Coupling Constant and P-Wave Phase Shifts in Low-Energy Pion-Nucleon Scattering†

ARCHIBALD W. HENDRY AND BERTHOLD STECH*
University of California, San Diego, La Jolla, California
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By means of a model for low-energy pion-nucleon scattering, it is shown that analyticity, unitarity, and the crossing properties are sufficiently strong conditions to determine the πN coupling constant, the P-wave phase shifts [including the position and width of the $(\frac{3}{2},\frac{3}{2})$ resonance] and to give restrictions on the $\pi\pi$ cut. In addition, the solution for the $(\frac{1}{2},\frac{1}{2})$ scattering amplitude turns out to be of the form describing the nucleon as a bound state of the πN system in close analogy to a bound state in potential theory. In the model, the $\pi\pi$ cut is represented by a pair of conjugate poles, and crossing is approximated by the static crossing relations. No cutoff parameter has to be introduced, and the crossing relations are fulfilled to a high degree.

1. INTRODUCTION

In recent years, much progress has been made in the understanding of strong interactions. However, the calculations of the π-meson nucleon-scattering phase shifts using partial-wave dispersion relations[1–3] are still in an unsatisfactory state. They rely on detailed experimental information. An arbitrary cutoff parameter has to be adjusted even to give a qualitative fit to the (3,3) resonance and its position. Another major difficulty is to find a solution which fulfills the crossing relations. Furthermore, in these calculations, the integrals involved have their main contributions from regions far above the validity of the elastic unitarity condition, due to kinematical factors arising from threshold behavior. These difficulties unfortunately also remain in the interesting treatments[4,5] initiated by Chew,[4] where the suggestion is explored that the (3,3) resonance provides the force (by means of a bootstrap mechanism) to make the nucleon a bound state. Many authors therefore feel that a knowledge of the contributions from inelastic channels and about the high-energy behavior is vital for a successful calculation. This is certainly true for any precise treatment. It is the purpose of this paper to show, however, that the entire qualitative behavior of the P-wave scattering amplitudes for small energies may very well be determined by singularities which lie relatively close to the physical threshold. The cutoff in the static Chew-Low theory is needed because of the strong increase of the P-wave amplitudes due to the q^{2l+1} threshold behavior. It seems plausible to us, however, that this increase will be reduced through nearby singularities in a more important way than through the effect of relativistic nucleon motion at much higher energies. We postulate, therefore, that the nearby portion of the π-π cut is precisely of a form leading to such a reduction. This assumption automatically leads to interesting restrictions for this cut which can no longer be chosen completely arbitrarily.

In order to study these ideas, we have solved numerically a model which contains the essential features of the π-meson nucleon system at low energies, but which is still simple enough to allow a thorough investigation of its solution. In particular, we wish to incorporate unitarity and crossing to a high degree in order to calculate the P waves and the πN coupling constant.

Our model consists of representing the $\pi\pi$ cut crudely by a pair of conjugate poles, and using the static crossing relations.[1] We find that in the framework of this model, the πN coupling constant, the P waves [including the (3,3) resonance position and width], the position and residues of the $\pi\pi$ poles, are so intricately interwoven that they determine one another. Also, without additional assumptions the nucleon turns out to be a bound state of the πN system in much the same way as bound states occur in potential theory. There is no need for a cutoff parameter.

In Sec. 2, we describe the singularities of the πN partial-wave scattering amplitudes. The coupled integral equations for the process are written down in Sec. 3, and the method of solution indicated. Lastly, the results for our model are stated and discussed briefly in Sec. 4.

2. ANALYTIC STRUCTURE

The analytic structure of the pion-nucleon partial-wave scattering amplitudes can be deduced from the Mandelstam representation. In our model, which is relevant to low-energy phenomena, we shall take a simplified version of the singularities in the vicinity of the physical threshold, as shown in Fig. 1 for the E plane, where E is the total energy of the π meson. The right-hand cut is the physical cut; the poles at the origin are the usual static nucleon poles. They consist of two parts: One corresponds to a short cut of the full relativistic theory, which is a force arising from the nucleon exchange; the other part originates from the true nucleon pole in the energy variable and appears only in the $I=\frac{1}{2}$, $J=\frac{1}{2}$ state. The arc in the figure arises from $\pi\pi$

