approximately as strong as it is in even-parity states; and this suggests that a value of D close to 40 MeV may be expected.

The validity of the representation of the Λ-nucleon interaction by effective central potentials is doubtful at the higher energies we have considered. At these energies, the effects of possible noncentral components may be important[8,9,17,37] and quite different from their effects in hypernuclei. Moreover, there may be an appreciable effect from the presence of the Σ-production channel.[8,9] The cross sections reported here for the higher energies are, therefore, to be considered only as the contributions of those components of the Λ-nucleon interaction which can reasonably be represented by effective central potentials at low energies.

That the presence of the Σ-production channel can have a pronounced effect in Λ-nucleon scattering has been emphasized by de Swart *et al.*[8,9] In particular, de Swart and Dullemond[8] have calculated Λ-nucleon scattering cross sections with hyperon-nucleon potentials deduced from phenomenological nucleon-nucleon potentials under the assumption of a universal pion-baryon interaction. Their cross sections have a prominent peak in the neighborhood of the Σ-production threshold (about 76 MeV in the zero-momentum frame), and have values above that threshold which are appreciably larger than the average empirical cross sections of Groves[20] and of Alexander *et al.*[18] Although their cross sections are consistent with the average empirical cross section of Arbuzov *et al.*[19], the measured and calculated angular distributions appear to be inconsistent. The cross sections $\sigma^{(a)}$ to which our effective central potentials lead for energies above the Σ-production threshold are closer to the empirical cross sections of Groves and of Alexander *et al.* than are those of de Swart and Dullemond. Considering the preliminary nature of the scattering data, however, it is probably too early to draw a conclusion from this comparison.

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Spin and Parity Analysis at All Production Angles*†

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Bohr's symmetry method is applied to an unstable spin-j state X, which is produced in a reaction $A+B\rightarrow C+X$ and then decays according to $X\rightarrow D+E$. Particles A, B, C, D are assumed to be spinless, and E is either a spinless particle or a gamma ray. Parity is conserved in production, but not necessarily in decay. The angular distribution of E, in the rest system of X, is $I(\theta)=\frac{1}{2}\Sigma a_L P_L(\cos\theta)$, where $L\leqslant 2j$ and the polar angle θ is measured from the normal to the production plane. The coefficients a_L depend upon the production angle δ and upon the dynamics of the production. It is proved here that the sign of the maximum-complexity coefficient a_{2j} depends only upon the parity of X, and that the magnitude of a_{2j} is not zero but lies between bounds which depend upon j and the parity alone. The implied test for j and the parity has the following advantages: (1) The spin j is equal to half the largest L in $I(\theta)$. Addition of a small amount of a higher P_L, which always improves the fit, is forbidden by the lower bound of a_{2j}. (2) The bounds of a_{2j} are independent of δ. Any (perhaps biased) average over δ may be performed before expanding $I(\theta)$ in the P_L. (3) All the data are condensed into a single test quantity a_{2j}, whose statistical error is reliably known.

1. INTRODUCTION

SUPPOSE an unstable particle or state X is produced in the reaction

$$A+B\rightarrow C+X \tag{1.1}$$

and then decays according to

$$X\rightarrow D+E, \tag{1.2}$$

where A, B, C, and D have spin zero, and E is either a spinless particle or a gamma ray. It was first pointed out by Bohr[1] that conservation of parity in the production reaction implies a symmetry condition for the spin state of X, and consequently also for its decay products. Bohr found that if the spin j of state X is equal to unity, then the angular distribution of its decay is given (for spinless E) by

$$I(\theta)=(3/4\pi)\cos^2\theta, \tag{1.3}$$

if the intrinsic parity is unchanged in the production process, and by

$$I(\theta)=(3/8\pi)\sin^2\theta, \tag{1.4}$$

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† This work was reported briefly at the Chicago Meeting of the American Physical Society [M. Peshkin, Bull. Am. Phys. Soc. 8, 514 (1963)].

1 A. Bohr, Nucl. Phys. 10, 486 (1959).

if the intrinsic parity is changed. The polar angle θ is measured between the momentum of E (in the rest frame of X) and the normal to the production plane. Distributions (1.3) and (1.4) have the virtue that they do not depend upon the direction in which X is produced.

In this report, Bohr's method is extended to all values of the spin j. It is demonstrated that the coefficient of the most complicated Legendre polynomial in $I(\theta)$ determines the spin and parity of X unambiguously.

