This is just the Lagrange bracket of x^l with p_i, and the above requirement is satisfied if the $Q_{\mathbf{k}}$, $P_{\mathbf{k}}$ form a canonical set (in involution).

If we had expressed Z^i as a function of $Q_{\mathbf{k}}$, and made a similar projection of $\Delta Q^\mu = \sum_\mu \delta Q^\mu$, we would have obtained instead of the Lagrange bracket relation, the Poisson bracket relations

$$\sum_i \left(\frac{\partial Q_{\mathbf{k}}}{\partial \mathbf{x}^i} \frac{\partial P_{\mathbf{k}'}}{\partial \mathbf{p}_i} - \frac{\partial Q_{\mathbf{k}}}{\partial \mathbf{p}_i} \frac{\partial P_{\mathbf{k}'}}{\partial \mathbf{x}^i} \right) = \delta_{\mathbf{k}\mathbf{k}'} , \qquad \text{(C.10)}$$

which also are satisfied if $Q_{\mathbf{k}}$, $P_{\mathbf{k}}$, form a canonical set of functions.

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Meissner Effect and Flux Quantization in the Quasiparticle Picture*

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A superconductor in a magnetic field is studied by means of a generalized quasiparticle transformation. Using the minimum principle for the thermodynamic grand potential, nonlinear equations are derived for the superconductor, which can be considered valid at finite temperature. For small magnetic potential these equations are linearized and shown to imply the London formula and the Meissner effect. For a multiply-connected superconductor the nonlinear equations are shown to be consistent with Maxwell's equation only if the magnetic flux is quantized in the units predicted by Onsager.

1. INTRODUCTION

In the basic work of Bardeen, Cooper, and Schrieffer,[1] the energy spectrum of a superconductor is described in terms of independent quasiparticle excitations. The natural mathematical tool for this description is the quasiparticle canonical transformation of Bogolyubov[2] and Valatin.[3] The presence of external electromagnetic fields, however, invalidates the simple pairing of the original Bogolyubov-Valatin transformation. In order to study this situation one can introduce a generalized quasiparticle transformation which makes no such pairing assumptions. This was done again by Bogolyubov[2,4] and also, in slightly different forms, by a number of other authors.[5] Using these methods a fully gauge invariant treatment of the Meissner effect has been achieved.

These discussions of the Meissner effect are usually limited to the case of absolute zero temperature. The generalization to finite temperatures is obviously desirable in order to study electromagnetic properties up to the transition temperature. However, it is far from obvious how one should proceed in order to generalize to finite temperatures some of the above mentioned discussions. It is convenient to follow rather closely Bogolyubov's work[4] which can be generalized to finite temperature without too much difficulty.[6] We make an ansatz for the density matrix in terms of independent quasiparticles and use the principle of minimization for the grand potential. In this way we obtain a set of nonlinear equations for a superconductor in a magnetic field which are valid at finite temperature. The Meissner effect is discussed in a fully gauge invariant way by linearizing the equations with respect to the vector potential. The current is seen to vanish at the critical temperature except for residual Landau-type diamagnetism. In the simple special case of a factorizable interaction one recovers the temperature dependence of the penetration depth given originally by BCS.[1]

Once the Meissner effect is established, one can use the nonlinear equations to study the phenomenon of magnetic flux quantization in a multiply-connected superconductor. This phenomenon, which was predicted theoretically by London[7] and Onsager,[8] has been verified experimentally by Deaver and Fairbank and by Doll and Näbauer.[9] For mathematical simplicity

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[1] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957).

[2] N. N. Bogolyubov, J. Phys. (U.S.S.R.) **11**, 23 (1947); Zh. Eksperim. i Teor. Fiz. **34**, 58 (1958) [translation: Soviet Phys.—JETP **7**, 41 (1958)].

[3] J. Valatin, Nuovo Cimento **7**, 843 (1958).

[4] N. N. Bogolyubov, Usp. Fiz. Nauk **67**, 549 (1959) [translation: Soviet Phys.—Usp. **2**, 236 (1959)].

[5] See e.g., J. Valatin, Phys. Rev. **122**, 1012 (1961); and J. M. Blatt, Progr. Theoret. Phys. (Kyoto) **24**, 851 (1960).

[6] Actually we can make some simplifications, since we do not seek the energy spectrum of collective excitations. This latter problem can be studied separately and was actually solved by E. R. Velibekov, Dokl. Akademii Nauk SSSR **142**, 1265 (1962) [translation: Soviet Phys.—Doklady **7**, 134 (1962)].

[7] F. London, *Superfluids* (Dover Publications, Inc., New York, 1961), Vol. I.

[8] L. Onsager, Phys. Rev. Letters **7**, 50 (1961).

[9] B. S. Deaver and W. M. Fairbank, Phys. Rev. Letters **7**, 43 (1961); R. Doll and M. Näbauer, Phys. Rev. Letters **7**, 51 (1961); Z. Physik **169**, 526 (1962).

the doubly connected geometry is simulated in our treatment (following Yang[10]) by a straight geometry with periodicity conditions. In the bulk of the material the nonlinear equations are found to be almost identical to those for an infinite medium with no field. The requirement of consistency with Maxwell's equation eliminates all but the solutions corresponding to quantized flux. This study of the nonlinear equations reproduces a number of the physical results given by Byers and Yang[11] in their work on magnetic flux quantization and by Yang[10] in his discussion of quantum-mechanical long-range order. In particular, we obtain the parabolic dependence of the free-energy curve upon the flux in the neighborhood of the quantized values. Since our scheme is valid at finite temperature, we can observe how the free-energy curve flattens out as one approaches the critical temperature from below. We also see that, in the absence of an attractive interaction causing superconductivity, the free-energy curve would be flat at all temperatures. The results of this discussion are very similar to those obtained independently by Maki and Tsuneto[12] with a different method, that of the thermal Green's functions. Our formulas, however, are more general, since they do not employ the stylized effective interaction of Gor'kov, and in particular contain contributions of Hartree-Fock type which are not considered by these authors.

It will be shown in a separate paper that the basic nonlinear equations given here can be used to give a systematic derivation of the Landau-Ginzburg theory of superconductivity.[13] This provides an alternative derivation to that given by Gor'kov[14] and Werthamer[15] using the method of the thermal Green's functions and can be used to justify the work of several authors[16] who have discussed the Meissner effect and quantization of the fluxoid within the framework of the Landau-Ginzburg theory.

2. MODEL

The model used aims at describing the essential equilibrium features more or less in common to all soft, pure superconductors. The Hamiltonian[17]

$$H=K_{f^2f^1}a_{f^1}{}^\dagger a_{f^2}+\tfrac{1}{2}P_{f^1f^2f^3f^4}a_{f^1}{}^\dagger a_{f^2}{}^\dagger a_{f^3}a_{f^4} \tag{1}$$

describes a system of charged interacting fermions. As is customary, we include in K a term containing the chemical potential μ. The matrix K is given by

$$\begin{aligned}K_{ff'}&=(f'|(\mathbf{p}-\mathbf{A})^2-\mu|f)\\&=\delta_{\sigma\sigma'}[\delta(\mathbf{k}-\mathbf{k}')(\mathbf{k}^2-\mu)-(\mathbf{k}+\mathbf{k}')\cdot\mathbf{A}(\mathbf{k}'-\mathbf{k})\\&\qquad+(\mathbf{A}^2)(\mathbf{k}'-\mathbf{k})].\end{aligned} \tag{2}$$

The kinetic-energy part of H is clearly invariant under a gauge transformation of the second kind. Assuming a diagonal and factorizable spin dependence for P,[18] the requirements of velocity independence and gauge invariance lead to essentially just one possibility for P.

$$P_{f^1f^2f^3f^4}=\delta_{\sigma1\sigma3}\delta_{\sigma2\sigma4}\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)P_{\mathbf{k}}\,, \tag{3}$$

$$\mathbf{k}=\mathbf{k}_1-\mathbf{k}_2\,,\quad P_{\mathbf{k}}=P_{-\mathbf{k}}{}^*=P_{-\mathbf{k}}\,. \tag{4}$$

Finally the charge current density operator associated with H is given by

$$\begin{aligned}\mathbf{J}_{\mathbf{q}}&=\mathbf{J}_{\mathbf{q}}{}^p+\mathbf{J}_{\mathbf{q}}{}^d\\&=eV^{-1}\sum_{\mathbf{k}\sigma}(2\mathbf{k}+\mathbf{q})a_{\mathbf{k}\sigma}{}^\dagger a_{\mathbf{k}+\mathbf{q}\sigma}\\&\qquad-2eV^{-1}\sum_{\mathbf{k}\mathbf{k}'\sigma}\mathbf{A}(\mathbf{k}-\mathbf{k}'+\mathbf{q})a_{\mathbf{k}\sigma}{}^\dagger a_{\mathbf{k}'\sigma}\,,\end{aligned} \tag{5}$$

where p and d denote the paramagnetic and diamagnetic parts, respectively.

3. QUASIPARTICLE TRANSFORMATION

In order to avoid the simple pairing assumptions of the BCS scheme and to have the possibility of gauge-invariant equations, Bogolyubov[4] introduced the generalized quasiparticle canonical transformation (sum over repeated indices)[19]

$$\begin{aligned}a_f&=u_{ff'}\alpha_{f'}+v_{ff'}\alpha_{f'}{}^\dagger\,,\\a_f{}^\dagger&=u_{ff'}{}^*\alpha_{f'}{}^\dagger+v_{ff'}{}^*\alpha_{f'}\,.\end{aligned} \tag{6}$$

It turns out to be convenient to introduce supervectors and supermatrices

$$\mathbf{a}=\begin{pmatrix}a\\a^\dagger\end{pmatrix},\quad\boldsymbol{\alpha}=\begin{pmatrix}\alpha\\\alpha^\dagger\end{pmatrix},\quad\mathbf{C}=\begin{pmatrix}u&v\\v^*&u^*\end{pmatrix}, \tag{7}$$

such that Eq. (6) can be written concisely as

$$\mathbf{a}=\mathbf{C}\boldsymbol{\alpha}\,. \tag{6'}$$

[10] C. N. Yang, Rev. Mod. Phys. **34**, 694 (1962).

[11] N. Byers and C. N. Yang, Phys. Rev. Letters **7**, 46 (1961).

[12] K. Maki and T. Tsuneto, Progr. Theoret. Phys. (Kyoto) **27**, 228 (1962).

[13] V. L. Ginzburg and L. D. Landau, Zh. Eksperim. i Teor. Fiz. **20**, 1064 (1950).

[14] L. P. Gor'kov, Zh. Eksperim. i Teor. Fiz. **36**, 1918 (1959) [translation: Soviet Phys.—JETP **9**, 1364 (1959)].

[15] N. R. Werthamer, Phys. Rev. **132**, 663 (1963).

[16] V. L. Ginzburg, Zh. Eksperim. i Teor. Fiz. **42**, 299 (1962) [translation: Soviet Phys.—JETP **15**, 207 (1962)]; J. Bardeen, Phys. Rev. Letters **7**, 164 (1961); J. B. Keller and B. Zumino, New York University (unpublished).

[17] Here $f=(\mathbf{k},\sigma)$ denotes the combination of momentum space vector and spin. Summation (periodic boundary conditions) over repeated indices is implied. The second quantized fermion field a_f satisfies the usual fermion equal-time commutation relations. The Hermiticity of H implies $K_{f^2f^1}=K_{f^1f^2}{}^*$ for the kinetic-energy density and $P_{f^1f^2f^3f^4}=P_{f^2f^1f^4f^3}=P_{f^4f^3f^2f^1}{}^*$ for the potential-energy density. It is convenient to use units for which $\hbar=c=2m=1$ and to define the vector potential so that $e\mathbf{A}_{\text{conventional}}=\mathbf{A}$, where $e=-|e|$ is the charge and m the mass of the fermions.

