.3)$$

where m is the nucleon mass, μ is the pion mass, k_s and z_s are the center-of-mass momentum and $\cos\theta$, respectively.

If the diagram in Fig. 1 is considered to represent the annihilation of a nucleon p_1 with an antinucleon $(-p_2)$ into two pions $(-q_1)$ and q_2, then the same scalar variables have different meanings in the center-of-mass system of the nucleon-antinucleon pair. Here,

$$t=4(q^2+\mu^2)=4(p^2+m^2), \quad (2.4)$$

$$s=-p^2-q^2+2pqz_t, \quad (2.5)$$

$$u=2m^2+2\mu^2-s-t, \quad (2.6)$$

where p and q are the center-of-mass momentum of the nucleon and the pion, respectively, and z_t is the cosine of the scattering angle in this channel.

Again, Fig. 1 can also be interpreted as representing the scattering of a pion $(-q_2)$ by the nucleon p_1 resulting as $(-q_1)$ and p_2. The role of s and u interchanges here as compared with the original scattering process. Thus,

$$u=m^2+\mu^2+2k_u^2+2[(k_u^2+m^2)(k_u^2+\mu^2)]^{1/2}, \quad (2.7)$$

$$t=-2k_u^2(1-z_u), \quad (2.8)$$

$$s=2m^2+2\mu^2-t-u. \quad (2.9)$$

III. CROSSING SYMMETRY

The relation between the pion-nucleon scattering amplitude and the $N\bar{N}\rightarrow\pi\pi$ amplitude was given by Frazer and Fulco.[3] Following their notation, we use the A and B amplitudes for pion-nucleon scattering, $\mathfrak{F}_{++}$ and $\mathfrak{F}_{+-}$ for $N\bar{N}\rightarrow\pi\pi$ amplitudes. The crossing relation reads

$$\begin{aligned} A^{(1/2;3/2)}(s,t)&=\tfrac{1}{3}[(4;1)A^{(3/2)}(u,t)+(-1;2)A^{(1/2)}(u,t)] \\ &=-8\pi\left\{(1;1)\left[\left(\frac{t}{t-4m^2}\right)^{1/2}\mathfrak{F}_{++}{}^{(+)}(t,s)-\frac{2mz_t}{(t-4m^2)^{1/2}(1-z_t^2)^{1/2}}\mathfrak{F}_{+-}{}^{(+)}(t,s)\right]\right. \\ &\qquad\left.+(2;-1)\left[\left(\frac{t}{t-4m^2}\right)^{1/2}\mathfrak{F}_{++}{}^{(-)}(t,s)-\frac{2mz_t}{(t-4m^2)^{1/2}(1-z_t^2)^{1/2}}\mathfrak{F}_{+-}{}^{(-)}(t,s)\right]\right\}, \end{aligned} \quad (3.1)$$

$$\begin{aligned} B^{(1/2;3/2)}(s,t)&=-\tfrac{1}{3}[(4;1)B^{(3/2)}(u,t)+(-1;2)B^{(1/2)}(u,t)] \\ &=\frac{16\pi}{(t-4\mu^2)^{1/2}(1-z_t^2)^{1/2}}[(1;1)\mathfrak{F}_{+-}{}^{(+)}(t,s)+(2;-1)\mathfrak{F}_{+-}{}^{(-)}(t,s)], \end{aligned} \quad (3.2)$$

where the first variable always designates the center-of-mass energy squared in the appropriate channel and the second variable is the corresponding momentum transfer squared.

The inversion of (3.1) and (3.2) gives four equations for the $\mathfrak{F}$'s in terms of A and B

$$\begin{aligned} \mathfrak{F}_{++}{}^{(+;-)}(t,s)&=\frac{1}{24\pi}\cdot\left\{(1;1)\left[-\left(\frac{t-4m^2}{t}\right)^{1/2}A^{(1/2)}(s,t)+mz_t\left(\frac{t-4\mu^2}{t}\right)^{1/2}B^{(1/2)}(s,t)\right]\right. \\ &\quad\left.+(2;-1)\left[-\left(\frac{t-4m^2}{t}\right)^{1/2}A^{(3/2)}(s,t)+mz_t\left(\frac{t-4\mu^2}{t}\right)^{1/2}B^{(3/2)}(s,t)\right]\right\}=+\frac{(1;-1)}{24\pi}\{s\rightarrow u\}, \end{aligned} \quad (3.3)$$

$$\mathfrak{F}_{+-}{}^{(+;-)}(t,s)=\frac{(t-4\mu^2)^{1/2}(1-z_t^2)^{1/2}}{48\pi}[(1;1)B^{(1/2)}(s,t)+(2;-1)B^{(3/2)}(s,t)]=(1;-1)[s\rightarrow u]. \quad (3.4)$$

Since A and B are invariant amplitudes free of kinematical singularities,[3] it is clear that $\mathfrak{F}_{++}$ and $\mathfrak{F}_{+-}$ will contain kinematical singularities as explicitly shown in Eqs. (3.3) and (3.4). These must, of course, be considered when we construct the ρ-meson exchange term.