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* Present address: Institute for Theoretical Physics, University of Heidelberg, Philosophenweg, Germany.
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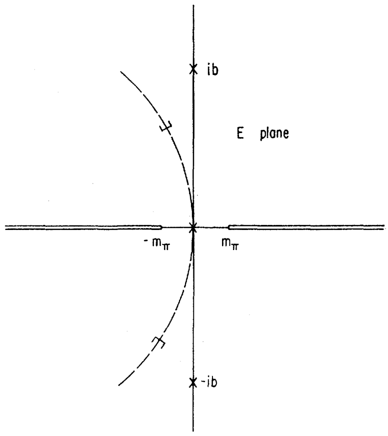

FIG. 1. Singularities of $T_\alpha(E)$ in the E plane.

interactions, and the contribution from the ρ meson (mass$=750$ MeV) starts from the positions indicated. This effect we approximate in a crude way by two poles on the imaginary axis at the points $E=\pm ib$.

We shall further use the crossing relations which follow from the correct expressions by letting the nucleon mass become large, namely the static crossing relations.[1] These relations do not mix states of different orbital angular momentum. In particular, we shall deal with P waves, which are important because the nucleon has the same quantum numbers as the $I=\frac{1}{2}$, $J=\frac{1}{2}$ scattering state, and because of the well-known strong resonance in the $I=\frac{3}{2}$, $J=\frac{3}{2}$ state.

3. FORMULATION AND SOLUTION PROCEDURE

In our treatment we shall use the amplitudes

$$T_\alpha(E)=E(1+E^2/b^2)e^{i\delta_\alpha}\sin\delta_\alpha/q^3(E). \tag{1}$$

The label α denotes $(2I,2J)$ and $\alpha=1, 2, 3$ corresponds to $(1,1)$, $(1,3)=(3,1)$ and $(3,3)$, respectively. The amplitudes $T_\alpha(E)$ therefore have only the right- and left-hand cuts of Fig. 1 along the real axis. The cuts of the pion momentum[6] $q(E)=(E^2-1)^{1/2}$ have been taken along the real axis for $|E|\geq 1$, and thus $T_\alpha(E)=T_\alpha^*(E^*)$. The asymptotic behavior of $T_\alpha(E)$ is given directly by the asymptotic behavior of the phase shifts.

The crossing property for T is

$$T_\alpha(-E)=-\sum_{\beta=1}^{3} c_{\alpha\beta}T_\beta(E) \tag{2}$$

in the cut E plane, where

$$c_{\alpha\beta}=\frac{1}{9}\begin{pmatrix} 1 & -8 & 16 \\ -2 & 7 & 4 \\ 4 & 4 & 1 \end{pmatrix}.$$

Putting $T_\alpha(E)=N_\alpha(E)/D_\alpha(E)$, where N contains the left-hand cut and D the right-hand cut,[7] we obtain the coupled integral equations

$$N_\alpha(E)=g_\alpha+\frac{E}{\pi}\int_1^\infty dE'\frac{A_\alpha(E')}{E'(E'+E)}, \tag{3}$$

$$D_\alpha(E)=1-\frac{E}{\pi}\int_1^\infty dE'\frac{h(E')N_\alpha(E')}{E'(E'-E)}, \tag{4}$$

where $h(E)=q^3(E)/E(1+E^2/b^2)$, and $D_\alpha(E)$ has been normalized to 1 at the origin. It may be seen that the $(1+E^2/b^2)$ factor arising from the $\pi\pi$ poles reduces the effect of the $q^3(E)$ increase in the integrand of (4). We have assumed that $N_\alpha(E)/E$ and $D_\alpha(E)/E\to 0$ as $E\to\infty$. The quantity $A_\alpha(E)$ is the discontinuity of $N_\alpha(E)$ across the left-hand cut; g_α is given by

$$g_\alpha=\tfrac{1}{3}f^2\begin{pmatrix} -8 \\ -2 \\ +4 \end{pmatrix} \tag{5}$$

with f^2 the renormalized unrationalized pion-nucleon coupling constant. The quantity $g_1=-(8/3)f^2$ can be written $g_1=+(1/3)f^2-(9/3)f^2$ where the second term originates from the true nucleon pole. The physical phase shifts can be calculated from