[18] Different spin dependences lead to minor modifications of what follows.

[19] The ansatz, Eq. (6), includes the simple pairing case for the special choice

$$u_{ff'}=\delta_{ff'}u_f\,,\quad v_{ff'}=\delta_{\bar{f}f'}v_f,$$

where $\bar{f}=-f$ and $u_f=u_{-f}$, $v_f=-v_{-f}$.

It is shown in Appendix I that if one requires the transformation $\mathbf{C}$ to be canonical and invertible, then $\mathbf{C}$ must be unitary. Then the matrix $\mathbf{C}$ is characterized by[20]

$$\mathbf{C}=\mathbf{C}^m=(\mathbf{C}^\dagger)^{-1}. \tag{7'}$$

It is clear that the set of all generalized quasiparticle canonical transformations forms a group.

Later we shall use the dyadic particle and quasiparticle density matrix operators[21] $\mathbf{aa}^\dagger$ and $\boldsymbol{\alpha\alpha}^\dagger$ as well as their thermodynamic expectation values in the slightly modified forms

$$\mathbf{G}=\mathbf{G}^\dagger=-\mathbf{G}^m=\langle\mathbf{aa}^\dagger-\tfrac{1}{2}\mathbf{I}\rangle=\begin{pmatrix}-G^* & F\\ -F^* & G\end{pmatrix}, \tag{8}$$

$$\boldsymbol{\Gamma}=\boldsymbol{\Gamma}^\dagger=-\boldsymbol{\Gamma}^m=\langle\boldsymbol{\alpha\alpha}^\dagger-\tfrac{1}{2}\mathbf{I}\rangle=\begin{pmatrix}-\Gamma^* & \Phi\\ -\Phi^* & \Gamma\end{pmatrix}, \tag{9}$$

such that

$$\begin{matrix}\langle aa^\dagger\rangle=G+\tfrac{1}{2}I=\bar{G}\\ \langle\alpha\alpha^\dagger\rangle=\Gamma+\tfrac{1}{2}I=\bar{\Gamma}\end{matrix},\quad \mathsf{G}=\begin{pmatrix}-\bar{G}^* & F\\ -F^* & \bar{G}\end{pmatrix}. \tag{9'}$$

It is seen immediately from the definitions that $\mathbf{G}$ and $\boldsymbol{\Gamma}$ are Hermitian and m antisymmetric. Using Eqs. (6′), (8), and (9), one immediately verifies the relation

$$\mathbf{G}=\mathbf{C}\boldsymbol{\Gamma}\mathbf{C}^\dagger. \tag{10}$$

The ansatz for the statistical operator W corresponding to independent quasiparticles is

$$\begin{aligned}W&=\textstyle\prod_f[\bar{\Gamma}_f\alpha_f{}^\dagger\alpha_f+(1-\bar{\Gamma}_f)\alpha_f\alpha_f{}^\dagger]\\ &=\exp[\mathrm{Tr}(\boldsymbol{\alpha\alpha}^\dagger\ln(\boldsymbol{\Gamma}+\tfrac{1}{2}\mathbf{I}))],\end{aligned} \tag{11}$$

and it follows, in complete analogy to the free-particle case, that $\boldsymbol{\Gamma}$ is diagonal. The entropy S is computed as the free-quasiparticle entropy according to

$$\begin{aligned}-TS&=\beta^{-1}\langle\ln W\rangle\\ &=(2\beta)^{-1}\,\mathrm{Tr}[(\tfrac{1}{2}\mathbf{I}+\boldsymbol{\Gamma})\ln(\tfrac{1}{2}\mathbf{I}+\boldsymbol{\Gamma})\\ &\qquad+(\tfrac{1}{2}\mathbf{I}-\boldsymbol{\Gamma})\ln(\tfrac{1}{2}\mathbf{I}-\boldsymbol{\Gamma})].\end{aligned} \tag{12}$$

As derived in Appendix II the ansatz (11) implies the complete factorization of the two-particle correlation function according to

$$\langle a_{f^1}{}^\dagger a_{f^2}{}^\dagger a_{f^3}a_{f^4}\rangle=\bar{G}_{f^1f^4}\bar{G}_{f^2f^3}-\bar{G}_{f^1f^3}\bar{G}_{f^2f^4}+F_{f^1f^2}{}^\dagger F_{f^3f^4}. \tag{13}$$

The three terms appearing here correspond, loosely speaking, to the approximations associated with the names Harteee, Fock, and BCS, respectively.

Now $\langle H\rangle$ can be expressed as

$$\langle H\rangle=\tfrac{1}{2}\,\mathrm{Tr}[(E+K)\bar{G}+F^\dagger D], \tag{14}$$

with the following definitions[22] for the self-consistent energy E and the pair potential D:

$$Q_{f^1f^2f^3f^4}=P_{f^1f^3f^4f^2}-P_{f^1f^3f^2f^4}, \tag{15}$$

$$E=K+Q\cdot\bar{G},\quad D=P\cdot F. \tag{16}$$

In complete analogy with the density matrices $\mathbf{G}$ and $\boldsymbol{\Gamma}$ one defines the energy matrices $\mathbf{E}$ and $\boldsymbol{\varepsilon}$ by

$$\mathbf{E}=\mathbf{E}^\dagger=-\mathbf{E}^m=\begin{pmatrix}-E^* & D\\ -D^* & E\end{pmatrix} \tag{17}$$

and

$$\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}^\dagger=-\boldsymbol{\varepsilon}^m=\begin{pmatrix}-\mathcal{E}^* & \Delta\\ -\Delta^* & \mathcal{E}\end{pmatrix} \tag{17'}$$

such that

$$\mathbf{E}=\mathbf{C}\boldsymbol{\varepsilon}\mathbf{C}^\dagger. \tag{17''}$$

We shall finally abbreviate Eq. (16) according to Eq. (17) as

$$\mathbf{E}=\mathbf{E}[\mathsf{G}]=\mathbf{K}+\mathbf{P}\cdot\mathsf{G}. \tag{16'}$$

With this notation the grand potential Ω can be written by adding Eqs. (12) and (14)

$$\begin{aligned}\Omega=\tfrac{1}{4}\,\mathrm{Tr}\{(\mathbf{E}+\mathbf{K})\mathsf{G}+2\beta^{-1}[(\tfrac{1}{2}\mathbf{I}+\boldsymbol{\Gamma})\ln(\tfrac{1}{2}\mathbf{I}+\boldsymbol{\Gamma})\\ +(\tfrac{1}{2}\mathbf{I}-\boldsymbol{\Gamma})\ln(\tfrac{1}{2}\mathbf{I}-\boldsymbol{\Gamma})]\},\end{aligned} \tag{18}$$

from which it is seen immediately that Ω is real.

4. MINIMIZATION OF THE GRAND POTENTIAL

We can now make use of the minimum principle for the grand potential.[23] We allow variations of $\boldsymbol{\Gamma}$ and $\mathbf{C}$ with the restriction, however, that $\mathbf{C}$ be unitary and m symmetric, and set $\delta\Omega=0$. It is easy to see that $\delta\langle H\rangle=\tfrac{1}{2}\,\mathrm{Tr}[\mathbf{E}\delta\mathbf{G}]$, where, according to Eq. (10)

$$\delta\mathbf{G}=\delta\mathbf{C}\boldsymbol{\Gamma}\mathbf{C}^\dagger+\mathbf{C}\delta\boldsymbol{\Gamma}\mathbf{C}^\dagger+\mathbf{C}\boldsymbol{\Gamma}\delta\mathbf{C}^\dagger. \tag{19}$$

Let us first vary $\boldsymbol{\Gamma}$, keeping $\mathbf{C}$ fixed. We obtain

$$\delta\Omega=\tfrac{1}{2}\,\mathrm{Tr}[\mathbf{C}^\dagger\mathbf{E}\mathbf{C}\delta\boldsymbol{\Gamma}]+(2\beta)^{-1}\,\mathrm{Tr}[\ln(\boldsymbol{\Gamma}/\mathbf{I}-\boldsymbol{\Gamma})\delta\boldsymbol{\Gamma}]$$

and therefore

$$\boldsymbol{\Gamma}=-\tfrac{1}{2}\tanh(\tfrac{1}{2}\beta\boldsymbol{\varepsilon}_d), \tag{20}$$

where the matrix $\boldsymbol{\varepsilon}$ is defined in (17′) and the subscript d means "diagonal part."

Varying $\mathbf{C}$ with $\boldsymbol{\Gamma}$ fixed, we have

$$\delta\Omega=\tfrac{1}{2}\,\mathrm{Tr}[\mathbf{E}(\delta\mathbf{C}\boldsymbol{\Gamma}\mathbf{C}^\dagger+\mathbf{C}\boldsymbol{\Gamma}\delta\mathbf{C}^\dagger)].$$

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[20] To characterize the special structure of $\mathbf{C}$ and similar matrices it is expedient to define the m adjoint by

$$\mathbf{A}^m=\begin{pmatrix}a_1 & a_2\\ a_3 & a_4\end{pmatrix}^m=\begin{pmatrix}a_4{}^* & a_3{}^*\\ a_2{}^* & a_1{}^*\end{pmatrix}$$

and

$$\mathbf{a}^m=\begin{pmatrix}a_1\\ a_2\end{pmatrix}^m=\begin{pmatrix}a_2{}^*\\ a_1{}^*\end{pmatrix}.$$

[21] In general the symbols will be defined such that Latin symbols refer to particles and Greek symbols to quasiparticles.

[22] It is useful to introduce a special associative product (dot) operation by

$$A_{f^1f^2}B_{f^2f^1}=A\cdot B=\mathrm{Tr}(AB),$$
$$A_{f^1f^2f^3f^4'}B_{f^4f^3}=(A'\cdot B)_{f^1f^2},$$
$$A_{f^1f^2}B_{f^2f^1f^3f^4'}=(A\cdot B')_{f^3f^4}.$$

[23] See, e.g., H. Koppe, *Variations-methoden in der Quantenstatistik, W. Heisenberg und die Physik unserer Zeit* (Vieweg, Braunschweig, 1961), p. 182.

Because $\mathbf{C}$ is unitary, we find

$$\delta\Omega = \tfrac{1}{2}\,\mathrm{Tr}[(\mathbf{\Gamma C^\dagger E C} - \mathbf{C^\dagger E C \Gamma})\mathbf{C}^\dagger\delta\mathbf{C}].$$

Taking properly into account the symmetry of all matrices involved, we conclude

$$[\mathbf{\Gamma},\boldsymbol{\mathcal{E}}]=0 \quad \text{or also} \quad [\mathbf{G},\mathbf{E}]=0. \tag{21}$$

It is important at this point to note a certain ambiguity in the matrix $\mathbf{C}$ which was essentially already observed for the zero temperature case by Valatin.[5] If one considers a different set of quasiparticles $\boldsymbol{\alpha}' = \mathbf{D}^{-1}\boldsymbol{\alpha}$ with a unitary m-symmetric $\mathbf{D}$ which commutes with $\mathbf{\Gamma}$, then it follows that $W'=W$, or that the statistical operator is unchanged. All physical quantities remain unchanged under this transformation. In particular, we have

$$\mathbf{G} = \mathbf{C\Gamma C}^\dagger = (\mathbf{CD})\mathbf{\Gamma}(\mathbf{CD})^\dagger.$$

On the other hand, it is well known that an arbitrary Hermitian matrix $\mathbf{M}$ can always be diagonalized by a unitary transformation $\mathbf{U}$. If, furthermore, $\mathbf{M}$ is m antisymmetric, $\mathbf{U}$ can be chosen m symmetric; from which it follows that $\mathbf{M}_{\text{diag}}$ is also m antisymmetric. The proof is very simple. If $\mathbf{g}$ is an eigenvector of $\mathbf{M}$ with (real) eigenvalue λ then $\mathbf{g}^m$ is an eigenvector of $\mathbf{M}$ with eigenvalue $-\lambda$. If now $\mathbf{U}$ is defined as $(\mathbf{g}_1{}^m\mathbf{g}_2{}^m\cdots\mathbf{g}_1\mathbf{g}_2\cdots)$ then the m symmetry of $\mathbf{U}$ is obvious.