[3] W. R. Frazer and J. R. Fulco, Phys. Rev. **117**, 1603 (1960); **119**, 1420 (1960).

IV. PARTIAL-WAVE AMPLITUDES

We relate the A and B amplitudes to the partial-wave amplitudes for the s channel, i.e., $\pi-N$ scattering, as follows:

$$A(s,t)=\tfrac{1}{2}\sum_{l=0}^{\infty}(2l+1)A_l(s)P_l(z_s), \quad (4.1)$$

$$B(s,t)=\tfrac{1}{2}\sum_{l=0}^{\infty}(2l+1)B_l(s)P_l(z_s). \tag{4.2}$$

The phase shifts are then related to A_l and B_l as follows:

$$\begin{aligned} h_J(W) &\equiv \frac{s^J}{(E+m)k_s^{2J}}\sin\delta_{l=J-1/2}(W)\exp[i\delta_{l=J-1/2}(W)] \\ &= -\frac{s^J}{(E+m)k_s^{2J}}\sin\delta_{l=J+1/2}(-W)\exp[i\delta_{l=J+1/2}(-W)] \\ &= \frac{s^{J-1/2}}{16\pi k_s^{2J-1}}\Bigg\{A_{J-1/2}(s)+(W-m)B_{J-1/2}(s) \\ &\qquad +\left(\frac{E-m}{k_s}\right)^2[-A_{J+1/2}(s)+(W+m)B_{J+1/2}(s)]\Bigg\}; \end{aligned} \tag{4.3}$$

$$J=\tfrac{1}{2},\tfrac{3}{2},\cdots,$$

where $W=s^{1/2}$ and $E=(s+m^2-\mu^2)/(2W)$. The factor $(E+m)^{-1}$ is introduced so that $h_J(W)$ has a constant behavior at the threshold points $W=(m\pm\mu)$ and $W=-(m\pm\mu)$. The factor s^J is included so that all the h's are bounded by $O(1/W)$ at infinity and, in principle, satisfies a no-subtraction dispersion relation.

The invariant amplitudes A and B can also be expressed in terms of partial-wave amplitudes h_J as follows:

$$A(s,t)=4\pi\sum_{J=1/2}^{\infty}\Bigg\{(k_s^{2J-1}/s^J)h_J(W)\Bigg[(W+m)P'_{J+1/2}(z_s) +\left(\frac{E+m}{E-m}\right)(W-m)P'_{J-1/2}(z_s)\Bigg] +[W\to -W]\Bigg\}, \tag{4.4}$$

$$B(s,t)=4\pi\sum_{J=1/2}^{\infty}\Bigg\{(k_s^{2J-1}/s^J)h_J(W)\Bigg[P'_{J+1/2}(z_s) -\left(\frac{E+m}{E-m}\right)P'_{J-1/2}(z_s)\Bigg]+[W\to -W]\Bigg\}. \tag{4.5}$$

The expansion of $\mathfrak{F}_{++}$ and $\mathfrak{F}_{+-}$ in terms of $N\bar{N}\to\pi\pi$ partial-wave amplitudes is given by[3]

$$\mathfrak{F}_{++}{}^{(+;-)}(t,s)=(1/q)\sum_J(J+\tfrac{1}{2})T_+{}^{J(+;-)}(t)P_J(z_t), \tag{4.6}$$

and

$$\mathfrak{F}_{+-}{}^{(+;-)}(t,s)=\frac{(1-z_t^2)^{1/2}}{q}\times\sum_J\frac{J+\frac{1}{2}}{[J(J+1)]^{1/2}}T_-{}^{J(+;-)}(t)P_J'(z_t). \tag{4.7}$$

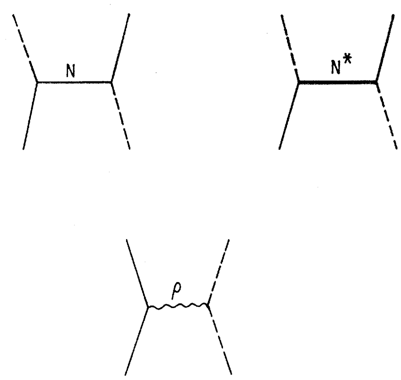

FIG. 2. The diagrams representing the exchange of a single nucleon and the N^* in the u channel and the exchange of the ρ in the t channel.