$$h(E)\cot\delta_\alpha(E)=\mathrm{Re}D_\alpha(E)/N_\alpha(E). \tag{6}$$

Further, on taking the imaginary part of the crossing relation (2), we obtain

$$\mathrm{Im}T_\alpha(-E)=-\sum_{\beta=1}^{3} c_{\alpha\beta}\,\mathrm{Im}T_\beta(E), \tag{7}$$

which yields a further equation involving $A_\alpha(E)$:

$$h(E)A_\alpha(E)/D_\alpha(-E)=\sum_{\beta=1}^{3} c_{\alpha\beta}\sin^2\delta_\beta(E). \tag{8}$$

To obtain a solution for $T_\alpha(E)$, we will use only the imaginary part of the crossing relation (7) and (8). This relation should be a good approximation to the true situation for low energies.

It is convenient to define the function $\mathfrak{D}_\alpha(E)$ which is closely related to $D_\alpha(E)$:

$$\mathfrak{D}_\alpha(E)=\exp\left[-\frac{E}{\pi}\int_1^\infty dE'\frac{\delta_\alpha(E')}{E'(E'-E)}\right]. \tag{9}$$

$\mathfrak{D}_\alpha(E)$ has the same analytic properties, the same phase along its cut, and normalization, as $D_\alpha(E)$. They may, however, differ by their asymptotic behavior and therefore by a factor which is a real polynomial in E with value 1 at $E=0$. The asymptotic behavior of $\mathfrak{D}_\alpha(E)$ is

$$\mathfrak{D}_\alpha(E)\to E^{\delta_\alpha(\infty)/\pi}$$

apart from logarithmic terms.[8]

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[6] We take the units $\hbar=c=m_\pi=1$.

[7] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

[8] The phase shifts $\delta_\alpha(E)$ are defined to be zero at threshold.

An interation procedure was adopted to solve Eqs. (3), (4), (6), and (8). To start, $A_\alpha^1(E)$ was taken to be zero, that is $N_\alpha^1(E)=g_\alpha$ which is obtained from (5) with an input value of f^2. In subsequent iterations, first $A_\alpha(E)$ is calculated from Eq. (8) using the values of δ_α and D_α (or $\mathfrak{D}_\alpha$ as described later) of the previous iteration. This provides new values for N_α and D_α which are then used to calculate new phase shifts.

It follows immediately from (4) and (6) that the first iteration phase shifts go logarithmically for large E as $\delta_1^1\to-\pi$, $\delta_2^1\to-\pi$, and $\delta_3^1\to 0$. For $\alpha=3$, $\mathfrak{D}_\alpha^1(E)$ as calculated from (9) with these first iteration phase shifts is therefore equal to $D_\alpha^1(E)$. However, for the cases $\alpha=1,2$, $\mathfrak{D}_\alpha^1(E)$ and $D_\alpha^1(E)$ are not equal, but differ by a polynomial of degree one in E, thus illustrating their different asymptotic behaviors. Indeed, for $\alpha=1,2$, $D_\alpha^1(E)$ has a zero for a negative value of E [corresponding to a ghost in $T_\alpha^1(E)$]. In subsequent iterations, however, the values of $D_2(E)$ and $\mathfrak{D}_2(E)$ rapidly became identical, showing that the final amplitude $T_2(E)$ has no ghost. Also for large energies, $\delta_2(E)$ like $\delta_3(E)$ approaches zero rather than $-\pi$. The amplitude $T_1(E)$, on the other hand, behaves differently. The phase $\delta_1(E)$ continues to go asymptotically to $-\pi$ and a ghost remains in this amplitude. It is not difficult to see how this ghost originates. The denominator $D_1(E)$ behaves asymptotically (again apart from logarithmic terms) as $1/E\times$polynomial in E. The ghost will disappear only if $N_1(E)$ likewise contains this same polynomial factor so that it cancels out exactly in (6). This cancellation does not occur and a ghost persists in $T_1(E)$.

This situation is not discouraging. The behavior $\delta_1(E)\to-\pi$ and $D_1(E)\to 1/E$ opens the way to use unsubtracted equations for this phase. Since (3) and (4) simplify, no ghost will appear. If successful, this approach will lead to physically significant results. The nucleon can then be considered as a bound state and the residue of the (1,1) pole can be calculated and compared with the original coupling constant used. In this treatment, the nucleon pole is automatically kept at $E=0$, which fixes the binding-energy to be equal to m_π.