From the Hermitian nature of $\mathbf{G}$ and $\mathbf{E}$ and Eq. (21) one concludes that both $\mathbf{G}$ and $\mathbf{E}$ can be diagonalized by the same unitary transformation. Since $\mathbf{C}$ already diagonalizes $\mathbf{G}$ into $\mathbf{\Gamma}$ but is only determined up to a $\mathbf{D}$ transformation and furthermore $\mathbf{E}$ can always be diagonalized by an m symmetric unitary transformation, it it possible to find an m-symmetric unitary $\mathbf{C}$ matrix which diagonalizes $\mathbf{G}$ and $\mathbf{E}$ simultaneously into $\mathbf{\Gamma}$ and $\boldsymbol{\mathcal{E}}$. Then Eq. (21) is automatically satisfied. Since $\boldsymbol{\mathcal{E}}$ is diagonal, we can rewrite Eq. (20) by dropping the subscript $_d$.

Our problem can now be rephrased as follows. Find $\mathbf{C}$ and $\mathbf{\Gamma}$ satisfying Eqs. (7′), (10), (17″), (16′), and (20). We shall refer to this problem summarily as the "$\mathbf{C}$ problem."

5. THE CASE OF ZERO EXTERNAL FIELD

Let us first consider the familiar case of a vanishing magnetic vector potential. This case can be treated simply in the momentum representation if one introduces a "barring" operation for coordinates by

$$\bar{f} = (\bar{\mathbf{k}},\bar{\sigma}) = (-\mathbf{k},\,-\sigma). \tag{22}$$

With the spin-space matrices

$$S_{\sigma\sigma'}{}^{(1)} = \delta_{\sigma\sigma'} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad S_{\sigma\sigma'}{}^{(2)} = \sigma\delta_{\bar{\sigma}\sigma'} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \tag{22′}$$

and the momentum-space matrices

$$I_{\mathbf{kk}'}{}^{(1)} = \delta(\mathbf{k}-\mathbf{k}'),\quad I_{\mathbf{kk}'}{}^{(2)} = \delta(\bar{\mathbf{k}}-\mathbf{k}'), \tag{22″}$$

the solution of the $\mathbf{C}$ problem is of the form

$$\left.\begin{matrix} u=u_d\\ G=G_d\\ \Gamma=\Gamma_d\\ E=E_d\\ \mathcal{E}=\mathcal{E}_d \end{matrix}\right\}\cdot I^{(1)}S^{(1)},\quad \left.\begin{matrix} v=v_d\\ F=F_d\\ \phi=\phi_d\\ D=D_d\\ \Delta=\Delta_d \end{matrix}\right\}\cdot I^{(2)}S^{(2)}, \tag{23}$$

where the subscript d again indicates diagonal matrices and is dropped below for simplicity. All the diagonal matrices can be taken as even under the "barring" operation and real.

Since $\mathbf{\Gamma}$ and $\boldsymbol{\mathcal{E}}$ must be real and diagonal, it follows that ϕ and Δ are equal to zero. Equation (7′) implies that $u^2+v^2=1$ which can be parametrized by $u=\cos\frac{1}{2}\gamma$ and $v=\sin\frac{1}{2}\gamma$ while Eq. (20) demands that $\Gamma = -\frac{1}{2}\tanh(\frac{1}{2}\beta\mathcal{E})$. The relations (10) and (17″) then lead to

$$\begin{pmatrix}\Gamma\\ 0\end{pmatrix} = R\begin{pmatrix}G\\ F\end{pmatrix},\quad \begin{pmatrix}\mathcal{E}\\ 0\end{pmatrix} = R\begin{pmatrix}E\\ D\end{pmatrix}, \tag{24}$$

with the rotation matrix

$$R = \begin{pmatrix}u^2-v^2 & 2uv\\ -2uv & u^2-v^2\end{pmatrix} = \begin{pmatrix}\cos\gamma & \sin\gamma\\ -\sin\gamma & \cos\gamma\end{pmatrix}.$$

One derives immediately from Eq. (24) that

$$\tilde{R}\begin{pmatrix}\Gamma\mathcal{E}\\ 0\end{pmatrix} = \begin{pmatrix}\Gamma E\\ \Gamma D\end{pmatrix} = \begin{pmatrix}\mathcal{E}G\\ \mathcal{E}F\end{pmatrix},$$

as well as $\mathcal{E}^2 = E^2+D^2$ and $\Gamma^2 = G^2+F^2$. Eliminating the spin dependence from Eq. (16′) and substituting the last results lead to the final two coupled integral equations for E and D

$$\begin{aligned} E_{\mathbf{k}} &= \mathbf{k}^2 - \mu + \textstyle\sum_{\mathbf{k}'}(P_{\mathbf{k}-\mathbf{k}'} - 2P_0)[\mathcal{E}_{\mathbf{k}'}{}^{-1}\Gamma_{\mathbf{k}'}E_{\mathbf{k}'} + \frac{1}{2}],\\ D_{\mathbf{k}} &= -\textstyle\sum_{\mathbf{k}'} P_{\mathbf{k}-\mathbf{k}'}\mathcal{E}_{\mathbf{k}'}{}^{-1}\Gamma_{\mathbf{k}'}D_{\mathbf{k}'}, \end{aligned} \tag{25}$$

together with Eq. (20) and $\mathcal{E}_{\mathbf{k}}{}^2 = E_{\mathbf{k}}{}^2 + D_{\mathbf{k}}{}^2$.

The quantity $\mathcal{E}$ is the quasiparticle excitation energy. The system of equations (25) has the general structure well known from the work of BCS and reduces to the single BCS gap equation if one neglects the Hartree-Fock contributions (containing $P_{\mathbf{k}-\mathbf{k}'} - 2P_0$) in the equation for $E_{\mathbf{k}}$ and uses a factorizable (nongauge-invariant) interaction in the gap equation. They have a trivial solution $D\equiv 0$ and may admit a nontrivial solution if the potential is sufficiently attractive. In their present general form, they could be studied asymptotically by a slight generalization of a method employed by Zubarev[24] for the gap equation. We shall

[24] D. N. Zubarev, Usp. Fiz. Nauk **71**, 71 (1960) [translation: Soviet Phys.—Usp. **3**, 320 (1960)].

not carry out this study here. For our later work it is sufficient to assume that a nontrivial solution of (25) has been found somehow. The expressions

$$u^2=\{\cos[\tfrac{1}{2}\arctan(E^{-1}D)]\}^2=\tfrac{1}{2}(1+\mathcal{E}^{-1}E),$$
$$v^2=\{\sin[\tfrac{1}{2}\arctan(E^{-1}D)]\}^2=\tfrac{1}{2}(1-\mathcal{E}^{-1}E)$$

then give the solution of the C problem.

The current calculated from this solution vanishes trivially as is expected. One can also easily derive for the grand potential, according to Eq. (18)

$$\Omega=\sum_{\mathbf{k}}\{(K_{\mathbf{k}}-E_{\mathbf{k}})(\tfrac{1}{2}+\mathcal{E}_{\mathbf{k}}^{-1}\Gamma_{\mathbf{k}}E_{\mathbf{k}})+E_{\mathbf{k}}-\mathcal{E}_{\mathbf{k}}-\mathcal{E}_{\mathbf{k}}^{-1}\Gamma_{\mathbf{k}}D_{\mathbf{k}}^2-2\beta^{-1}\ln[1+\exp(-\beta\mathcal{E}_{\mathbf{k}})]\}. \quad (18')$$

6. RESPONSE OF THE SYSTEM TO A SMALL PERTURBATION

Let the "kinetic energy" part of the Hamiltonian be $\mathbf{K}=\mathbf{K}^{(0)}+\mathbf{K}^{(1)}$ where $\mathbf{K}^{(1)}$ is small of first order and assume that the solution for $\mathbf{K}^{(0)}$ is known. One is lead to consider the perturbation ansatz $\mathbf{C}=\mathbf{C}^{(0)}(\mathbf{I}+\mathbf{B})$ where $\mathbf{B}$ is taken to be of first order of smallness. The matrix $\mathbf{I}+\mathbf{B}$ must be unitary and therefore $\mathbf{B}$ must be anti-Hermitian. Denoting first order differences of the kind $\mathbf{M}(\mathbf{K})-\mathbf{M}(\mathbf{K}^{(0)})$ by $\mathbf{M}^{(1)}$ one finds that $\mathbf{\Gamma}^{(1)}$ and $\boldsymbol{\varepsilon}^{(1)}$ are diagonal and related by the equation following from (20)

$$\mathbf{\Gamma}^{(1)}=-\tfrac{1}{4}\beta[\cosh(\tfrac{1}{2}\beta\boldsymbol{\varepsilon})]^{-2}\boldsymbol{\varepsilon}^{(1)}. \quad (26)$$

Furthermore, it follows from Eqs. (10) and (17″) that (writing $\mathbf{C}$ again for $\mathbf{C}^{(0)}$)

$$\mathbf{G}^{(1)}=\mathbf{C}^{(1)}\mathbf{\Gamma}\mathbf{C}^{\dagger}+\mathbf{C}\mathbf{\Gamma}^{(1)}\mathbf{C}^{\dagger}+\mathbf{C}\mathbf{\Gamma}\mathbf{C}^{\dagger(1)}=\mathbf{G}^{(1)\dagger},$$
$$\mathbf{E}^{(1)}=\mathbf{C}^{(1)}\boldsymbol{\varepsilon}\mathbf{C}^{\dagger}+\mathbf{C}\boldsymbol{\varepsilon}^{(1)}\mathbf{C}^{\dagger}+\mathbf{C}\boldsymbol{\varepsilon}\mathbf{C}^{\dagger(1)}=\mathbf{E}^{(1)\dagger},$$

or, since $\mathbf{C}^{(1)}=\mathbf{C}\cdot\mathbf{B}$, that $\mathbf{G}^{(1)}=\mathbf{C}\mathbf{\Gamma}'\mathbf{C}^{\dagger}$ and $\mathbf{E}^{(1)}=\mathbf{C}\boldsymbol{\varepsilon}'\mathbf{C}^{\dagger}$, with

$$\mathbf{\Gamma}'=\mathbf{\Gamma}'^{\dagger}=-\mathbf{\Gamma}'^{m}=\mathbf{\Gamma}^{(1)}+[\mathbf{B},\mathbf{\Gamma}],$$
$$\boldsymbol{\varepsilon}'=\boldsymbol{\varepsilon}'^{\dagger}=-\boldsymbol{\varepsilon}'^{m}=\boldsymbol{\varepsilon}^{(1)}+[\mathbf{B},\boldsymbol{\varepsilon}].$$

