Excited States of the Nucleon*

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The quantum numbers of the higher resonances in pion-nucleon scattering belonging to the nucleon and 3–3 isobar trajectories are shown to arise from the exchange of the nucleon and pion-nucleon resonances themselves. In particular the members of the nucleon trajectory are created principally by forces due to the exchange of members of the 3–3 isobar trajectory and vice versa. The quantum numbers obtained from dynamical considerations agree completely with those conjectured on the basis of Regge pole considerations. The dynamical scheme is approximately symmetrical under interchange of the two trajectories.

INTRODUCTION

TWO dominant trends are discernible in recent efforts to unify the theoretical discription of elementary particles and resonances. One approach[1,2] groups particles of different angular momentum but identical internal quantum numbers together into families, the members thereof constituting the observable points on "Regge trajectories." On the other hand, the similarities of particles of the same spin and parity (e.g., the eight baryons) suggests the existence of an underlying symmetry, as exemplified[3] by the "eightfold way." While both of these approaches are very promising, there has not been much understanding of the role of dynamics in these theories, although some suggestive work has been done by Capps,[4,5] Cutkosky,[6,7] Sakurai[8] and others. It seems very important to discover which, if any, symmetries and regularities of Regge trajectories are determined dynamically.

The present work is a continuation of two previous papers[9,10] in which a scheme for the dynamics relating the members of the nucleon and 3–3 isobar trajectories was proposed. There it was shown that the observed pattern of quantum numbers can be understood qualitatively by means of the constructive collaboration of the forces due to exchange of the baryonic states themselves. The forces responsible for the existence of the members of the nucleon trajectory arise principally from the exchange of the members of the 3–3 isobar trajectory, and vice versa. We thus obtain "coupled," mutually dependent trajectories, a circumstance we expect to find in the resonance spectra of other strongly interacting particles. To emphasize this interdependence we call the six observed physical states of the nucleon and isobar trajectories a *constellation* of resonances. One of the most interesting results is the qualitative invariance of the dynamical scheme underlying the constellation to reflection about the horizontal axis (Fig. 3). This result generalizes the "dynamical equivalence" of the nucleon and the 3–3 isobar discussed by Chew.[11]

It should perhaps be mentioned that none of the present analysis depends on continuations in the angular momentum, so that one really need not mention the concept of Regge trajectories. However, the physical situation is very similar to that suggested by the situation in potential scattering, which considerations led to the present work. In a sense, the existence of Regge trajectories in the direct channel is a trivial, *a posteriori*, result. We have not treated the exchanged states as Regge poles, mainly because of the enormous increase in the already burdensome computational work. Perhaps such a refinement eventually will be desirable.

In order to avoid being overwhelmed by the enormous number of angular momentum states demanding attention, we have pruned the theoretical apparatus to a minimum. We start from fixed-momentum transfer-dispersion relations[12] and examine the influence of resonances in the crossed channel. The observed set of quantum numbers is seen by inspection of the crossing matrix to be the most favorable situation so that when examining a given partial-wave amplitude we treat the crossed channel as experimentally known. We find then that the "forces" due to the exchange of baryonic resonances become large in the appropriate states at about the right energy for the above mentioned resonances. The proper ordering is obtained automatically, isospin-$\frac{1}{2}$ states lying lower in energy than isospin $\frac{3}{2}$, essentially because the exchange of the latter gives stronger forces in isospin $\frac{1}{2}$ than the other way round. In particular, the order of magnitude of the slope of the trajectories is understood automatically when it is recognized that centrifugal barriers associated with baryon exchange forces are involved. Nor is the approximate

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constancy of slopes a surprise although the precise calculation of resonance energies requires the specification of all significant forces. The calculation of resonance energies on the basis of the present model is deferred to a subsequent paper. The possibility that 3–3 isobar exchange gives rise to a $P_{1/2}$ isospin $\frac{1}{2}$ resonance near 900 MeV suggested by Feld and Layson[13] is discussed in Sec. IV. This resonance, if substantiated by further work, would signal the beginning of a second trajectory with the quantum numbers of the nucleon.