In a new treatment, therefore, after the first iteration we used for $\alpha=1$ the unsubtracted equations

$$N_1(E)=-\frac{1}{\pi}\int_1^\infty dE'\frac{A_1(E')}{E'+E}, \qquad (10)$$

$$D_1(E)=-\frac{1}{\pi}\int_1^\infty dE'\frac{h(E')N_1(E')}{E'-E}, \qquad (11)$$

$$h(E)A_1(E)/\mathfrak{D}_1(-E)=\sum_{\beta=1}^{3} c_{1\beta}\sin^2\delta_\beta. \qquad (12)$$

Equation (12) is different from (8) since $D_1(-E)$ is replaced by $\mathfrak{D}_1(-E)$. This is appropriate for the second iteration since $\mathfrak{D}_1^1(-E)$ has the asymptotic behavior necessary for unsubtracted dispersion relations and does not produce a ghost. Then in the second iteration, $D_1^2(E)$ as calculated from (11) is identical to $\mathfrak{D}_1^2(E)$ apart from a multiplicative constant. Subsequent iterations showed that this proportionality property is maintained; $T_1(E)$ therefore no longer contains a ghost. The multiplicative constant, which is $D_1(E)/\mathfrak{D}_1(E)$, may be evaluated by taking $E=0$; it is just $D_1(0)$.

By using $\mathfrak{D}_1(E)$ in (12), we have tacitly chosen a particular normalization factor for $A_1(E)$ which is arbitrary in the unsubtracted Eqs. (10) and (11). But in order that Eq. (12) is consistent with (7), the two forms $D_1(E)$ and $\mathfrak{D}_1(E)$ should not only be proportional but equal. Thus $D_1(0)$ as calculated from (11) must be 1. We shall later make use of this condition to find the coupling constant f^2 for which the three crossing relations (7) are fulfilled.

For the other amplitudes $\alpha=2,3$, we continued to use the subtracted Eqs. (3) and (4). For $\alpha=2$, again the values of $D_2(E)$ and $\mathfrak{D}_2(E)$ in further iterations rapidly became identical, showing that $T_2(E)$ has no ghost and $\delta_2(E)\to 0$ as before. The amplitudes for the phases $\alpha=2,3$ are therefore markedly different from the amplitude for phase $\alpha=1$ where we found it possible to use unsubtracted equations.

Following the above procedure, we obtained a reasonably rapid convergence. Indeed, the eighth, ninth, and tenth iterations for $A_\alpha(E)$ were within 1% of the seventh; this was true for all the mesh values of E used. (In the computation, we used the equations with the range $1\leq E\leq\infty$ mapped on to the finite interval $0\leq z\leq 1$ where $z=1/E$, and employed 70 mesh points. Using 100 mesh points did not produce any noticeable change in the results.)

This convergence implies that the crossing relations (8) and (7) for the imaginary part of T_α are satisfied precisely for $\alpha=2,3$ along the cuts. Equation (12) is also satisfied. However, the relation (7) for $\alpha=1$ is satisfied only if $D_1(0)$ comes out to be exactly equal to 1. If this is indeed the case, then it follows that the computed amplitudes $T_\alpha(E)$ fulfill besides unitarity the dispersion representation

$$T_\alpha(E)=g_\alpha+\frac{E}{\pi}\int_1^\infty dE' \times\left\{\frac{\mathrm{Im}T_\alpha(E')}{E'(E'-E)}+\sum_{\beta=1}^{3}\frac{c_{\alpha\beta}\,\mathrm{Im}T_\beta(E')}{E'(E'+E)}\right\}. \qquad (13)$$

Here g_2 and g_3 are given by (5) with f^2 having its original input value. However, since we used unsubtracted equations for $\alpha=1$, g_1 in Eq. (13) is given by the formula

$$g_1=\frac{-1}{\pi}\int_1^\infty dE'\frac{A_1(E')}{E'}\Big/D_1(0). \qquad (14)$$

The value of g_1 computed from (14) is of course not necessarily identical to $-8f^2/3$.