Finally, Eq. (16′) implies the relation

$$\mathbf{E}^{(1)}=\mathbf{K}^{(1)}+\mathbf{P}\cdot\mathbf{G}^{(1)}. \quad (27)$$

It is interesting to observe that the matrix $\mathbf{B}$ can be completely eliminated from the above equations. Observe that $\mathbf{\Gamma}$, $\mathbf{\Gamma}^{(1)}$, $\boldsymbol{\varepsilon}$, and $\boldsymbol{\varepsilon}^{(1)}$ are all diagonal and therefore commute. Then it follows with the Jacobi identity that

$$[\mathbf{\Gamma}',\boldsymbol{\varepsilon}]=[[\mathbf{B},\mathbf{\Gamma}],\boldsymbol{\varepsilon}]=[[\mathbf{B},\boldsymbol{\varepsilon}],\mathbf{\Gamma}]=[\boldsymbol{\varepsilon}',\mathbf{\Gamma}]. \quad (26')$$

Since the diagonal $\boldsymbol{\varepsilon}$ and $\mathbf{\Gamma}$ are given, this relation determines the nondiagonal matrix elements of $\boldsymbol{\varepsilon}'$ in terms of those of $\mathbf{\Gamma}'$. On the other hand $\boldsymbol{\varepsilon}_d'=\boldsymbol{\varepsilon}^{(1)}$ and $\mathbf{\Gamma}_d'=\mathbf{\Gamma}^{(1)}$ so that from Eq. (26)

$$\boldsymbol{\varepsilon}_d'=-4\beta^{-1}[\cosh(\tfrac{1}{2}\beta\boldsymbol{\varepsilon})]^2\mathbf{\Gamma}_d'.$$

Equation (27) can be written in terms of $\boldsymbol{\varepsilon}'$ and $\mathbf{\Gamma}'$ and becomes

$$\boldsymbol{\varepsilon}'=\mathbf{C}^{\dagger}\mathbf{K}^{(1)}\mathbf{C}+\mathbf{C}^{\dagger}[\mathbf{P}\cdot(\mathbf{C}\mathbf{\Gamma}'\mathbf{C}^{\dagger})]\mathbf{C}. \quad (27')$$

Finally, the linearized problem, with $\mathbf{B}$ eliminated from it, can be stated as follows: Find $\mathbf{\Gamma}'$ and $\boldsymbol{\varepsilon}'$ satisfying Eqs. (26), (26′), and (27′). The matrices $\mathbf{C}$, $\mathbf{\Gamma}$, $\boldsymbol{\varepsilon}$, $\mathbf{P}$, and $\mathbf{K}^{(1)}$ are to be considered known.

The Meissner effect is then established by examining the relation between the current and the vector potential. From Eq. (5) one has to first order in $\mathbf{A}$

$$\langle\mathbf{J}_{\mathbf{q}}\rangle=eV^{-1}\sum_{\mathbf{k}\sigma}(2\mathbf{k}+\mathbf{q})G_{\mathbf{k}\sigma\mathbf{k}+\mathbf{q}\sigma}^{(1)}-2eNV^{-1}\mathbf{A}(\mathbf{q}), \quad (28)$$

where $G^{(1)}$ is given in terms of Γ'. One can now remove all the spin dependence by factorization of the spin matrices appearing in the formulas. The matrix $\mathbf{C}$ is known as

$$\mathbf{C}=\begin{pmatrix} u_dI^{(1)}S^{(1)} & v_dI^{(2)}S^{(2)} \\ v_dI^{(2)}S^{(2)} & u_dI^{(1)}S^{(1)} \end{pmatrix}, \quad (29)$$

while

$$\mathbf{K}^{(1)}=\begin{pmatrix} -K^{(1)*} & 0 \\ 0 & K^{(1)} \end{pmatrix} \quad (29')$$

is with the matrix elements from Eq. (2)

$$K_{ff'}^{(1)}=-\delta_{\sigma\sigma'}(\mathbf{k}+\mathbf{k}')\cdot\mathbf{A}(\mathbf{k}'-\mathbf{k}).$$

It is convenient to denote matrices like u, v which have their spin dependence factored away by the same symbols as before, if there is no risk of confusion. We also set

$$\mathbf{\Gamma}'=\begin{pmatrix} -S^{(1)}\Gamma'^* & S^{(2)}\Phi' \\ -S^{(2)}\Phi'^* & S^{(1)}\Gamma' \end{pmatrix}, \quad \Gamma'^{\dagger}=\Gamma', \quad \tilde{\Phi}'=\Phi'$$

and

$$\boldsymbol{\varepsilon}'=\begin{pmatrix} -S^{(1)}\mathcal{E}'^* & S^{(2)}\Delta' \\ -S^{(2)}\Delta'^* & S^{(1)}\mathcal{E}' \end{pmatrix}, \quad \mathcal{E}'^{\dagger}=\mathcal{E}', \quad \tilde{\Delta}'=\Delta'.$$

Making use of these expressions in Eq. (27) one obtains easily

$$G^{(1)}=-v\Gamma'^*v-u\Phi'^*v-v\Phi'u+u\Gamma'u,$$
$$F^{(1)}=u\Gamma'^*v-v\Phi'^*v+u\Phi'u+v\Gamma'u.$$

On the other hand, substituting into Eq. (27′), one also obtains, after some calculations, equations not containing any more spin variables.

$$\mathcal{E}'+[vQ\cdot G^{(1)*}v+vP\cdot F^{(1)}u+uP\cdot F^{(1)*}v-uQ\cdot G^{(1)}u]=uK^{(1)}u-vK^{(1)*}v,$$

$$\Delta'^*+[uQ\cdot G^{(1)}v+uP\cdot F^{(1)*}u-vP\cdot F^{(1)}v+vQ\cdot G^{(1)*}u]=-uK^{(1)}v-vK^{(1)*}u.$$

There are, of course, two more equations obtained from these by complex conjugation.

The spin separated equations we have obtained contain the unknown quantities together with their complex conjugates. It is possible to extricate the quantities themselves by taking suitable linear combinations of the equations. For any quantity $M_{\mathbf{kk}'}$ define

$$M_{\mathbf{kk}'}{}^{\tau}=\tfrac{1}{2}(M_{\mathbf{kk}'}+\tau M_{-\mathbf{k}-\mathbf{k}'}{}^{*}),\quad \tau=\pm 1,\qquad (30)$$

so that $M_{\mathbf{kk}'}=\sum_\tau M_{\mathbf{kk}'}{}^{\tau}$. Due to the linearity of the equations and the fact that they have real coefficients, we can take the appropriate τ combination. Using the abbreviations

$$\begin{aligned} g_{\mathbf{kk}'}{}^{\tau}&=u_{\mathbf{k}}u_{\mathbf{k}'}-\tau v_{\mathbf{k}}v_{\mathbf{k}'},\\ f_{\mathbf{kk}'}{}^{\tau}&=u_{\mathbf{k}}v_{\mathbf{k}'}+\tau v_{\mathbf{k}}u_{\mathbf{k}'},\\ Q_{\mathbf{kk}'\mathbf{mm}'}&=\delta(\mathbf{k}'-\mathbf{k}+\mathbf{m}'-\mathbf{m})Q_{\mathbf{km}}{}^{\mathbf{q}},\\ P_{\mathbf{k}-\mathbf{k}'\mathbf{m}-\mathbf{m}'}&=\delta(\mathbf{k}'-\mathbf{k}+\mathbf{m}'-\mathbf{m})P_{\mathbf{km}}{}^{\mathbf{q}}, \end{aligned}\qquad (31)$$

one obtains immediately that[25]

$$G_{\mathbf{kk}'}{}^{(1)}=g_{\mathbf{kk}'}\Gamma_{\mathbf{kk}'}{}'-f_{\mathbf{kk}'}\Phi_{\mathbf{k}-\mathbf{k}'}{}',\qquad (31')$$

$$F_{\mathbf{k}-\mathbf{k}'}{}^{(1)}=f_{\mathbf{kk}'}\Gamma_{\mathbf{kk}'}{}'+g_{\mathbf{kk}'}\Phi_{\mathbf{k}-\mathbf{k}'}{}',$$

$$\mathcal{E}_{\mathbf{kk}'}{}'+[-g_{\mathbf{kk}'}Q_{\mathbf{kk}'\mathbf{m}'\mathbf{m}}g_{\mathbf{mm}'}+f_{\mathbf{kk}'}P_{\mathbf{k}-\mathbf{k}'\mathbf{m}-\mathbf{m}'}f_{\mathbf{mm}'}]\Gamma_{\mathbf{mm}'}{}' +[g_{\mathbf{kk}'}Q_{\mathbf{kk}'\mathbf{m}'\mathbf{m}}f_{\mathbf{mm}'}+f_{\mathbf{kk}'}P_{\mathbf{k}-\mathbf{k}'\mathbf{m}-\mathbf{m}'}g_{\mathbf{mm}'}]\Phi_{\mathbf{m}-\mathbf{m}'}{}' =g_{\mathbf{kk}'}K_{\mathbf{kk}'}{}^{(1)},\qquad (31'')$$

$$\Delta_{\mathbf{k}-\mathbf{k}'}{}'+[f_{\mathbf{kk}'}Q_{\mathbf{kk}'\mathbf{m}'\mathbf{m}}g_{\mathbf{mm}'}+g_{\mathbf{kk}'}P_{\mathbf{k}-\mathbf{k}'\mathbf{m}-\mathbf{m}'}f_{\mathbf{mm}'}]\Gamma_{\mathbf{mm}'}{}' +[-f_{\mathbf{kk}'}Q_{\mathbf{kk}'\mathbf{m}'\mathbf{m}}f_{\mathbf{mm}'}+g_{\mathbf{kk}'}P_{\mathbf{k},-\mathbf{k}'\mathbf{m}-\mathbf{m}'}g_{\mathbf{mm}'}]\Phi_{\mathbf{m}-\mathbf{m}'}{}' =-f_{\mathbf{kk}'}K_{\mathbf{kk}'}{}^{(1)}.$$

From Eq. (31), it follows that $\mathbf{k}'-\mathbf{k}=\mathbf{m}'-\mathbf{m}=\mathbf{q}$ enters into Eq. (31″) only parametrically, and that the summation over $\mathbf{m}'$ can be performed due to the delta functions in Eq. (31). The resulting equations can be written concisely for each fixed value of $\mathbf{q}$ in vector-matrix notation by introducing the vectors $\Gamma'=(\Gamma_{\mathbf{k}}'(\mathbf{q}))=(\Gamma_{\mathbf{kk}+\mathbf{q}}{}')$ and $\Phi'=(\Phi_{\mathbf{k}}'(\mathbf{q}))=(\Phi_{\mathbf{k}-\mathbf{k}-\mathbf{q}}{}')$; (similarly for $\mathcal{E}'$, Δ'), the diagonal matrices $g=(I_{\mathbf{kl}}{}^{(1)}g_{\mathbf{kk}+\mathbf{q}})$ (similarly for f) and the matrices $P_{\mathbf{kl}}=P_{\mathbf{kl}}{}^{\mathbf{q}}$ (similarly for Q).