We shall use the following combinations of partial-wave amplitudes when considering the interaction due to the exchange of the ρ meson

$$\Gamma_1(t)=(mE_t/p^2q^2)\times[T_+{}^{1(-)}(t)-(E_t/2^{1/2}m)T_-{}^{1(-)}(t)], \tag{4.8}$$

$$\Gamma_2(t)=(E_t/2p^2q^2)\times[-T_+{}^{1(-)}(t)+(m/2^{1/2}E_t)T_-{}^{1(-)}(t)], \tag{4.9}$$

where $E_t=\frac{1}{2}t^{1/2}$, Γ_1 and Γ_2 are the Frazer-Fulco amplitudes which enter into the calculation of the charge and the magnetic moment form factor, respectively.

V. INTERACTION TERMS

In this section we will consider the $\pi-N$ interaction generated by the exchange of resonant- and single-particle intermediate states in the t and u channel.

We will first take the renormalized Born terms corresponding to the diagrams in Fig. 2 where the resonances have been treated as single-particle states. Unitarity will later be enforced on the partial-wave amplitudes in the s channel.

The nucleon pole term contributes only to the amplitude B,

$$B_N{}^{1/2}(u,t)=3g_r{}^2/(m^2-u), \tag{5.1}$$

where $g_r{}^2$ is the rationalized $\pi-N$ coupling constant. The contribution of the $3-3$ resonance is given by

$$A_{33}{}^{3/2}(u,t)=8\pi\gamma_{33}\left\{\frac{a_{33}-\frac{3}{2}(m_{33}+m)s}{m_{33}{}^2-u}\right\}, \tag{5.2}$$

$$B_{33}{}^{3/2}(u,t)=8\pi\gamma_{33}\left\{\frac{b_{33}-\frac{3}{2}s}{m_{33}{}^2-u}\right\}, \tag{5.3}$$

where

$$\begin{aligned} a_{33}&=[(m_{33}-m)(E_{33}+m)^2 \\ &\quad -\tfrac{3}{2}(m_{33}+m)(\tfrac{1}{2}m_{33}{}^2-m^2-\mu^2-(m^2-\mu^2)^2/2m_{33}{}^2)^2], \\ b_{33}&=[-(E_{33}+m)^2 \\ &\quad +\tfrac{3}{2}(m^2+\mu^2-\tfrac{1}{2}m_{33}{}^2+(m^2-\mu^2)^2/2m_{33}{}^2)], \end{aligned}$$

and E_{33} is the nucleon energy at the $3-3$ resonance, m_{33} is the mass of the N^*. The coupling parameter γ_{33} is normalized as follows:

$$\mathrm{Im}\{h_{3/2}(W)/s\}\simeq\pi\gamma_{33}\delta(W-m_{33}) \qquad (5.4)$$

and can be obtained from the experimental width of the N^*.

The exchange of a ρ meson of mass m_ρ gives rise to poles in the invariant amplitudes as a function of the t variable. In terms of the $N\bar{N}\rightarrow\pi\pi$ amplitudes $\mathfrak{F}_{++}$ and $\mathfrak{F}_{+-}$, we find

$$-8\pi\left[\left(\frac{t}{t-4m^2}\right)^{1/2}\mathfrak{F}_{++,\rho}{}^{(-)}(t,s) -\frac{2mz_t}{(t-4m^2)^{1/2}(1-z_t{}^2)^{1/2}}\mathfrak{F}_{+-,\rho}{}^{(-)}(t,s)\right] =\frac{6\pi\gamma_2}{m_\rho{}^2-t}(2s+t-2m^2-2\mu^2)\,, \qquad (5.5)$$

$$\frac{16\pi}{(t-4\mu^2)^{1/2}(1-z_t{}^2)^{1/2}}[\mathfrak{F}_{+-,\rho}{}^{(-)}(t,s)] =-12\pi\left[\frac{\gamma_1}{m_\rho{}^2-t}+\frac{2m\gamma_2}{m_\rho{}^2-t}\right], \qquad (5.6)$$

where $\gamma_{1,2}$ are the residues which are normalized so that $\mathrm{Im}\Gamma_{1,2}(t)\simeq\pi\gamma_{1,2}\delta(t-m_\rho{}^2)$. Furthermore, if these Γ's are used to calculate the nucleon-vector form factors, the following ratio is determined:

$$\gamma_1/\gamma_2\simeq\frac{G_{1\rho}{}^v(0)}{G_{2\rho}{}^v(0)}\simeq\frac{0.5e}{1.83(e/2m)}\,, \qquad (5.7)$$

where the ρ subscript indicates the ρ contribution to the nucleon form factors. The fact that the contributions of the ρ are nearly equal to the total charge and magnetic moment is the result of experiment[4] as well as our previous analysis of the nucleon form factors.[5]