The situation in the D states is considerably more complicated owing to the presence of forces of opposite signs and different ranges. In particular, the possibility[10] of a bootstrap operating between the 600-MeV resonance and the 850-MeV shoulder should be viewed with suspicion pending a more thorough investigation of the inelasticity that is so prominent in these phenomena. It has been thought for some time that the strong inelasticity ($\pi N \rightarrow 2\pi N$) is responsible for the 600-MeV maximum. Since it is difficult to treat properly the inelasticity (except of course in the crossed channel) in the present framework we do not discuss this state (nor the very inelastic 850-MeV "shoulder") in any detail in this paper, although in Sec. V we shall review previous theoretical work in the light of our results.

Thus the present work is incomplete in that proper account has not been made for the strongly coupled three-particle channels. We believe, though, that for the even parity maxima the *essential* forces are created by the exchange of baryonic states. The details, and in particular the resonance energies may be changed somewhat by inelasticity and the exchange of mesonic systems. Recent progress in the mathematical description of the inelastic channels by Cook and Lee,[14] Ball, Frazer, and Nauenberg,[15] Mandelstam *et al.*,[16] and Hwa[17] can be utilized for this aspect of the problem. One refinement, the use of the Mandelstam representation (along the lines sketched by Frautschi and Walecka[18] for the 3–3 resonance) instead of the more primitive fixed momentum transfer dispersion relations, is already in progress.[19] We should like to emphasize that the results of the present investigation cast doubt on any theory of production amplitudes which neglects the exchange of baryonic states. Previously such forces usually have been neglected on the basis of their short range. We find, though, that when these short-range forces (which are after all quite strong) act in consort they are quite important.

All the members of the constellation are compatible with the "spin-orbit rule" discussed by Kycia and Riley.[20] However for odd parity resonances the present theory is almost certainly incomplete so that no fair comparison can be made. The finer details of the rule do not agree with our results, although this fact is scarcely significant. For example, although $F_{7/2}$ isospin $\frac{1}{2}$ is repulsive in accordance with the rule, $F_{5/2}$ isospin $\frac{3}{2}$ is slightly attractive. The origin of such results is explained in Sec. III.

A most interesting question, now under investigation, is whether the pion-hyperon resonances can be similarly correlated. One can, in optimistic moments, hope that by such calculations the meaning of such concepts as isotopic spin will be dynamically determined in as simple a way as (for example) the symmetry of crystals. (Of course to calculate the symmetry of a given crystal is often very difficult, but nobody is suprised that crystals exist, or that their symmetry has a decisive influence on their physical behavior.) For example from the present work one can see that isospin $\frac{3}{2}$ is energetically unlikely for a $P_{1/2}$ nucleon. Moreover, without isospin one would not have a 3–3 resonance to "bind" the nucleon via Chew's reciprocal bootstrap.

Readers uninterested in computational details are advised to skim Sec. II and then go directly to Sec. IV.

II. BASIC EQUATIONS

The kinematical formulas required for the analysis of pion-nucleon scattering have been given in many places.[12,18] Because our work depends in a crucial way on the detailed isospin, angular momentum structure of the problem we shall summarize here enough results to make the paper reasonably self-contained. The initial and final pion four-momenta are labeled q_1 and q_2; for the initial and final nucleon four-momenta we write p_1, p_2. From these four vectors one forms the usual invarient variables $s=(p_1+q_1)^2$, $u=(p_1-q_2)^2$ and $t=(q_1-q_2)^2$. s, u, and $-t$ are, respectively, the squares of the c.m. total energy, "crossed" total energy, and momentum transfer. We express all quantities except laboratory pion kinetic energy in terms of the positive pion mass, μ. [We suppose all pions to have the same mass (μ) and take both nucleons to have the proton's mass $M=(6.72\mu)$.] For computational purposes it is useful to note that $M^2=45.16\mu^2$, $(M^2-\mu^2)^2=1950.0\,\mu^2$, and $s=(59.60+0.0963\,T_L)\,\mu^2$, where T_L is the lab pion kinetic energy in MeV. Introducing the c.m. momentum k and the cosine of the scattering angle $x=\cos\theta$, one finds the following useful relations:

$$s+t+u=2(M^2+\mu^2)\,, \tag{2.1}$$

$$t=-2k^2(1-x)\,, \tag{2.2}$$

$$k^2=\tfrac{1}{4}s-\tfrac{1}{2}(M^2+\mu^2)+(M^2-\mu^2)^2/4s\,. \tag{2.3}$$

The total c.m. energy $(s)^{1/2}$ is called W. The c.m.