It has been pointed out above that $\boldsymbol{\mathcal{E}}'$ is determined by $\boldsymbol{\Gamma}'$ from Eqs. (26′). These equations, after eliminating the spin matrices give rise to

$$\mathcal{E}_{\mathbf{k}}'(\mathbf{q})(\Gamma_{\mathbf{k}}-\Gamma_{\mathbf{k}+\mathbf{q}})=(\mathcal{E}_{\mathbf{k}}-\mathcal{E}_{\mathbf{k}+\mathbf{q}})\Gamma_{\mathbf{k}}'(\mathbf{q}),$$
$$\Delta_{\mathbf{k}}'(\mathbf{q})(\Gamma_{\mathbf{k}}+\Gamma_{\mathbf{k}+\mathbf{q}})=(\mathcal{E}_{\mathbf{k}}+\mathcal{E}_{\mathbf{k}+\mathbf{q}})\Phi_{\mathbf{k}}'(\mathbf{q}),$$

together with the diagonal relation $\mathcal{E}_{\mathbf{k}}'(\mathbf{o})=-4\beta^{-1}\times\cosh^2(\tfrac{1}{2}\beta\mathcal{E}_{\mathbf{k}})\cdot\Gamma_{\mathbf{k}}'(\mathbf{o})$. We also observe that the relation for $\mathbf{q}=\mathbf{o}$ is the limit of the relation for $\mathbf{q}\neq\mathbf{o}$ as $\mathbf{q}\to\mathbf{o}$. Setting

$$\Gamma_{\mathbf{k}}{}^{\pm}(\mathbf{q})=\Gamma_{\mathbf{k}}\pm\Gamma_{\mathbf{k}+\mathbf{q}},\quad \mathcal{E}_{\mathbf{k}}{}^{\pm}(\mathbf{q})=\mathcal{E}_{\mathbf{k}}\pm\mathcal{E}_{\mathbf{k}+\mathbf{q}},\qquad (32)$$
$$\rho^{\pm}=(\Gamma^{\pm})^{-1}\mathcal{E}^{\pm},$$

we can write this in vector form

$$\mathcal{E}'=\rho^{-}\Gamma'\quad\text{and}\quad \Delta'=\rho^{+}\Phi'\qquad (32')$$

which can be taken as valid for all values of $\mathbf{q}$.

Making use of Eq. (32′), we can eliminate $\mathcal{E}'$ and Δ' from Eq. (31″). The resulting equations, written in compact vector notation appear as

$$\mathbf{L}\begin{pmatrix}\Gamma'\\ \Phi'\end{pmatrix}=\begin{pmatrix}gK^{(1)}\\ -fK^{(1)}\end{pmatrix},\qquad (33)$$

where $\mathbf{L}$ is the symmetric operator

$$\mathbf{L}=\begin{pmatrix}\rho^{-}+fPf-gQg & fPg+gQf\\ gPf+fQg & \rho^{+}+gPg-fQf\end{pmatrix}.\qquad (33')$$

Following up the steps taken one finds for $G^{(1)}$

$$G^{(1)}=S^{(1)}\textstyle\sum_\tau(g\Gamma'-f\Phi')$$

and from Eq. (28) with $\Lambda^{-1}=2e^2NV^{-1}$

$$\langle\mathbf{J}_{\mathbf{q}}\rangle=(e\Lambda)^{-1}[N^{-1}\textstyle\sum_{\mathbf{k}\tau}(2\mathbf{k}+\mathbf{q})\times(g_{\mathbf{k}}\Gamma_{\mathbf{k}}'-f_{\mathbf{k}}\Phi_{\mathbf{k}}')-\mathbf{A}(\mathbf{q})].\qquad (33'')$$

One should observe at this point that with our choice of the perturbation (magnetic vector potential) $K^{(1)\tau}$ vanishes for $\tau=+1$. Therefore, Eq. (33) has a vanishing inhomogeneous term for $\tau=1$ and, since the linear operator is nonsingular for $\tau=1$, it is in fact possible to conclude that the unknown quantities Γ' and Φ' vanish for $\tau=1$. Only the combination for $\tau=-1$ survives and has to be found. Therefore we shall assume in the following that $\tau=-1$. If we had chosen a different perturbation, the situation might have been different. For instance, if $K^{(1)}$ corresponds to a weak external *electric* potential, then it is easy to see that $K^{(1)\tau}$ vanishes for $\tau=-1$ and one has to study in detail the equations for the value $\tau=+1$. In the case of a general perturbation, both contributions would be present.

7. DERIVATION OF THE LONDON FORMULA

We first consider the behavior of our linearized equations (33) under gauge transformations. An infinitesimal gauge transformation can be written in momentum space as $\mathbf{A}(\mathbf{k})\to\mathbf{A}(\mathbf{k})+\mathbf{k}\cdot\eta_{\mathbf{k}}$ and $a_{\mathbf{k}}\to a_{\mathbf{k}}+\sum_{\mathbf{k}'}\eta_{\mathbf{k}-\mathbf{k}'}a_{\mathbf{k}'}$. The corresponding first-order change in K is

$$K_{ff'}{}^{(1)}=-\delta_{\sigma\sigma'}(\mathbf{k}'^2-\mathbf{k}^2)\eta_{\mathbf{k}'-\mathbf{k}}\qquad (34)$$

and the first-order changes in F and G are $F^{(1)}=\eta F-F\eta^*$ and $G^{(1)}=[\eta^*,G]$. We are therefore led to expect that, if one introduces an infinitesimal purely longitudinal perturbation, Eq. (34), the solution of our linearized problem will be given by these expressions, or on comparing terms by

$$\mathbf{B}=\mathbf{C}^{\dagger}\begin{pmatrix}\eta & 0\\ 0 & \eta^*\end{pmatrix}\mathbf{C}.\qquad (35)$$

It is verified in Appendix III that this expected solution is indeed a solution of the linearized problem for a

[25] The superscript τ is dropped and $\mathbf{k}$, $\mathbf{k}'$ are not summed over.

purely longitudinal perturbation and does not give rise to any current. For Γ' and Φ' one obtains

$$\Gamma'=\Gamma^- g^+\eta\,,\quad \Phi'=-\Gamma^+ f^+\eta\,. \tag{36}$$

By factoring η from the corresponding Eqs. (33) and (36), one therefore finds

$$\mathbf{L}\boldsymbol{\gamma}=\mathbf{L}\begin{pmatrix}-\Gamma^- g^+\\ \Gamma^+ f^+\end{pmatrix}=\begin{pmatrix}g(\mathbf{k}'^2-\mathbf{k}^2)\\ -f(\mathbf{k}'^2-\mathbf{k}^2)\end{pmatrix}=\boldsymbol{\zeta}\,. \tag{37}$$

It is now easy to see that the operator $\mathbf{L}$ has a zero eigenvalue, if $\mathbf{q}=\mathbf{o}$ and the gap equation has a nontrivial solution. Indeed for $\mathbf{q}=\mathbf{o}$ the vector $\boldsymbol{\zeta}$ is the nullvector and (37) then implies $\mathbf{L}(\mathbf{q}=\mathbf{o})\boldsymbol{\gamma}(\mathbf{q}=\mathbf{o})=0$. Here the various quantities appearing tend to the following limits for $\mathbf{q}=\mathbf{o}$:

$$\begin{aligned} &g^-=g=1\,,\quad g^+=u^2-v^2=\mathcal{E}^{-1}E\,,\quad \rho^+=\Gamma^{-1}\mathcal{E}\,,\\ &f^-=f=0\,,\quad f^+=2uv=\mathcal{E}^{-1}D\,,\qquad \rho^-=(d\mathcal{E}/d\Gamma)\,. \end{aligned} \tag{38}$$

Written out, Eq. (37) therefore requires $(\Gamma^{-1}\mathcal{E}+P)\mathcal{E}^{-1}\Gamma D=0$ to be fulfilled. However, this equation will have a nontrivial solution $\mathcal{E}^{-1}\Gamma D=F$ exactly if the gap equation $D+PF=0$ has a nontrivial solution. Thus the operator, which is in general nonsingular, tends to a singular operator in the limit of $\mathbf{q}=\mathbf{o}$. It is to be noted however that for spherically symmetric P and Q, the eigenfunction corresponding to the eigenvalue zero will be spherically symmetric also. In particular it will be even under the parity operation $\mathbf{k}\leftrightarrow-\mathbf{k}$. This remark will be important later on.

A general vector potential $\mathbf{A}(\mathbf{q})$ can always be decomposed into transverse and longitudinal parts and, similarly, the perturbation K can be split according to

$$K_{\mathbf{k}}(\mathbf{q})=K_{\mathbf{k}}{}^t+K_{\mathbf{k}}{}^l=-2\mathbf{k}\cdot\mathbf{A}^t(\mathbf{q})-(2\mathbf{k}+\mathbf{q})\cdot\mathbf{A}^l(\mathbf{q})\,. \tag{39}$$

We introduce a parity operation in $\mathbf{k}$-space Π, which reverses the $\mathbf{k}$ vector components in the direction of $\mathbf{A}^t$. Clearly quantities like $\mathbf{k}^2$, $(\mathbf{k}+\mathbf{q})^2$ and $P_{\mathbf{k}-\mathbf{k}'}$ (by assumption) are even under Π while $\mathbf{k}\cdot\mathbf{A}^t$ is odd. Since $\mathbf{L}$ is constructed from quantities depending on $\mathbf{k}^2$, $(\mathbf{k}+\mathbf{q})^2$, and $P_{\mathbf{k}-\mathbf{k}'}$, it follows that $[\mathbf{L},\Pi]=0$. On the other hand, the longitudinal part of the perturbation in Eq. (39) is even under Π while the transverse part is odd under Π. Hence it follows that if $\mathbf{q}\neq\mathbf{o}$ ($\mathbf{L}$ nonsingular), the solution of Eq. (33) will be the sum of two parts, one even under Π, the other odd. As $\mathbf{q}\to\mathbf{o}$ one then expects $\mathbf{L}$ to be nonsingular in the subspace of odd Π parity while it becomes singular, with the eigensolution (37) in the orthogonal subspace of even Π parity. This illustrates the characteristic features of the Meissner effect, i.e., the completely different nature of the response of the system to longitudinal and to transverse vector potential perturbations.