VI. DISPERSION RELATIONS

Before writing down partial-wave dispersion relations in the s channel, we first summarize the contribution to $A(s;t)$ and $B(s;t)$ obtained when all of the interaction terms are inserted into the crossing relations Eqs. (3.1) and (3.2). We use the superscript L to indicate that these terms have singularities only outside of the physical region in the s channel. We have

$$A^{L(1/2;3/2)}(s,t)=(2;\,-1)(6\pi)\cdot\frac{\gamma_2}{m_\rho{}^2-t}[2s+t-2m^2-2\mu^2]+(\tfrac{4}{3};\,\tfrac{1}{3})(8\pi\gamma_{33}) \cdot\left\{\frac{(m_{33}-m)(E_{33}+m)^2}{m_{33}{}^2-u}+\frac{3(m_{33}+m)}{2(m_{33}{}^2-u)}\cdot\left[m^2+\mu^2-\tfrac{1}{2}m_{33}{}^2-s+\frac{(m^2-\mu^2)^2}{2m_{33}{}^2}\right]\right\} \qquad (6.1)$$

and

$$B^{L(1/2;3/2)}(s,t)=(1;\,-2)\frac{g_r{}^2}{m^2-u}+(2;\,-1)(-12\pi)\left[\frac{\gamma_1}{m_\rho{}^2-t}+\frac{2m\gamma_2}{m_\rho{}^2-t}\right]+(-\tfrac{4}{3};\,-\tfrac{1}{3})(8\pi\gamma_{33}) \times\left\{-\frac{(E_{33}+m)^2}{m_{33}{}^2-u}+\frac{2}{2(m_{33}{}^2-u)}\left[m^2+\mu^2-\tfrac{1}{2}m_{33}{}^2-s+\frac{(m^2-\mu^2)^2}{2m_{33}{}^2}\right]\right\}. \qquad (6.2)$$

Partial-wave projections of $A^L(s;t)$ and $B^L(s;t)$ are given by Eqs. (4.1), (4.2), and (4.3). For the remainder of this section, we will suppress the I-spin index. Explicitly, we rewrite Eq. (4.3):

$$h_J{}^L(W)=\frac{s^{J-1/2}}{16\pi k_s{}^{2J-1}} \times\left\{A_{J-1/2}{}^L(s)+(W-m)B_{J-1/2}{}^L(s)+\left(\frac{E-m}{k_s}\right)^2 \times[-A_{J+1/2}{}^L(s)+(W+m)B_{J+1/2}{}^L(s)]\right\}. \qquad (6.3)$$

Since $A^L(s;t)$ and $B^L(s;t)$ are analytic in the s-channel physical region, $h_J{}^L(W)$ is also analytic in the physical region; $W\geq(m+\mu)$ for the $l=J-\frac{1}{2}$ amplitudes and $W\leq-(m+\mu)$ for the $l=J+\frac{1}{2}$ amplitudes. It is easily seen that singularities of $h_J{}^L(W)$ appear in the center region of the W plane including branch cuts on parts of the real axis, all of the imaginary axis and the familiar circle of radius $(m^2-\mu^2)^{1/2}$. By evaluating the exchange terms directly, we have avoided considerations of cuts on such complex domains.

Throughout this paper, we shall use dispersion relations only as a tool to impose the unitarity condition on partial-wave amplitudes in the physical region while keeping all the singularities in the exchange terms unchanged. The elastic unitarity condition on $h_J(W)$ is

$$\lim_{\epsilon\rightarrow0+}\mathrm{Im}h_J(W+i\epsilon) =\left(\frac{E+m}{W^{2J}}\right)|k_s{}^{2J}h_J{}^2(W)|\,;\quad W\geq(m+\mu)$$
$$=\left(\frac{E+m}{W^{2J}}\right)|k_s{}^{2J}h_J{}^2(W)|\,;\quad W\leq-(m+\mu). \qquad (6.4)$$

[4] C. de Vries, R. Hofstadter, and R. Herman, Phys. Rev. Letters 8, 381 (1962).
[5] J. S. Ball and D. Y. Wong, Phys. Rev. 130, 2112 (1963).