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nucleon total energy is then $E=(s+M^2-\mu^2)/2W$. The invariant amplitude $\bar{u}(p_2)Tu(p_1)$ may be expressed in terms of the two invariant functions $A(s,t,u)$ and $B(s,t,u)$ by

$$T=-A+\tfrac{1}{2}\gamma\cdot(q_1+q_2)B. \tag{2.4}$$

It is customary to regard T as a matrix in the nucleon isospin space. The simplest crossing properties are possessed by the amplitudes $A^{\pm}$, $B^{\pm}$ defined in terms of the amplitude for a pion of charge state i scattering to charge state j (i, $j=1$, 2, 3) by

$$A_{ji}=\delta_{ij}A^{+}+\tfrac{1}{2}[\tau_j,\tau_i]A^{-}, \tag{2.5}$$

$$B_{ji}=\delta_{ij}B^{+}+\tfrac{1}{2}[\tau_j,\tau_i]B^{-}. \tag{2.6}$$

The τ_i are the 2×2 Pauli isospin matrices.

The crossing symmetry connecting the channels where s and u are physical is then expressed by

$$A^{\pm}(s,t,u)=\pm A^{\pm}(u,t,s), \tag{2.7}$$

$$B^{\pm}(s,t,u)=\mp B^{\pm}(u,t,s). \tag{2.8}$$

We find it more useful to work with isospin-diagonal amplitudes. The appropriate projection operators are

$$I_{ji}^{1/2}=\tfrac{1}{3}\tau_j\tau_i, \tag{2.9}$$

$$I_{ji}^{3/2}=\delta_{ji}-\tfrac{1}{3}\tau_j\tau_i. \tag{2.10}$$

The connection between the plus-minus amplitudes and the isospin amplitudes A^T (T signifies isospin) is

$$A^{+}=\tfrac{1}{3}(A^{1/2}+2A^{3/2}), \tag{2.11}$$

$$A^{-}=\tfrac{1}{3}(A^{1/2}-A^{3/2}). \tag{2.12}$$

The differential cross section is given by $d\sigma/d\Omega=\frac{1}{2}\,\mathrm{Tr}(f^{+}f)$ where f is given in terms of the "direct" amplitude f_1+xf_2 and the spin flip complitude f_2 by

$$f=f_1+\boldsymbol{\sigma}\cdot\hat{q}_2\boldsymbol{\sigma}\cdot\hat{q}_1f_2, \tag{2.13}$$

where $\hat{q}_j\equiv\mathbf{q}_i/q_i$. In terms of the partial-wave amplitudes $f_{l\pm}=[\exp(2i\delta_{l\pm})-1]/2ik$ for states of angular momentum $j=l\pm\frac{1}{2}$ and parity $(-1)^l$ we have

$$f_1=\sum_l[f_{l+}P_{l+1}'(x)-f_{l-}P_{l-1}'(x)], \tag{2.14}$$

$$f_2=\sum_l(f_{l-}-f_l)P_l'(x). \tag{2.15}$$

Equations (2.14) and (2.15) are inverted by

$$f_{l\pm}=\frac{1}{2}\int_{-1}^{1}dx[f_1P_l(x)+f_2P_{l\pm1}(x)]. \tag{2.16}$$

f_1 and f_2 are related linearly to A and B (of the appropriate isospin) by the matrix $\alpha(s)$:

$$\begin{pmatrix}f_1\\f_2\end{pmatrix}=\alpha(s)\begin{pmatrix}A\\B\end{pmatrix}; \tag{2.17}$$

$$\alpha(s)=\frac{1}{8\pi W}\begin{pmatrix}(E+M) & (E+M)(W-M)\\-(E-M) & (E-M)(W+M)\end{pmatrix}. \tag{2.18}$$

We adhere to the inaccurate custom of calling l the "orbital momentum" although the label $l\pm$ signifies more precisely the total angular momentum and parity of the state. (Although $\mathbf{l}^2$ is not conserved, the centrifugal barrier of a state is determined by l rather than J.) For each value of l, then, one has to consider two values of j and two values of isospin T. To keep track of all these states, we use a "spectroscopic" notation $(l)_{2T,2j}$ in accordance with established practice in low-energy pion physics. For instance the low-energy $j=T=\frac{3}{2}$, p-wave resonance is in our notation labeled P_{33}.