We proceed now to discuss the Buckingham sum rule and the current conservation law. It is expedient to introduce

$$\boldsymbol{\Gamma}_\alpha=\begin{pmatrix}\Gamma_\alpha\\ \Phi_\alpha\end{pmatrix},\quad \mathbf{K}_\alpha=\begin{pmatrix}-g(2k_\alpha+q_\alpha)\\ f(2k_\alpha+q_\alpha)\end{pmatrix}, \tag{40}$$

such that $\mathbf{L}\boldsymbol{\Gamma}_\alpha=\mathbf{K}_\alpha$. Now according to Eqs. (33″) and (40), the expectation value of the current can be written as

$$\langle \mathbf{J}_\alpha(\mathbf{q})\rangle=(e\Lambda)^{-1}[S_{\alpha\beta}(\mathbf{q})-\delta_{\alpha\beta}]A_\beta(\mathbf{q})\,, \tag{41}$$

where with Eq. (40) and the symmetry of $\mathbf{L}$,

$$\begin{aligned} S_{\alpha\beta}(\mathbf{q})&=N^{-1}\textstyle\sum_{\mathbf{k}}(2k_\alpha+q_\alpha)(g^-\Gamma_\beta-f^-\Phi_\beta)\,,\\ &=-N^{-1}\langle\boldsymbol{\Gamma}_\beta|\,\mathbf{L}\,|\boldsymbol{\Gamma}_\alpha\rangle\\ &=S_{\beta\alpha}(\mathbf{q})\,. \end{aligned} \tag{41′}$$

Thus the paramagnetic response tensor $S_{\alpha\beta}$ of Eq. (41′) is seen to be symmetric. Furthermore it is seen with the aid of Eq. (37) that

$$\begin{aligned} q_\alpha S_{\alpha\beta}&=-N^{-1}\langle\boldsymbol{\Gamma}_\beta|\,\mathbf{L}\,|q_\alpha\boldsymbol{\Gamma}_\alpha\rangle\\ &=N^{-1}\langle\boldsymbol{\Gamma}_\beta|\,\mathbf{L}\,|\boldsymbol{\gamma}\rangle=N^{-1}\langle\boldsymbol{\gamma}|\,\mathbf{L}\,|\boldsymbol{\Gamma}_\beta\rangle\\ &=N^{-1}\langle\boldsymbol{\gamma}|\,\mathbf{K}_\beta\rangle=N^{-1}\textstyle\sum_{\mathbf{k}}(2k_\beta+q_\beta)(\Gamma^- g^+ g+\Gamma^+ f^+ f)\\ &=N^{-1}\textstyle\sum_{\mathbf{k}}(2k_\beta+q_\beta)(\bar{G}_{\mathbf{k}}-\bar{G}_{\mathbf{k+q}})=q_\beta\,. \end{aligned} \tag{42}$$

Thus we obtain the *Buckingham sum rule*

$$[S_{\alpha\beta}(\mathbf{q})-\delta_{\alpha\beta}]q_\beta=0\,, \tag{42′}$$

which insures that only the transverse vector potential gives rise to any current. From Eqs. (41) and (41′) it then follows immediately that the current is conserved.

$$q_\alpha\langle J_\alpha(\mathbf{q})\rangle=0.$$

Since the effect of a longitudinal vector potential has been studied explicitly and shown not to give rise to any current, we can assume from now on that the perturbation appearing on the right-hand side of Eq. (33) is purely transverse. In general it will not be possible to invert the operator $\mathbf{L}$ and solve the equation. However, it is of interest to consider the limiting case $\mathbf{q}\to\mathbf{o}$ (London limit) and to study the temperature dependence of the current in this limit. Using Eq. (38), Eqs. (33) become

$$\left(\frac{d\mathcal{E}}{d\Gamma}-Q\right)\Gamma'=K'^t,\quad \left(\frac{\mathcal{E}}{\Gamma}+P\right)\Phi'=0\,. \tag{43}$$

Since Φ' must also have odd Π parity, it must vanish. Hence the current response is given from Eq. (33″) for spherically symmetric Q as

$$\begin{aligned} &\langle\mathbf{J}(\mathbf{q}\to\mathbf{o})\rangle\\ &=(e\Lambda)^{-1}[N^{-1}\textstyle\sum_{\mathbf{k}}2\mathbf{k}\Gamma_{\mathbf{k}}'-\mathbf{A}]\\ &=(e\Lambda)^{-1}\left[N^{-1}\sum_{\mathbf{kk}'}2\mathbf{k}\left(1-\frac{d\Gamma}{d\mathcal{E}}Q\right)^{-1}\frac{d\Gamma}{d\mathcal{E}}(-2\mathbf{k}'\cdot\mathbf{A})-\mathbf{A}\right]\\ &=-2eV^{-1}N_s\mathbf{A}\,, \end{aligned} \tag{44}$$

with

$$N_s=4\sum_{\mathbf{k}\mathbf{k}'} k_3\left(1-\frac{d\Gamma}{d\mathcal{E}}Q\right)^{-1}\frac{d\Gamma}{d\mathcal{E}}k_3' +2\sum_{\mathbf{k}}(\mathcal{E}_{\mathbf{k}}^{-1}\Gamma_{\mathbf{k}}E_{\mathbf{k}}+\tfrac{1}{2}), \quad (45)$$

where k_3 is the component of the momentum along any axis. N_s can be interpreted as the number of superconducting electrons at the given temperature and has the expected temperature dependence. Indeed, as $T\to 0$ the first term in Eq. (45) vanishes, because $d\Gamma/d\mathcal{E}\to 0$, and we have $\lim_{T\to 0}N_s=N$, the total number of electrons. On the other hand, as $T\to T_c$, the critical temperature, or at any rate if $D=0$, we have successively

$$\mathcal{E}=|E|, \quad \sigma_{\mathbf{k}}=E_{\mathbf{k}}/|E_{\mathbf{k}}|,$$

$$N_s=4\sum k_3\left(1-\frac{d\Gamma}{d|E|}Q\right)^{-1}\frac{d\Gamma}{d|E|}k_3 +2\sum(\Gamma\sigma+\tfrac{1}{2}). \quad (45')$$

Now, it is easy to show that (45′) vanishes identically. Consider the derivative

$$\frac{\partial}{\partial k_3}[k_3(\Gamma_{\mathbf{k}}\sigma_{\mathbf{k}}+\tfrac{1}{2})]=\Gamma_{\mathbf{k}}\sigma_{\mathbf{k}}+\tfrac{1}{2}+k_3\frac{\partial}{\partial k_3}(\Gamma_{\mathbf{k}}\sigma_{\mathbf{k}}).$$

Now

$$\frac{\partial}{\partial k_3}(\Gamma_{\mathbf{k}}\sigma_{\mathbf{k}})=\frac{d\Gamma}{d|E|}\frac{\partial E_{\mathbf{k}}}{\partial k_3};$$

on the other hand, from Eq. (25), we have

$$\frac{\partial E_{\mathbf{k}}}{\partial k_3}=2k_3+\sum_{\mathbf{k}'}Q_{\mathbf{k}-\mathbf{k}'}\frac{\partial}{\partial k_3'}(\Gamma_{\mathbf{k}'}\sigma_{\mathbf{k}'}).$$

It follows that

$$\left(1-\frac{d\Gamma}{d|E|}Q\right)\frac{\partial(\Gamma\sigma)}{\partial k_3}=\frac{d\Gamma}{d|E|}\cdot 2k_3$$

and Eq. (45′) becomes

$$N_s=2\sum_{\mathbf{k}}\frac{\partial}{\partial k_3}[k_3(\Gamma_{\mathbf{k}}\sigma_{\mathbf{k}}+\tfrac{1}{2})]=0. \quad (46)$$

The expression, Eq. (45), for the number of superelectrons is a generalization of that given by BCS[1] and reduces to it if one neglects the contributions containing Q.

If we set $D=0$ but allow $\mathbf{q}$ to differ from zero, the total current does not vanish. For small $\mathbf{q}$ one obtains a contribution proportional to $\mathbf{q}^2$ which must be interpreted as a Landau-type diamagnetism. One can use the formulas given above to study the magnetic susceptibility including corrections due to the interaction between the electrons. This will be included in a forthcoming publication.

If one is willing to sacrifice the gauge invariance of the interaction at this late stage in order to be able to solve the general linear equations in a special case, one can consider the factorizable kernel $P_{\mathbf{k}-\mathbf{k}'}\to J_{\mathbf{k}}\cdot J_{\mathbf{k}'}$. This model includes the kernels of BCS and Gor'kov as special cases and $J_{\mathbf{k}}$ will be assumed to be a spherically symmetric function with a support of the approximate width $2\omega_{\mathrm{Debye}}$ centered around the Fermi surface. All that is needed for the validity of the results given below is that $J_{\mathbf{k}}$ be such that $J_{\mathbf{k}}$ and the basic quantities $\Gamma_{\mathbf{k}}$, $\Gamma_{\mathbf{k+q}}$, $u_{\mathbf{k}}$, $u_{\mathbf{k+q}}$, $v_{\mathbf{k}}$, $v_{\mathbf{k+q}}$ be even under the operation Π. The special form of $J_{\mathbf{k}}$ is not essential in the following and actually does not enter significantly into the final formula for the Pippard kernel. From Eq. (33′) we see that the operator $\mathbf{L}$ can now be written with the aid of dyadic notation

$$\mathbf{L}=\begin{pmatrix}\rho^- & 0\\ 0 & \rho^+\end{pmatrix}+2J_{\mathbf{q}}^2\begin{pmatrix}g^2 & -fg\\ -fg & f^2\end{pmatrix} +\begin{pmatrix}gJ\\ -fJ\end{pmatrix}(-gJ|fJ)+\begin{pmatrix}fJ\\ gJ\end{pmatrix}(fJ|gJ). \quad (47)$$

Since $J_{\mathbf{k}}$ is of even parity it follows that the part of $\mathbf{L}$ written in dyadic form gives zero when applied to the solution of Eq. (33), which must be of odd parity. Equation (33) can now be immediately solved with the result

$$\begin{aligned}\Gamma'&=\{1+2J_{\mathbf{q}}^2[(\rho^-)^{-1}g^2+(\rho^+)^{-1}f^2]\}^{-1}(\rho^-)^{-1}gK'^t,\\ \Phi'&=-\{1+2J_{\mathbf{q}}^2[(\rho^-)^{-1}g^2+(\rho^+)^{-1}f^2]\}^{-1}(\rho^+)^{-1}fK'^t.\end{aligned} \quad (48)$$

Furthermore, if we are interested in values of $\mathbf{q}$ small compared to the Fermi momentum, we can make the approximation $J_{\mathbf{q}}=0$ in Eq. (48), since the function $J_{\mathbf{q}}$ is assumed to be different from zero only in a shell around the Fermi surface.

For the paramagnetic part of the current one finds then

$$\begin{aligned}\langle\mathbf{J}_{\mathbf{q}}^p\rangle&=(e\Lambda)^{-1}N^{-1}\sum_{\mathbf{k}}(2\mathbf{k}+\mathbf{q})(2\mathbf{k}\cdot\mathbf{A}^t(\mathbf{q}))L_{\mathbf{k}}(\mathbf{q})\\ &=(e\Lambda)^{-1}[4N^{-1}\sum_{\mathbf{k}}k_3^2L_{\mathbf{k}}(\mathbf{q})]\mathbf{A}^t(\mathbf{q})\end{aligned} \quad (49)$$

with $L_{\mathbf{k}}(\mathbf{q})=-(\rho^+)^{-1}f^2-(\rho^-)^{-1}g^2$ where the evenness (oddness) of $L_{\mathbf{k}}(\mathbf{q})$ $(\mathbf{k}\cdot\mathbf{A}^t)$ under Π has been used in the last step. Therefore the total current is given by

$$\langle\mathbf{J}_{\mathbf{q}}\rangle=-2eV^{-1}N_s(\mathbf{q},T)\mathbf{A}^t(\mathbf{q}) \quad (50)$$

with

$$N_s(\mathbf{q},T)=N-4\sum_{\mathbf{k}}k_3^2L_{\mathbf{k}}(\mathbf{q}).$$

This relation is identical to the one derived by BCS[1] and hence all the further manipulations quoted there apply here, if one makes their assumptions about $J_{\mathbf{k}}$ so that the basic equilibrium solution used here coincides with theirs. The expression derived here is thus slightly more general.

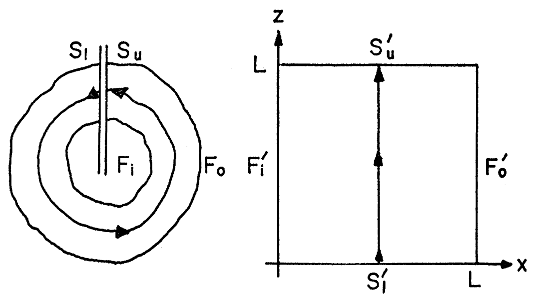

FIG. 1. Typical geometry.