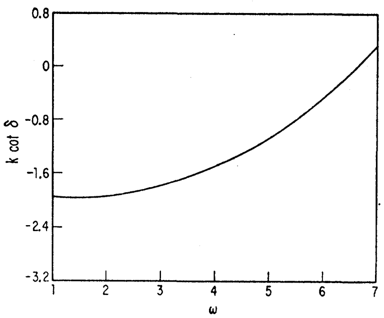

FIG. 3. The value of $k\cot\delta_1$ as a function of $\omega\equiv W-m$ for the $I=\frac{1}{2}$, s-wave solution obtained with $\gamma_{33}=0.06$, and $\gamma_1=-1.0$.

We have put the k_s^{2J} factor inside the absolute sign to avoid all possible confusion on the over-all sign.

If we assume that $h_J{}^L(W)$ is sufficiently convergent for large W, the dispersion relation for $h_J(W)$ takes the form

$$h_J(W)=h_J{}^L(W)+\frac{1}{\pi}\left[\int_{-\infty}^{-(m+\mu)}+\int_{(m+\mu)}^{\infty}\right]dW'\times\frac{(E'+m)\,|k_s'^{2J}h_J{}^2(W')|}{W'^{2J}(W'-W)}. \quad (6.5)$$

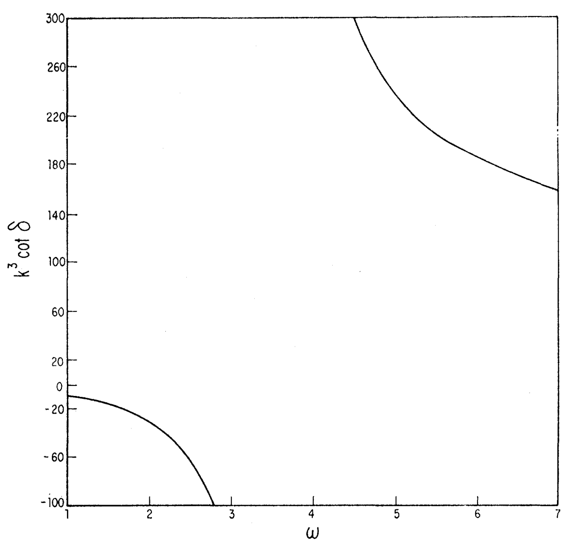

FIG. 4. The value of $k^3\cot\delta_{11}$ for the $I=\frac{1}{2}$; $J=\frac{1}{2}$ p-wave solution obtained with $\gamma_{33}=0.06$ and $\gamma_1=-1.0$.

Once $h_J{}^L(W)$ is known, Eq. (6.5) can be solved by the well-known N/D method:

$$h_J\equiv N/D,$$

where

$$D(W)=1-\frac{W}{\pi}\left[\int_{-\infty}^{-(m+\mu)}+\int_{(m+\mu)}^{\infty}\right]dW'\times\left(\frac{E'+m}{W'^{2J+1}}\right)\frac{|k_s'|^{2J}N(W')}{(W'-W)}, \quad (6.7)$$

and

$$N(W)=h_J{}^L(W)+\frac{1}{\pi}\left[\int_{-\infty}^{-(m+\mu)}+\int_{(m+\mu)}^{\infty}\right]dW'\times\left[\frac{W'h_J{}^L(W')-Wh_J{}^L(W)}{W'-W}\right]\left(\frac{E'+m}{W'^{2J+1}}\right)\times|k_s'|^{2J}N(W'). \quad (6.8)$$

Unfortunately, the $h_J{}^L(W)$ resulting when the exchange terms given in Eqs. (6.1) and (6.2) are substituted in Eq. (6.3) behave like W for large W. With this asymtotic behavior one can show that no solution exists for Eq. (6.5) in its present form. This is the well-known divergent behavior associated with forces arising from the exchange of particles of spin ≥ 1. To avoid this difficulty we modify Eq. (6.5) by introducing a cutoff for all dispersion integrals depending on a single parameter W_c. This consists of simply terminating the dispersion integrals in Eqs. (6.5)–(6.8), i.e., replace ∞ by W_c. This will yield a solution for all partial-wave amplitudes depending on one arbitrary parameter W_c. Numerical solutions for Fredholm equations of the type

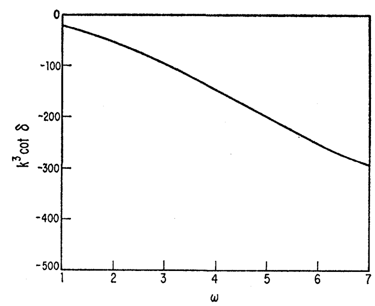

FIG. 5. The value of $k^3\cot\delta$ for the $I=\frac{1}{2}$, $J=\frac{3}{2}$ p-wave solution obtained with $\gamma_{33}=0.06$ and $\gamma_1=-1.0$.