It is also useful to express $f(\theta)$ in terms of angular momentum projection operators $\mathcal{G}_{l\pm}(\hat{q}_2,\hat{q}_1)$:

$$\mathcal{G}_{l+}(\hat{q}_2,\hat{q}_1)=(l+1)P_l(x)-i\boldsymbol{\sigma}\cdot\hat{q}_2\times\hat{q}_1P_l'(x), \tag{2.19}$$

$$\mathcal{G}_{l-}(\hat{q}_2,\hat{q}_1)=lP_l(x)+i\boldsymbol{\sigma}\cdot\hat{q}_2\times\hat{q}_1P_l'(x). \tag{2.20}$$

$f(\theta)$ is then given by

$$f(\theta)=\sum_l[f_{l+}\mathcal{G}_{l+}(\hat{q}_2,\hat{q}_1)+f_{l-}\mathcal{G}_{l-}(\hat{q}_2,\hat{q}_1)]. \tag{2.21}$$

We often use various crossing matrices. For isospin we have the two operators I^{α}, where $\alpha=1$, 2 corresponds to $T=\frac{1}{2}$, $\frac{3}{2}$ (i.e., isospins antiparallel or parallel). We can express I_{ij}^{α} (crossed indices) in terms of the I_{ji}^{β} with the indices in the proper order by means of the isospin crossing matrix $M_{\alpha\beta}$:

$$I_{ij}^{\alpha}=\sum_\beta M_{\alpha\beta}'I_{ji}^{\beta}, \tag{2.22}$$

where M' is the transpose of the matrix M

$$M=\frac{1}{3}\begin{pmatrix}-1 & 4\\2 & 1\end{pmatrix}. \tag{2.23}$$

We have written (2.22) in an unnatural way (transpose of M) so that in the dispersion relations M appears in a matrix product in a natural way. In the same way we consider Eqs. (2.19) and (2.20) for given l; write $\mathcal{G}_\alpha(\hat{q}_2,\hat{q}_1)$, where $\alpha=1$, 2 corresponds to $j=l-\frac{1}{2}$ and $j=l+\frac{1}{2}$ (antiparallel and parallel spin and orbital momentum). The "crossed" projection operator $\mathcal{G}_\alpha(\hat{q}_1,\hat{q}_2)$ is then given in terms of the $\mathcal{G}_\beta(\hat{q}_2,\hat{q}_1)$ (for the same l) by the "static" (see below) angular momentum crossing matrix $N_{\alpha\beta}$:

$$\mathcal{G}_\alpha(\hat{q}_1,\hat{q}_2)=\sum_\beta N_{\alpha\beta}'\mathcal{G}_\beta(\hat{q}_2,\hat{q}_1), \tag{2.24}$$

where N' is the transpose of

$$N=\frac{1}{2l+1}\begin{pmatrix}-1 & 2l+2\\2l & 1\end{pmatrix}. \tag{2.25}$$

We have called N the "static" crossing matrix because in the heavy mass limit, where exchange of an object with given j, l only couples with states of the same l, (2.25) is the complete crossing matrix.