8. FLUX QUANTIZATION

We consider now a doubly connected superconductor. In the stationary situation the bulk of the superconducting material will exhibit the Meissner effect and we can at first neglect surface effects. In order to obtain simpler equations, it is expedient to distort the typical doughnut geometry which occurs in the phenomenon of flux quantization by cutting it open and stretching it into a "straight" shape. (See Fig. 1.) It is apparent that the "natural" boundary conditions at the surfaces S_u' and S_l' are periodic ones. Furthermore, since surface effects are ignored here, it will be allowed in a reasonable limit to move the surfaces F_0 and F_i, respectively, to the right and left and straighten them out into planes. It is then convenient to move them in such a fashion as to construct an L-periodic box within which the usual box quantization description with periodic-boundary conditions is adopted. It is emphasized however that these periodicity requirements are not on the same footing as far as their physical significance is concerned. The x, y periodicity is sheer mathematical convenience bringing about the discreteness of the quantum numbers k_x, k_y while the z periodicity imitates a doubly-connected 3-dimensional volume. We wish to study the nonlinear **C** problem in the geometry just described.

The flux passing through the interior (to the left of F_i) is measured by the line integral of the vector potential along the arrowed path in Fig. 1:

$$\varphi = \int\int d\mathbf{F}\cdot\mathbf{B} = e^{-1}\int \mathbf{A}\cdot d\mathbf{s}.$$

The path can be deformed arbitrarily as long as it stays within the superconducting material and clear of the boundary (where flux and fluxoid differ because of the surface currents) without changing φ. In the modified geometry, a change of gauge does not affect φ and therefore it is convenient to choose the gauge so that $\mathbf{A}$ is a constant vector in the z direction. Then integrating along a straight line from S_l' to S_u', $\varphi = e^{-1}LA$ and

$$K_{ff'} = \delta_{\sigma\sigma'}\delta(\mathbf{k}-\mathbf{k}')[(\mathbf{k}-\mathbf{A})^2-\mu]. \quad (51)$$

We decompose a general $\mathbf{A}$ according to[26]

$$\mathbf{A} = \mathbf{A}' + \mathbf{A}'', \quad (52)$$

$$A' = L^{-1}\pi[\pi^{-1}LA + \tfrac{1}{2}], \quad A'' = A - A'. \quad (53)$$

Since $A' = nL^{-1}\pi$ (n integer), it is seen that for any "allowed" lattice vector $\mathbf{k}$, the vector $-\mathbf{k}+2\mathbf{A}'$ is also an "allowed" lattice vector. This suggests a modified "barring" operation corresponding to a modified pairing,

$$\bar{f} = (\bar{\mathbf{k}}, \bar{\sigma}) = (-\mathbf{k}+2\mathbf{A}', -\sigma), \quad (54)$$
$$\overline{A_{ff'}} = \bar{A}_{ff'} = A_{\bar{f}\bar{f}'}$$

and leads to the modification of $I^{(2)}$ into

$$I^{(2,A)} = \delta(\mathbf{k}+\mathbf{k}'-2\mathbf{A}').$$

It is apparent that of the matrices $S^{(1)}$, $S^{(2)}$, $I^{(1)}$, and $I^{(2,A)}$, only $S^{(2)}$ has odd "barring" parity, all others being even. The **C** problem is then solved in perfect analogy with the solution given in Sec. 5 except that $I^{(2)}$ is replaced by $I^{(2,A)}$ and the modified "barring" symmetries are used. Let us examine the various equations of the **C** problem.[27]

The relation (7′) is satisfied if

$$uI^{(1)}S^{(1)}\tilde{S}^{(1)}\tilde{I}^{(1)}u + vI^{(2,A)}S^{(2)}\tilde{S}^{(2)}\tilde{I}^{(2,A)}v = 1,$$
$$uI^{(1)}S^{(1)}\tilde{S}^{(2)}\tilde{I}^{(2,A)}v + vI^{(2,A)}S^{(2)}\tilde{S}^{(1)}\tilde{I}^{(1)}u = 0,$$

or equivalently $u^2+v^2=1$, $-u\bar{v}+v\bar{u}=0$. These relations can be solved by setting $u=\cos\frac{1}{2}\gamma$, $v=\sin\frac{1}{2}\gamma$ and $\bar{u}=u$, $\bar{v}=v$. The antisymmetry of D and F implies $[D,I^{(2,A)}]=[F,I^{(2,A)}]=0$ which is equivalent to

$$D=\bar{D}, \quad F=\bar{F}. \quad (55)$$

The relations (10) and (17″) then demand after separating the spin dependence and observing Eq. (55) that

$$\Gamma = u^2G - v^2\bar{G} + 2uvF \quad (56)$$
$$0 = -uv(G+\bar{G}) + (u^2-v^2)F$$
$$\mathcal{E} = u^2E - v^2\bar{E} + 2uvD \quad (57)$$
$$0 = -uv(E+\bar{E}) + (u^2-v^2)D.$$

The remaining equations are identical with Eqs. (20) and (25), since the barring operation commutes with the difference kernels P and Q in Eq. (25), except that the kinetic energy is modified according to

$$K_{\mathbf{k}} = (\mathbf{k}-\mathbf{A}'-\mathbf{A}'')^2-\mu, \quad \bar{K}_{\mathbf{k}} = (\mathbf{k}-\mathbf{A}'+\mathbf{A}'')^2-\mu. \quad (58)$$

Introducing the prime and double prime operations, which split a quantity O into parts of even and odd "barring" parity, by

$$O' = \tfrac{1}{2}(O+\bar{O}), \quad O'' = \tfrac{1}{2}(O-\bar{O}),$$

[26] Here the symbol $[x]$ denotes the largest integer smaller than x.

[27] The subscript d denoting diagonal matrices is dropped for brevity.

one obtains the equations

$$\begin{pmatrix}\Gamma'\\0\end{pmatrix}=R\begin{pmatrix}G'\\F\end{pmatrix},\quad \begin{pmatrix}\mathcal{E}'\\0\end{pmatrix}=R\begin{pmatrix}E'\\D\end{pmatrix}, \tag{57'}$$

$$\Gamma''=G'',\quad \mathcal{E}''=E'', \tag{57''}$$

$$\Gamma'=-\tfrac{1}{4}\{\tanh[\tfrac{1}{2}\beta(\mathcal{E}'+\mathcal{E}'')]+\tanh[\tfrac{1}{2}\beta(\mathcal{E}'-\mathcal{E}'')]\},$$
$$\Gamma''=-\tfrac{1}{4}\{\tanh[\tfrac{1}{2}\beta(\mathcal{E}'+\mathcal{E}'')]-\tanh[\tfrac{1}{2}\beta(\mathcal{E}'-\mathcal{E}'')]\}, \tag{59}$$

and

$$K_{\mathbf{k}}'=(\mathbf{k}-\mathbf{A}')^2+\mathbf{A}''^2-\mu,\quad K_{\mathbf{k}}''=-2(\mathbf{k}-\mathbf{A}')\cdot\mathbf{A}''. \tag{60}$$

One observes that except for the coupling, which occurs in Eq. (59) through $\mathcal{E}'$ and $\mathcal{E}''$ the prime and double prime systems of equations are disconnected.

It is then immediately seen that $\mathcal{E}'G'=\Gamma'E'$, $\mathcal{E}'F=\Gamma'D$ and $\Gamma'^2=G'^2+F^2$, $\mathcal{E}'^2=E'^2+D^2$ so that finally

$$E'=K'+Q[\mathcal{E}'^{-1}\Gamma'E'+\tfrac{1}{2}],\quad E''=K''+Q\cdot\Gamma''$$
$$D=-P[\mathcal{E}'^{-1}\Gamma'D]. \tag{60'}$$

Together with Eq. (59), the Eqs. (60′) form a set of 3 coupled integral equations for the quantities E', E'' and D, from which all the other quantities can be reconstructed.

In the special case that $A''=0$, i.e., $A=A'=n\pi L^{-1}$ the flux is quantized in half the London units φ_L

$$\varphi=n\pi e^{-1}=\tfrac{1}{2}n\varphi_L\quad (n\text{ integer}). \tag{61}$$

Since all the double primed quantities vanish, the primed and original quantities coincide and the equations are identical to the equations of Sec. 5 except for the added $\mathbf{A}'$ in the kinetic-energy expression (58). Since P and Q are difference kernels, it is easy to see that the solution of these equations for $A'\neq 0$ is related to the solution of the equations for $A'=0$ studied in Sec. 5 by the identity

$$E_{\mathbf{k}+\mathbf{A}'}(\mathbf{A}'\neq 0)\equiv E_{\mathbf{k}}(\mathbf{A}'=0), \tag{62}$$

and similar identities for all the other quantities. In particular it follows then from Eq. (62) that all these solutions have zero current and exactly the same grand-potential value as the solution for $A'=0$ of Sec. 5, in contrast to the solutions for which the double primed variables differ from zero as shown below.

The general system with $A\neq A'$ can be attacked by several methods. For one thing one can expand the equations in powers of the deviation from the nearest equilibrium point A', i.e., in powers of A''. One can, however, also solve an approximate system of equations which obtains by dropping the $Q\Gamma''$ term in Eq. (60′).

$$\mathcal{E}''=K'',$$
$$\Gamma''=-\tfrac{1}{4}\{\tanh[\tfrac{1}{2}\beta(\mathcal{E}'+K'')]-\tanh[\tfrac{1}{2}\beta(\mathcal{E}'-K'')]\}. \tag{63}$$

Then all the double primed quantities have disappeared from the equations for the primed quantities and only the latter have to be found from Eq. (60′) with Γ' from Eqs. (63) and (59). This set of equations is again identical to the set derived for $A''=0$ except for the different Γ' and the added A''^2 appearing in K'.

As far as the linear current response is concerned it suffices to linearize Eqs. (59), (60), and (60′) with respect to the perturbation A'' for fixed A'. In this way one studies the behavior of the system in the neighborhood of the quantized flux solutions. From Eqs. (57″), (59), and (60) it is seen that to first order in A''

$$\bar{G}_{\mathbf{k}}''=\Gamma_{\mathbf{k}}''\cong\frac{d\Gamma_{\mathbf{k}}}{d\mathcal{E}_{\mathbf{k}}}\cdot\mathcal{E}_{\mathbf{k}}''=\frac{d\Gamma_{\mathbf{k}}}{d\mathcal{E}_{\mathbf{k}}}(K_{\mathbf{k}}''+Q_{\mathbf{k}-\mathbf{k}'}\bar{G}_{\mathbf{k}}'')$$

and hence

$$\bar{G}_{\mathbf{k}}''=\sum_{\mathbf{k}'}\left(1-\frac{d\Gamma}{d\mathcal{E}}Q\right)^{-1}\frac{d\Gamma}{d\mathcal{E}}[-2\mathbf{A}''\cdot(\mathbf{k}'-\mathbf{A}')]. \tag{64}$$

Substituting this into the expression (5) for the current and using the "barring" parity of the unperturbed solution for fixed A' one finds

$$\begin{aligned}\langle\mathbf{J}_{\mathbf{q}}\rangle&=(e\Lambda)^{-1}\delta(\mathbf{q})[N^{-1}\textstyle\sum_{\mathbf{k}}(\mathbf{k}\bar{G}_{\mathbf{k}}+\bar{\mathbf{k}}\bar{G})_{\bar{\mathbf{k}}}-\mathbf{A}]\\&=(e\Lambda)^{-1}\delta(\mathbf{q})[N^{-1}\textstyle\sum_{\mathbf{k}}(2\mathbf{k}-4\mathbf{A}')\bar{G}_{\mathbf{k}}''-\mathbf{A}'']\\&=-2eV^{-1}\left[N+4\textstyle\sum(k_A-2A')\left(1-\frac{d\Gamma}{d\mathcal{E}}Q\right)^{-1}\times\frac{d\Gamma}{d\mathcal{E}}(k_A-A')\right]\delta(\mathbf{q})\mathbf{A}''\\&=-2eV^{-1}N_s\cdot\delta(\mathbf{q})\mathbf{A}'',\end{aligned} \tag{65}$$

or $\langle\mathbf{J}\rangle=-2eV^{-1}N_s\mathbf{A}''$. In the last step the isotropy of the current response for spherically symmetric Q and the identity (62) have been used in order to employ the definition (45) of N_s. This factor very clearly displays the expected temperature behavior of the supercurrent. Furthermore, since the basic solution for fixed A' is a solution of the full nonlinear equations, it is permissible to integrate Eq. (65) in order to find the dependence of the grand potential Ω on A'',

$$\Omega(A'')=\Omega(0)+N_sA''^2. \tag{66}$$

Since $\Omega(0)$ is equal to the grand potential for the equilibrium state of the zero-field case it is apparent that the behavior of Ω in the vicinity of the stationary points corresponding to the quantized flux solutions is parabolic as predicted by Byers and Yang.[11] Furthermore, the factor N_s displays the anticipated temperature dependence of the term peculiar to superconductors in Eq. (66) and, as shown above, causes this term to vanish above the critical temperature or at any rate if the gap vanishes.