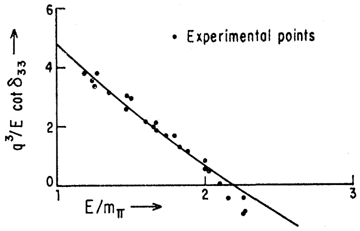

FIG. 2. Chew-Low plot with $b/m_\pi=5$ and $f^2=0.087$.

As mentioned earlier in our iteration procedure, we have made use of only the crossing relations (7) for the imaginary parts of the amplitudes. From (13), however, it follows that the complete amplitudes obey the relations

$$T_\alpha(-E)=-c_{\alpha\beta}T_\beta(E)+d_\alpha \tag{15}$$

in the entire complex E plane, with $d_\alpha=g_\alpha+\sum_{\beta=1}^3 c_{\alpha\beta}g_\beta$. Equation (15) differs from (2) by the real additive constants d_α. The condition $d_\alpha=0$ implies complete crossing [Eq. (2)].

By way of checking the relation (15) numerically, we calculated $T_\alpha(E)$ for $\alpha=1,2,3$ at the two points $E=ib/2$ and ib on the imaginary axis, and found that (15) is indeed well satisfied (within 3%).

4. RESULTS

(a) For a fixed value[9] of $f^2=0.087$, it is possible to choose a value of b (namely $b\approx5.0$) such that the graph for $q^3\cot\delta_3(E)/E$ agrees well with experiment for energies below the (3,3) resonance, as shown in Fig. 2. We have also drawn the phase shifts $\delta_\alpha(E)$ in Fig. 3.

The solution so obtained with these values of b and f^2 satisfies Eq. (7) for $\alpha=2,3$ accurately for all values E along the cuts. However, for the small phase $\alpha=1$, the relation (7) is fulfilled to only 79%; this follows since the computed value of $D_1(0)$ turned out to be 0.79 instead of 1. The crossing relations (2) are then not satisfied precisely.

Equation (14) makes it possible to calculate the coupling constant f_c^2, defined by $f_c^2=-3g_1/8$, and to compare it with the value $f^2=0.087$. It turned out to be 0.075, deviating from f^2 by 13.5%. Such a good agreement for f^2 may be partially fortuitous since it is calculated from a ratio. Nevertheless, the qualitative agreement suggests that our basic assumptions and the underlying ideas are correct.

An interesting aspect of our model is that the residues of the poles representing the $\pi\pi$ cuts are not arbitrary parameters, but are automatically determined. In a more realistic treatment of the πN problem with a better approximation for the $\pi\pi$ cut, it therefore seems very likely that the physical requirements for the amplitudes impose strong restrictions on the $\pi\pi$ contribution. The residues $R_\alpha=-\frac{1}{2}T_\alpha(ib)$ for the amplitudes $e^{i\delta_\alpha}\sin\delta_\alpha/q^3$ at the pole position $E=5i$ were computed to be $R_1=0.020-0.055i$, $R_2=0.013-0.011i$, and $R_3=-0.027-0.029i$. For points on the imaginary axis, crossing requires

$$\mathrm{Re}R_2/\mathrm{Re}R_3=-0.5,\quad \mathrm{Re}R_1/\mathrm{Re}R_3=-2.0,$$
$$\text{and}\quad \mathrm{Im}(R_1+R_2-2R_3)=0. \tag{16}$$

It is seen that these equations are satisfied reasonably apart from the second (which is to be expected, since we have not attempted to obtain the best crossing in the present paragraph). The contribution from these $\pi\pi$ poles for physical values of E, namely $[R_\alpha/(E-ib)+\text{c.c.}]$, is smaller than that estimated directly[4] from information on the ρ meson. A comparison is, however, hardly possible since our static model is certainly not good at $E=\pm5i$.

(b) In this section, we pursue our model further. Here we take the crossing relations of the model more seriously in order to study their full implication, rather than try to obtain a fit to experimental data.