For the complete characterization of a state in terms of T, J, l we write the projection operators

$$\mathcal{P}_\mu(k',k)=I_{k',k}{}^T\mathcal{J}_{l\pm}(\hat{k}',\hat{k}), \tag{2.26}$$

where $\mu=1, 2, 3, 4$, corresponds to $T=\frac{1}{2}$, $j=l-\frac{1}{2}$; $T=\frac{1}{2}$, $j=l+\frac{1}{2}$, $T=\frac{3}{2}$, $j=l-\frac{1}{2}$; $T=\frac{3}{2}$, $j=l+\frac{1}{2}$, respectively. On the left of (2.26) k', k stand for all the relevant (charge and angle) variables. The complete static crossing matrix $A_{\mu\nu}$ is given by

$$\mathcal{P}_\mu(k,k')=\sum_\nu A_{\mu\nu}'\mathcal{P}_\nu(k',k), \tag{2.27}$$

where A' is the transpose of A:

$$A=\frac{1}{3(2l+1)}\begin{bmatrix} 1 & -(2l+2) & -4 & 4(2l+2) \\ -2l & -1 & 8l & 4 \\ -2 & 2(2l+2) & -1 & 2l+2 \\ 4l & 2 & 2l & 1 \end{bmatrix}. \tag{2.28}$$

The contribution of crossed πN scattering to the amplitude $f_\mu(\omega)$ is then, in the no-recoil limit

$$\sum_\nu A_{\mu\nu}\frac{1}{\pi}\int_\mu^\infty\left(\frac{k}{k'}\right)^{2l}\frac{\operatorname{Im}f_\nu(\omega')}{\omega'+\omega}d\omega', \tag{2.29}$$

a straightforward generalization of the familiar crossed term of the Chew-Low theory.[21] In Sec. III we discuss in detail the physical consequences of Eq. (2.28), or more precisely, its relativistic refinement. Here we only note that from (2.29) one can find the effect of a sharp resonance in state ν_1 on the scattering in the states μ (of the same l) by inspection of the ν_1th column of (2.28). For example, exchange of the 33 resonance gives the following coefficients $A_{\mu 4}$ for the states P_{11}, P_{13}, P_{31}, P_{33}, respectively: 16/9, 4/9, 4/9, 1/9.

The entire discussion is based on the fixed-momentum transfer dispersion relations

$$A^\pm(s,t)=\frac{1}{\pi}\int\frac{ds'}{s'-s}\operatorname{Im}A^\pm(s,'t) \pm\frac{1}{\pi}\int\frac{du'}{u'-u}\operatorname{Im}A^\pm(u',t), \tag{2.30}$$

$$B^\pm(s,t)=g^2\left(\frac{1}{M^2-s}\mp\frac{1}{M^2-u}\right)+\frac{1}{\pi}\int\frac{ds'}{s'-s}\operatorname{Im}B^\pm(s',t) \mp\frac{1}{\pi}\int\frac{du'}{u'-u}\operatorname{Im}B^\pm(u',t). \tag{2.31}$$

g^2 is the renormalized $\pi-N$ coupling constant $g^2/4\pi=15$. In these equations the integrations run from $(M+\mu)^2$ to ∞. We suppose that the t-dependent terms which must be added to [(2.30)–(2.31)] are adequately represented[22,23] by the ρ exchange contribution (and possibly some $T=0$ $\pi-\pi$ exchange). Since we are interested mainly in high partial waves (D through H), the ρ exchange contribution, estimated in Appendix B, turns out to be small. We ignore the $T=0$ $\pi\pi$ interaction entirely, since exchange of such states cannot discriminate between $T=\frac{1}{2}$ and $\frac{3}{2}$ $\pi-N$ scattering.

The nucleon pole in $1/(M^2-s)$ (2.31) contributes only to the $J=\frac{1}{2}$ amplitudes and will be dropped except insofar as it is germane to our discussion of bootstrap mechanisms.

The pole at $u=M^2$, due to nucleon exchange, is very important. In terms of isospin amplitudes $B_{\text{pole}}{}^{1/2}=g^2/(M^2-u)$, $B_{\text{pole}}{}^{3/2}=-2g^2/(M^2-u)$. In order to obtain simple formulas for the partial-wave contributions of the pole and u-channel terms it is helpful to rewrite the denominators of the form $w-u$ using Eqs. (2.1)–(2.3) as

$$w-u\equiv 2k^2(y+x), \tag{2.32}$$

$$y=1+(w-u_0(s))/2k^2. \tag{2.33}$$

x is again the cosine of the scattering angle and $u_0(s)\equiv(M^2-\mu^2)^2/s$ is the value u takes along the line $\cos\theta_s=-1$. We shall make frequent use of the expansion[24]

$$\frac{1}{w-u}=\frac{1}{2k^2}\sum_{l=0}^\infty(2l+1)P_l(-x)Q_l(y). \tag{2.34}$$

The centrifugal barrier find its mathematical realization in terms of the Q functions.