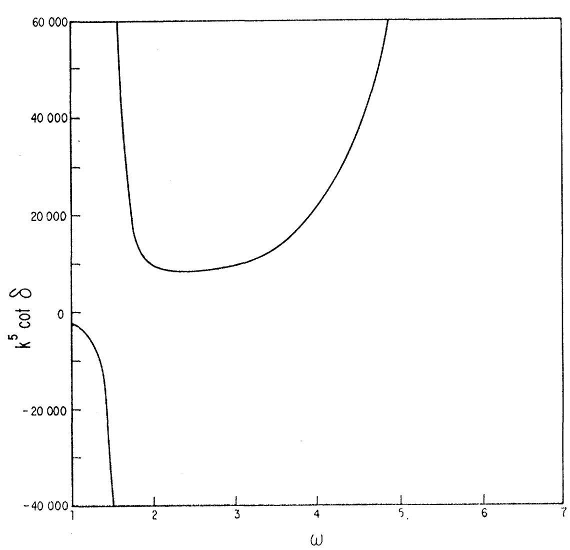

FIG. 6. The value of $k^5\cot\delta$ for the $I=\frac{1}{2}$, $J=\frac{3}{2}$ d-wave solution obtained with $\gamma_{33}=0.06$ and $\gamma_1=-1.0$.

TABLE I. The results obtained by solving the $\pi-N$ dispersion relations for various values of the input parameters γ_{33} and γ_1. The tabulated quantities are calculated values defined as follows: m is the nucleon mass; g^2 is the pion-nucleon coupling constant; Γ_{33} is the width of the N^*; a_1 and a_3 are the s-wave scattering lengths; the $c_{2I,2J}$ are the p-wave scattering lengths; the $d_{2I,2J}$ are the d-wave scattering lengths. The value of the cutoff parameter used to produce $M_{33}=8.8$ is W_c. All quantities are given in units in which $\hbar=c=\mu=1$.

γ_{33}	γ_1	m	g^2	Γ_{33}	a_1	a_3	c_{11}	c_{31}	c_{13}	c_{33}	d_{13}	d_{33}	W_c
0.05	−1.0	5.6	21.8	0.96	−0.511	−0.137	−0.068	−0.094	−0.047	0.334	−0.0003	0.0007	16.8
0.05	−0.5	5.6	18.6	0.97	−0.459	−0.176	−0.056	−0.066	−0.032	0.324	−0.0013	0.0010	17.5
0.05	0.0	6.0	16.5	0.84	−0.390	−0.196	−0.080	−0.046	0.081	0.317	−0.0043	0.0013	18.3
0.06	−1.0	6.0	23.6	1.00	−0.504	−0.153	−0.105	−0.086	−0.045	0.336	−0.0005	0.0006	16.6
0.06	−0.5	6.1	20.7	0.98	−0.448	−0.184	−0.108	−0.061	−0.022	0.324	−0.0017	0.0010	17.3
0.06	0.0	6.9	18.5	0.98	−0.372	−0.197	−0.311	−0.043	+0.301	0.316	−0.0068	0.0013	18.0
0.07	−1.0	6.2	25.0	0.97	−0.497	−0.165	−0.147	−0.080	−0.042	0.336	−0.0006	0.0006	16.4
0.07	−0.5	6.4	22.2	0.95	−0.439	−0.191	−0.176	−0.057	−0.007	0.321	−0.0021	0.0009	17.0
0.07	0.0	7.6	24.7	1.04	−0.359	−0.198	−2.68	−0.041	+4.25	0.313	−0.0106	0.0012	17.7

(6.8) are readily obtainable by the use of a computer. The solution for $N(W)$ is then substituted in (6.7) and then into (6.6).

VII. NUMERICAL PROGRAM

The value of m_ρ, m_{33}, γ_{33}, and g_r^2 for the exchange terms are all taken from experimental data

$$\begin{aligned} m_\rho&=5.4,\\ m_{33}&=8.8,\\ \gamma_{33}&\simeq 0.06,\\ g_r^2/4\pi&=14. \end{aligned} \tag{7.1}$$

On the basis of a calculation of the nucleon-isovector form factors we estimate that $\gamma_1\simeq-1.0$, but to illustrate the effect of this parameter we will give solutions for $\gamma_1=-1.0$, -0.5, and 0.0. Furthermore, while we expect the uncertainty in γ_{33} is of the order 10%, we will give solutions for $0.05\leq\gamma_{33}\leq 0.07$.