The method given here retains the severe restrictions on particles A–E. However, many cases of practical importance are in fact covered. The target B must always be a spinless nucleus, such as an alpha particle. The projectile A may be another spinless nucleus or a pion or kaon. Examples of X include nuclear and hypernuclear states and new mesons. A new meson would have to be accompanied by a spinless nuclear state C, but it appears that such processes are likely to be important at very high energies.[2]

2. MAXIMUM-COMPLEXITY COEFFICIENT

Consider first the case in which E is a spinless particle. Let δ represent the production angle, i.e., the angle between the momentum of A and that of X in the center-of-mass system for reaction (1.1). The spin function $X(\delta)$ for particles X moving in a given direction may be expressed as

$$X(\delta)=\sum_m \beta_m(\delta)X_{jm}\,, \tag{2.1}$$

where the quantization (z) axis is taken as the normal to the production plane. The coefficients $\beta_m(\delta)$, which depend upon the dynamics, are normalized so that

$$\sum_m |\beta_m(\delta)|^2=1\,. \tag{2.2}$$

The maximum-complexity method is based on Bohr's observation that conservation of parity in the production process (1.1) implies that only even or only odd m contribute to the sum (2.1). Even m appear if there is no change in the intrinsic parity in (1.1), i.e., if $P_AP_B=P_CP_X$. Odd m appear if there is change ("yes") in the intrinsic parity.[3]

After decay, the angular part of the wave function for E, in the rest system of X, becomes

$$\psi_\delta(\theta,\phi)=\sum_m{}' \beta_m(\delta)Y_{jm}(\theta,\phi)\,, \tag{2.3}$$

where the notation $\sum'$ is used as a reminder that only even m, or only odd m, contribute. The angular distribution

$$I_\delta(\theta)=\int_0^{2\pi}|\psi_\delta(\theta,\phi)|^2d\phi \tag{2.4}$$

[2] S. M. Berman and S. D. Drell, Phys. Rev. Letters **11**, 220 (1963). These authors also give useful angular distribution tests which depend upon the azimuthal as well as the polar angle.

[3] Bohr proves this statement by considering the operation R, which is space inversion followed by 180° rotation about the z axis. When R acts on the initial state of reaction (1.1), it merely multiplies the wave function by the intrinsic parity P_AP_B. However, when it acts on a spin function X_{jm}, it multiplies it by $(-1)^m$. Then $X(\delta)$ can contain only those m for which $(-1)^m=P_AP_BP_CP_X$.

TABLE I. Bounds of the maximum-complexity coefficient.

j	Parity change	E = spinless particle	E = gamma ray
0	no	$a_0=1$	forbidden
0	yes	forbidden	forbidden
1	no	$a_2=2$	$a_2=-1$
1	yes	$a_2=-1$	$a_2=\frac{1}{2}$
2	no	$3/7\leq a_4\leq 18/7$	$-12/7\leq a_4\leq -2/7$
2	yes	$a_4=-12/7$	$a_4=8/7$
3	no	$30/33\leq a_6\leq 100/33$	$-100/44\leq a_6\leq -30/44$
3	yes	$-75/33\leq a_6\leq -5/33$	$5/44\leq a_6\leq 75/44$
4	no	$7/143\leq a_8\leq 490/143$	$-1960/715\leq a_8\leq -28/715$
4	yes	$-392/143\leq a_8\leq -56/143$	$224/715\leq a_8\leq 1568/715$

is expressed in terms of the Legendre polynomials through

$$I_\delta(\theta)=\tfrac{1}{2}\sum_{L=0}^{2j} a_L(\delta)P_L(\cos\theta)\,. \tag{2.5}$$

The factor $\frac{1}{2}$ is included to give the normalization condition

$$a_0(\delta)=1\,. \tag{2.6}$$

Equations (2.3)–(2.5) are easily combined to give

$$a_L(\delta)=[(2L+1)4\pi]^{\frac{1}{2}}\langle Y_j\|Y_L\|Y_j\rangle \times\sum_m{}' |\beta_m(\delta)|^2 C(jLj;m0m)\,, \tag{2.7}$$

where the function C is the vector coupling coefficient,[4] and the reduced matrix element is given by[5]

$$\langle Y_j\|Y_L\|Y_j\rangle=[(2L+1)/4\pi]^{\frac{1}{2}}C(jLj;000)\,. \tag{2.8}$$

In the usual case, the value of j cannot be deduced directly from the experimental $I_\delta(\theta)$ because any $a_L(\delta)$ may vanish by numerical accident in the sum (2.7). Then it can only be ascertained that j is not less than half the maximum L for which $a_L(\delta)$ is different from zero. However, the Bohr restriction to only even or only odd m makes it possible to take advantage of a special feature of the vector coefficients for $L=2j$, which have the simple form[6]

$$C(j,2j,j;m0m)=(-1)^{j-m}\left[\frac{2j+1}{(4j+1)!}\right]^{\frac{1}{2}} \times\frac{(2j)!(2j)!}{(j+m)!(j-m)!}\,. \tag{2.9}$$

Since all the vector coefficients in the sum (2.7) for

[4] In angular-momentum quantities, the notation is that of M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957).

[5] Ref. 4, p. 88.