If one requires that in addition to the electron field equations resulting from H also Maxwell's equation,

$$\text{curl}\mathbf{B}=\langle\mathbf{J}\rangle, \tag{67}$$

be satisfied, it is seen from Eq. (65) that almost all solutions parametrized by the one-dimensional continuous variable A are ruled out as consistent solutions since they have a bulk current. Only the solutions corresponding to $A''=0$, i.e., quantized flux values, are also electromagnetically consistent in that they assume a vector potential corresponding to a full Meissner effect and have no bulk current. Thus in the infinite cubic box geometry the Meissner effect drastically reduces the possible stable thermodynamic states and only allows solutions to the $\mathbf{C}$ problem with quantized flux parameter. The fact that more than one solution is possible is due to the periodicity requirement which imitates the doubly connected geometry. For the singly connected case there is just the solution corresponding to $n=0$, i.e., the equilibrium state. The actual experimental situation is somewhat different, since the geometry is neither infinite in extent nor straight, but typically finite and doubly connected as depicted in Fig. 1. If the superconducting specimen is sufficiently thick so that the full Meissner effect $\mathbf{B}=\mathbf{o}$ can develop in the bulk of the material, and sufficiently large so that the actual curved nature of the surface can be ignored as far as the microscopic properties are concerned, the solutions described above will afford a reasonably good description of the bulk properties of the system.

In the experiment described in the papers quoted in Ref. 9, the system is placed into an external homogeneous field parallel to the cylinder axis and then cooled below the transition temperature. The bulk of the cylinder goes over into one of the preferred states with $A''=0$, since this is the most economical arrangement as far as the volume contribution to the grand potential is concerned. However, two surface currents are needed to bring this preferred state into existence. One of these persistent currents flows along the outer surface and is such as to cancel the applied field inside the superconductor and thus enables the Meissner effect to be set up. The other surface current flows along the inner surface and creates the quantized amount of flux through the hole in order that the bulk of the material have a vector potential corresponding to one of the preferred values consistent with Maxwell's equation.

The theoretical discussion of the surface transition regions and the behavior of the field quantities there is rather complicated and essentially unsolved as far as a truly microscopic treatment is concerned. However within the framework of the Ginzburg-Landau theory several such discussions have been given.[16]

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APPENDIX I

Unitary of C

With the dichotomic index $\nu=(\pm)$ such that $a_f=a_{f-}$, $a_f^\dagger=a_{f+}$, $\alpha_f=\alpha_{f-}$, $\alpha_f^\dagger=\alpha_{f+}$, and

$$u_{ff'}=\mathbf{C}_{f-f'-}\quad u_{ff'}^*=\mathbf{C}_{f+f'+}$$
$$v_{ff'}=\mathbf{C}_{f-f'+}\quad v_{ff'}^*=\mathbf{C}_{f+f'-}$$

Eq. (6′) can be written as $a_{f\nu}=\mathbf{C}_{f\nu f'\nu'}\alpha_{f'\nu'}$. This transformation is canonical if and only if

$$\{a_{f\nu},a_{f'\nu'}\}=\delta_{ff'}\delta_{\nu,-\nu'}$$

implies

$$\{\alpha_{f\nu},\alpha_{f'\nu'}\}=\delta_{ff'}\delta_{\nu,-\nu'}$$

and vice versa. The m symmetry of $\mathbf{C}$ leads to

$$\mathbf{C}_{f\nu f'\nu'}=\mathbf{C}^*{}_{f-\nu f'-\nu'}$$

and one finds

$$\begin{aligned}\delta_{ff'}\delta_{\nu\nu'}&=\{a_{f\nu},a_{f'-\nu'}\}\\&=\mathbf{C}_{f\nu g\rho}\mathbf{C}_{f'-\nu' g'\rho'}\{\alpha_{g\rho},\alpha_{g'\rho'}\}\\&=\mathbf{C}_{f\nu g\rho}\mathbf{C}^\dagger{}_{g\rho f'\nu'}\end{aligned}$$

or $\mathbf{I}=\mathbf{C}\cdot\mathbf{C}^\dagger$. If Eq. (6′) is required to have an inverse then $\mathbf{C}$ has a unique $\mathbf{C}^{-1}$ and

$$\mathbf{C}=(\mathbf{C}^\dagger)^{-1}, \tag{A1}$$

i.e., $\mathbf{C}$ is unitary.

APPENDIX II

Two-Particle Correlation Function

It is seen from Eqs. (9′) and (10) that

$$\begin{aligned}\bar{G}&=u^*\bar{\Gamma}\tilde{u}+v^*(1-\bar{\Gamma})\tilde{v},\\F&=v\bar{\Gamma}\tilde{u}+u(1-\bar{\Gamma})\tilde{v},\\F^\dagger&=u^*\bar{\Gamma}v^\dagger+v^*(1-\bar{\Gamma})u^\dagger.\end{aligned}$$

Together with Eqs. (6) and (9′) this implies

$$\begin{aligned}&\langle a_{f^1}{}^\dagger a_{f^2}{}^\dagger a_{f^3}a_{f^4}\rangle\\&\quad=\langle(u_{f^1g^1}{}^*\alpha_{g^1}{}^\dagger+v_{f^1g^1}{}^*\alpha_{g^1})(u_{f^2g^2}{}^*\alpha_{g^2}{}^\dagger+v_{f^2g^2}{}^*\alpha_{g^2})\\&\qquad\times(u_{f^3g^3}\alpha_{g^3}+v_{f^3g^3}\alpha_{g^3}{}^\dagger)(u_{f^4g^4}\alpha_{g^4}+v_{f^4g^4}\alpha_{g^4}{}^\dagger)\rangle\\&=[u_{f^1g^1}{}^*u_{f^4g^1}\bar{\Gamma}_{g^1}+v_{f^1g^1}{}^*v_{f^4g^1}(1-\bar{\Gamma}_{g^1})]\\&\qquad\times[u_{f^2g^2}{}^*u_{f^3g^2}\bar{\Gamma}_{g^2}+v_{f^2g^2}{}^*v_{f^3g^2}(1-\bar{\Gamma}_{g^2})]\\&\qquad-[u_{f^1g^1}{}^*u_{f^3g^1}\bar{\Gamma}_{g^1}+v_{f^1g^1}{}^*v_{f^3g^1}(1-\bar{\Gamma}_{g^1})]\\&\qquad\times[u_{f^2g^2}{}^*u_{f^4g^2}\bar{\Gamma}_{g^2}+v_{f^2g^2}{}^*v_{f^4g^2}(1-\bar{\Gamma}_{g^2})]\\&\qquad+[u_{f^1g^1}{}^*v_{f^2g^1}{}^*\bar{\Gamma}_{g^1}+v_{f^1g^1}{}^*u_{f^2g^1}{}^*(1-\bar{\Gamma}_{g^1})]\\&\qquad\times[u_{f^3g^3}v_{f^4g^3}\bar{\Gamma}_{g^3}+v_{f^3g^3}u_{f^4g^3}(1-\bar{\Gamma}_{g^3})]\\&\quad=\bar{G}_{f^1f^4}\bar{G}_{f^2f^3}-\bar{G}_{f^1f^3}\bar{G}_{f^2f^4}+F_{f^1f^2}{}^\dagger F_{f^3f^4}.\end{aligned}$$

This corresponds to the results of a generalized Wick's theorem.

APPENDIX III

Proof of Equation (37)

The expected solution

$$\mathbf{B}=\mathbf{C}^{\dagger}\begin{pmatrix}\eta & 0\\ 0 & \eta^*\end{pmatrix}\mathbf{C} \tag{35}$$

leads to

$$\mathbf{\Gamma}'=[\mathbf{B},\mathbf{\Gamma}]=\begin{pmatrix}-\Gamma'^* & \Phi'\\ -\Phi'^* & \Gamma'\end{pmatrix}.$$

After both the factorization of the spin matrices and the τ separation, one finds $\Gamma'=\Gamma^- g^+\eta$ and $\Phi'=-\Gamma^+ f^+\eta$. Writing out Eq. (33) with the use of Eqs. (33′), (34), and (35), one obtains after factoring η

$$\mathcal{E}^- g^+ - gQ(\Gamma^+ ff^+ + \Gamma^- gg^+) - fP(\Gamma^+ gf^+ - \Gamma^- fg^+) = g(k^2-k'^2)\,,$$

$$\mathcal{E}^+ f^+ + gP(\Gamma^+ gf^+ - \Gamma^- fg^+) - fQ(\Gamma^+ ff^+ + \Gamma^- gg^+) = f(k^2-k'^2)\,,$$

which reduces to[28]

$$\begin{aligned}\mathcal{E}^- g^+ - g(E-E') + f(D+D') &= 0\,,\\ \mathcal{E}^+ f^+ - g(D+D') - f(E-E') &= 0\,,\end{aligned} \tag{36′}$$

if one uses

$$\Gamma^+ ff^+ + \Gamma^- gg^+ = \Gamma(u^2-v^2) - \Gamma'(u'^2-v'^2) = G-G'\,,$$

$$\Gamma^+ gf^+ - \Gamma^- fg^+ = \Gamma 2uv + \Gamma' 2u v' = F+F'\,,$$

$$\mathbf{k}^2-\mathbf{k}'^2 = E-E'-Q(G-G')\,,$$

and

$$0 = D+D'+P(F+F')\,.$$

Multiplying Eq. (36′) by f^+ and g^+, and adding and subtracting the resulting equations, one has

$$\begin{aligned}0 &= -\mathcal{E}\sin(\gamma+\gamma') + \sin\gamma'(E-E') + \cos\gamma'(D+D')\,,\\ 0 &= -\mathcal{E} + \mathcal{E}'\cos(\gamma+\gamma') + \cos\gamma(E-E') + \sin\gamma(D+D')\,,\end{aligned} \tag{36″}$$

and two equations with primed and unprimed quantities interchanged. Since, according to Eq. (24), $E=\mathcal{E}\cdot\cos\gamma$ and $D=\mathcal{E}\cdot\sin\gamma$ it is seen immediately that Eq. (36″) is satisfied.

[28] In this section the prime denotes a change of argument from $\mathbf{k}$ to $\mathbf{k}'$.