Our procedure is the following: For each set of parameters the value of W_c is adjusted to give the $3-3$ resonance at $W=8.8$, we then calculate all of the $J\leq\frac{3}{2}$ partial waves from Eqs. (6.6) to (6.8) using that value of W_c. For each set of parameters used a zero appeared in the D function for the $J=\frac{1}{2}$, $I=\frac{1}{2}$ amplitude, thus producing the nucleon as a bound state of the $\pi-N$ system. It should be emphasized that the nucleon pole was included only in the u-channel exchange term and that the s-channel pole was not included. For each set of parameters, we calculate the resulting nucleon mass, $\pi-N$ coupling constant ($-\frac{2}{3}$ times and the residue of the pole appearing in $h_{1/2}$), the width of the N^*, and the scattering lengths for each partial wave. These results are given in Table I. For the parameters $\gamma_{33}=0.06$ and $\gamma_1=-1.0$ we have plotted in Figs. 3–10 the $k^l\cot\delta_l$ for each partial wave. For comparison, the scattering lengths obtained from experimental data by Woolcock[6]

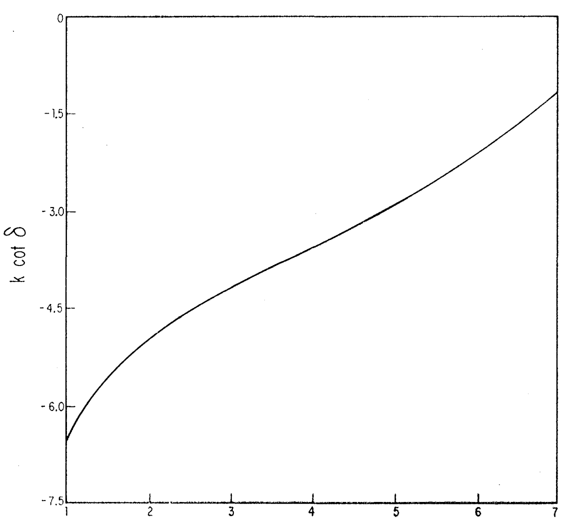

FIG. 7. The value of $k\cot\delta$ for the $I=\frac{3}{2}$ s-wave solution obtained with $\gamma_{33}=0.06$ and $\gamma_1=-1.0$.

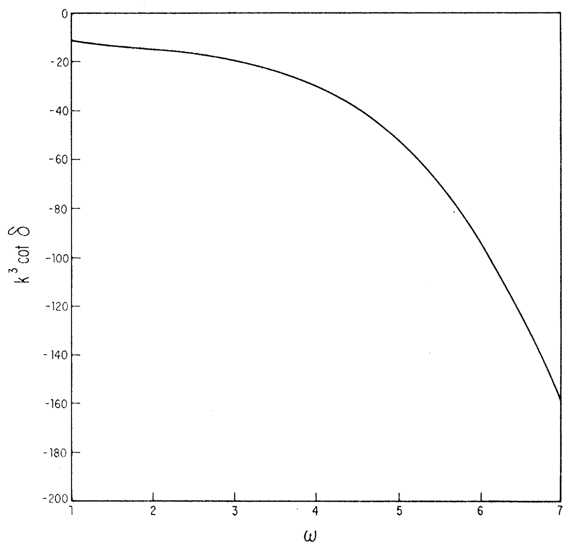

FIG. 8. The value of $k^3\cot\delta_{11}$ for the $I=\frac{3}{2}$, $J=\frac{1}{2}$ p-wave solution obtained with $\gamma_{33}=0.06$ and $\gamma_1=-1.0$.

[6] W. S. Woolcock, *Proceedings of the Aix-en-Provence Conference on Elementary Particles, 1961* (Centre d'Etudes Nucléaires de Saclay, Seine et Dise, 1961), p. 459.

are given: $a_1=0.170\pm0.005$, $a_3=-0.089\pm0.004$, $c_{11}=-0.104\pm0.006$, $c_{13}=-0.030\pm0.005$, $c_{31}=-0.040\pm0.004$, and $c_{33}=0.215\pm0.004$. The experimental width of the N^* is $\Gamma_{33}\simeq0.7$. It should be emphasized, however, that we have made no attempt to adjust the parameters γ_{33} and γ_1 to produce a good fit to scattering.