[6] This expression is easily obtained from Racah's formula [Ref. 4, p. 40], which reduces to one term in the case of maximum complexity.

$L=2j$ have the same sign,[7] $a_{2j}(\delta)$ cannot vanish and its sign determines the parity of X. The numerical value of $a_{2j}(\delta)$ must lie between bounds determined by the largest and the smallest vector coefficients for given j and parity. These bounds are obtained from Eq. (2.7) by substituting the extreme values of the vector coefficients for the weighted average over possible values of m. Numerical results for small values of j are given in Table I. In cases in which the sum contains only a single term, or no term, $a_{2j}(\delta)$ is determined exactly. In particular, the case $j=1$ is identical to that of Bohr and agrees with his result.

The case in which E represents a gamma ray instead of a spinless particle presents no difficulty. The intensity must be summed over polarization directions. This sum is easy to carry out. The expression for $a_L(\delta)$ is unchanged, except for the reduced matrix element. In particular, the reduced matrix element appearing in the expression for the maximum complexity coefficient is

$$\langle\gamma_j\|Y_{2j}\|\gamma_j\rangle=-[j/(j+1)]\langle Y_j\|Y_{2j}\|Y_j\rangle. \quad (2.10)$$

The bounds for $a_{2j}(\delta)$ in this case are shown in the last column of Table I.

3. COMMENTS

The coefficient of the most complex term in the experimental angular distribution (2.5) determines the spin and parity of X unambiguously. This maximum-complexity test has several important practical advantages.

First, the bounds of $a_{2j}(\delta)$ are independent of δ. It is therefore permissible to integrate the experimental $I_\delta(\theta)$ over all production angles δ before carrying out the expansion in the P_L. The only role of the production angle is to determine the z direction for the measurement of θ in each event. This advantage may be decisive in a situation in which there are perhaps a few hundred events in all. In that case, it is impossible to obtain an angular distribution of the decay at any one production angle, but the total angular distribution is still moderately well determined. It is also possible to integrate over a restricted range of δ, to avoid directions of high background. An experimental bias against some directions is acceptable as long as the bias depends only upon δ and not upon θ. It can even happen that the test for some spin and parity assignment may be statistically indecisive when all production angles are allowed, but that the assignment is clearly rejected by taking only a limited range of δ.

Second, the largest L in $I(\theta)$ is unambiguously identified by the bounds of a_{2j}, if the statistical accuracy is good enough. In practical cases, a good fit with a certain $L_{\max}$ can always be improved by adding a small amount of P_L with $L=L_{\max}+2$. However, the bounds of the coefficient of maximum complexity forbid adding a very small amount. Thus, the identification of $L_{\max}$ is likely to be unambiguous.

Third, the hypothesis that X has spin j is tested by a single test quantity a_{2j}, which summarizes all the data, and whose statistical uncertainty can be estimated reliably.

Fourth, although the maximum-complexity method applies only to special cases, they are just the cases in which angular correlations with other decays are not available, since all the other particles present are spinless.

The maximum-complexity method is evidently most useful in the type of experiment in which a small number of events is measured completely, regardless of the direction of the tracks, since $I(\theta)$ involves an integration over all azimuthal directions ϕ. However, it has some possibility of application to counter techniques as well. It is, unfortunately, necessary to move the E counter over the entire surface of a sphere. This is partially compensated by the feature that the C counter need not be moved at all, and may occupy a whole circle. Moreover, it is not necessary to have a statistically meaningful counting rate as a function of θ and ϕ; it is enough that the integral over ϕ for each θ should be meaningful.

The maximum-complexity method can be extended to a few cases not considered here, but there the analysis is not as powerful, and will not be given in detail. Quite generally, if A is a particle of integral spin different from zero, the results given here apply provided that A is aligned to give only an even or only an odd z component of its spin. If the z component is odd, the parity must be reversed in using Table I. Probably the most practical example is a gamma ray polarized either in the z direction or perpendicular to it. Aligned deuterons are another possibility. In either case, the production directions for which the analysis is valid are restricted to the plane perpendicular to the z axis, which is now determined by the characteristics of the initial state. Similar statements can be made when B represents a polarized proton target, but then $2j$ is odd and the maximum-complexity coefficient vanishes unless parity is mixed in the decay of X.

The maximum-complexity test cannot be generalized to include arbitrary spins of D and E, because the reduced matrix elements are usually unknown. When E has unit spin, the results for gamma rays apply if the spin state of E is known to be transverse. This condition is necessarily met if the parity change in production equals $(-1)^j$ and parity is conserved in decay.

[7] The alternation of the sign of the vector coefficient with m is of course no accident. The wave functions Y_{jm} are large on $(2j+1)$ more or less uniformly spaced cones. The polynomial $P_{2j}(\cos\theta)$ has $2j$ null points, also fairly uniformly spaced. Thus, P_{2j} changes sign between alternate cones.