The calculation of the amplitudes was carried out over a range of values of b and f^2. For a fixed b, it turned out that there is only one value of f^2 for which $D_1(0)=1$. The crossing relations (7) for the cuts are then fulfilled (to within 1%) for $\alpha=1,2,3$ and for all values of E along the cuts. The πN coupling constant is therefore

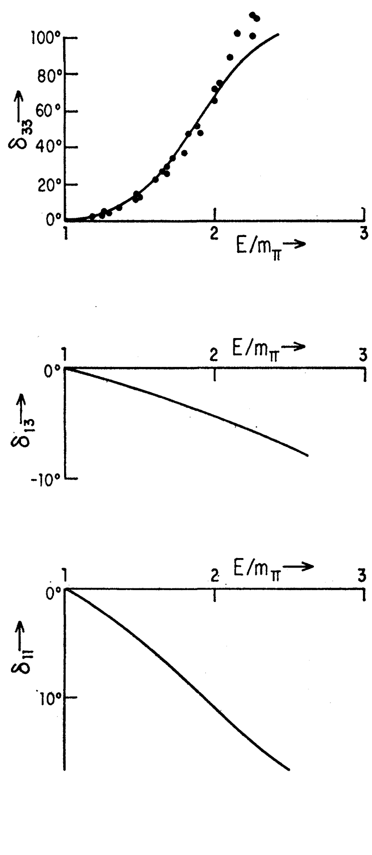

FIG. 3. Graph of phase shifts $\delta_\alpha(E)$ against E, with $b=5$ and $f^2=0.087$.

[9] S. W. Barnes, B. Rose, G. Giacomelli, J. Ring, K. Miyake, and K. Kinsey, Phys. Rev. 117, 226 (1960).

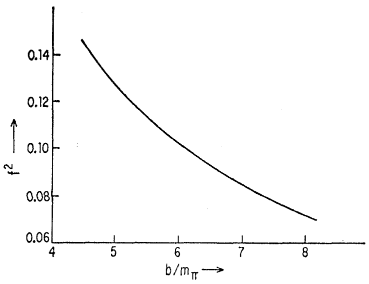

FIG. 4. Determination of f^2 by crossing. The line gives the values of the πN coupling constant as a function of the pole position b. For values of f^2 and b corresponding to this line, the crossing relation (7) is fulfilled.

not a free parameter but determined by crossing. The dependence of f^2 on b is shown in Fig. 4.[10]

As discussed in Sec. 3, the total amplitudes $T_\alpha(E)$ fulfill, in addition, the relations (15) throughout the whole complex E plane precisely. The coupling constant g_1 evaluated from the expression (14) turned out to be 77% of the corresponding input value along the line in Fig. 4. The constants d_α in (15) are therefore small but not zero; their values are $d_1=0.69f^2$, $d_2=-0.14f^2$, and $d_3=0.28f^2$. Thus, unitarity and the imaginary part of the crossing relations (7) can be fulfilled precisely, but the real part is only approximately correct.[11] This result again emphasizes our point that these general requirements are very restrictive.

It is worth noting that the spectrum conditions, together with unitarity and crossing, determine the solution of our model. Indeed, for a given pole position b, one is able to deduce the πN coupling constant as well as the phases and the residues R_α at these poles for all three amplitudes. It may seem surprising that the $\pi\pi$ poles should have such an important effect on the solutions for all the quantities, for example, f^2 and the position of the resonance. On the other hand, in our model at least, all these quantities depend very strongly on one another and their individual effects cannot be separated out.[12]

Another general feature coming out of all our calculations is that unsubtracted dispersion relations for $N_1(E)$ and $D_1(E)$ exist. Moreover, the coupling constant g_1 is negative and the phase δ_1 tends to $-\pi$ as E becomes large. This gives evidence that the nucleon can be considered as a bound state of the πN system, very much equivalent to the situation of a bound state occurring in potential theory.

Our findings enhance the viewpoint, particularly emphasized by Chew,[13] that elementary constants such as the πN coupling constant and the position of the (3,3) resonance are determined dynamically by a consistent application of general concepts.

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[10] There may be additional solutions of our model which do not appear in the iteration procedure. In the sense of a bootstrap mechanism, however, our solution should have more physical significance than others.

[11] The restrictions imposed on our model are slightly too stringent. Another pair of poles on the imaginary axis would almost certainly remove this difficulty.

[12] Indeed if we exclude the $\pi\pi$ poles [or equivalently enforce $T_\alpha(\pm ib)=0$] there does not seem to be any solution.

[13] G. F. Chew, *S-Matrix Theory of Strong Interactions* (W. A. Benjamin, Inc., New York, 1961).