VIII. CONCLUSIONS

The solutions we have obtained for the $\pi-N$ partial waves have the following interesting features: (a) They are solutions of relativistic dispersion relations satisfying the theoretical requirements of physical unitarity in the elastic-scattering region and containing what are believed to be the most important exchange terms. Also, the pion-nucleon partial-wave amplitudes that are not included as interaction terms in the u channel are in fact small when calculated in the s channel. Thus, crossing symmetry is approximately satisfied (failing to the extent that the calculated γ_{33} and g_r^2 are not identical to those used in the interaction terms). (b) The nucleon is produced as a bound state with approximately the correct mass. (c) The residue of the nucleon pole and the width of the $(3-3)$ resonance obtained from our solutions are typically within 30% compared with experimental values. (d) The rest of the partial-wave amplitudes are in reasonable agreement with experimental data except for the $I=\frac{1}{2}$ s wave which we will discuss later.

The disagreement between the values of c_{33} given in Table I and the value from experiment is due to some extent to the curvature of $k^3\cot\delta_{33}$ near threshold. If one uses a linear extrapolation from the resonance region to threshold (the dashed line in Fig. 9) one obtains a scattering length consistent with the Γ_{33} given in Table I. In our model this curvature appears only below 50-MeV lab kinetic energy and may, in fact, be

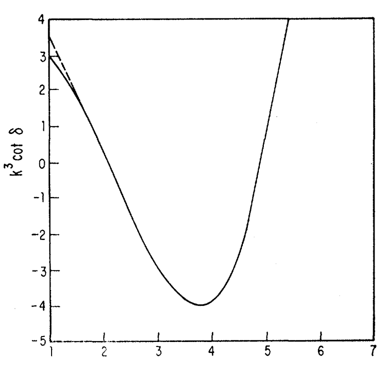

FIG. 9. The value of $k^3\cot\delta$ for the $I=\frac{3}{2}$, $J=\frac{3}{2}$ p-wave solution obtained with $\gamma_{33}=0.06$ and $\gamma_1=-1.0$.

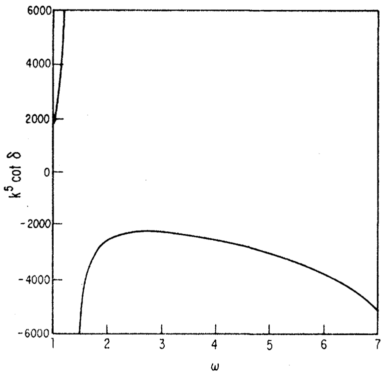

FIG. 10. The value of $k^5\cot\delta$ for the $I=\frac{3}{2}$, $J=\frac{3}{2}$ d-wave solution obtained with $\gamma_{33}=0.06$ and $\gamma_1=-1.0$.

present in the experimental data for $k^3\cot\delta_{33}$. For this reason Γ_{33} is a better measure of the agreement of our (3,3) solution with experiment.

The only result that seems to be in clear contradiction with the experimental analysis is the $I=\frac{1}{2}$ s-wave scattering length. One may argue that the value of a_1 should receive a substantial attractive contribution from the low-energy pion production where the pion-pair is likely to be produced in the $I=0$ state. In fact, if one examines the isotopic spin of the lowest energy inelastic final states, one finds that they are predominantly $I=\frac{1}{2}$; for example, the $I=0$ pion-pair plus a nucleon, $N-\eta$, $N-\omega$, and the $K-\Lambda$ are pure $I=\frac{1}{2}$, while the one-pion exchange term leading to $\rho-N$ favors $I=\frac{1}{2}$ to $I=\frac{3}{2}$ by a ratio of 4 to 1.[7] Thus, one would expect inelastic effects to be more pronounced for $I=\frac{1}{2}$ amplitudes. We have, however, made no attempt to include any inelastic states in our present treatment of the pion-nucleon problem. The value of a_3 seems to be in surprisingly good agreement with the observed value, particularly in view of the fact that it requires two orders of magnitude of "suppression" from the anomalously large nucleon pole term. We should mention that the only amplitudes that are sensitively dependent on the ρ-meson exchange contribution are the $I=\frac{1}{2}$, $J=\frac{3}{2}$, p- and d-wave amplitudes. Without the ρ-meson term, the repulsive interaction term arising from the N and N^* exchange is so strong that zeroes are created in the real part of the D functions above the threshold. These zeroes do not correspond to normal resonances, however, as the amplitudes have the sign opposite to that of the usual resonance amplitude. When the ρ-meson term is included these zeroes disappear, and the p- and d-wave phase shifts are small.

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[7] J. S. Ball and W. R. Frazer, Phys. Rev. Letters 7, 204 (1961).