For $w=M^2$ in Eq. (2.32) we find the "Born approximation" phase shifts from Eqs. (2.16), (2.17) for isospin $\frac{1}{2}$

$$\delta_{l\pm}{}^B=(g^2/16\pi kW)[(-1)^l(E+M)(W-M)Q_l(y) +(-1)^{l+1}(E-M)(W+M)Q_{l\pm1}(y)]. \tag{2.35}$$

For isotopic spin $\frac{3}{2}$, Eq. (2.35) is to be multiplied by -2. The sign of the phase shift is determined by the Q of lowest index, despite the difference in size of the two coefficients of the Q functions in (2.35). For odd l, states with minimum T and J ($\frac{1}{2}$ and $l-\frac{1}{2}$) and maximum T and J ($\frac{3}{2}$ and $l+\frac{1}{2}$) are attractive while the other states are repulsive. For even l the pattern is reversed. The largest phase shift occurs for the "stretched" configuration of angular momentum and isospin vectors since then the largest Q occurs with the largest coefficient. Thus the Born terms give strong attractions in P_{33}, F_{37}, $H_{3,11}\cdots$ and strong repulsions in D_{35}, G_{39}, $\cdots$. Further discussion and numerical results are given below.

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[23] J. Bowcock, W. N. Cottingham, and D. Lurie, Nuovo Cimento **16**, 918 (1960).

[24] E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge University Press, New York, 1952), 4th ed., p. 321.

In the following we do not write the pole terms explicitly. Using relations (2.11)–(2.12), one finds dispersion relations for the isospin amplitudes:

$$A^j(s,t)=\frac{1}{\pi}\int\frac{ds'}{s'-s}\,\mathrm{Im}A^j(s',t)+\sum_k M_{jk}\frac{1}{\pi}\int\frac{du'}{u'-u}\,\mathrm{Im}A^k(u',t)\,,\quad (2.36)$$

$$B^j(s,t)=\frac{1}{\pi}\int\frac{ds'}{s'-s}\,\mathrm{Im}B^j(s',t)-\sum_k M_{jk}\frac{1}{\pi}\int\frac{du'}{u'-u}\,\mathrm{Im}B^k(u',t)\,.\quad (2.37)$$

The coefficients of the isospin crossing matrix are given by Eq. (2.23). From these equations one can find f_1 and f_2 and in turn equations for the partial-wave amplitudes. This problem is discussed in the following section.

III. COUPLING OF PARTIAL-WAVE AMPLITUDES

In order to find the partial-wave dispersion relations we convert the simple equations for the invariant amplitudes to those for f_{1j}, f_{2j}. From Eqs. (2.17)–(2.18) and (2.36)–(2.37) one gets

$$\begin{pmatrix} f_{1j}(s,t)\\ f_{2j}(s,t)\end{pmatrix}=\frac{1}{\pi}\int\frac{ds'}{s'-s}C(s,s')\,\mathrm{Im}\begin{pmatrix} f_{1j}(s',t)\\ f_{2j}(s',t)\end{pmatrix}+\sum_k M_{jk}\frac{1}{\pi}\int\frac{du'}{u'-u}D(s,u')\,\mathrm{Im}\begin{pmatrix} f_{1k}(u',t)\\ f_{2k}(u',t)\end{pmatrix},\quad (3.1)$$

where the matrices C and D are given by

$$C(s,s')\equiv\alpha(s)\alpha^{-1}(s')\,,\quad (3.2)$$

$$D(s,s')\equiv\alpha(s)\sigma_3\alpha^{-1}(s')\,,\quad (3.3)$$

where σ_3 is the diagonal Pauli spin matrix. Explicitly $C(s,s')$ is given by

$$C(s,s')=\begin{bmatrix}\dfrac{(W+W')(E+M)}{2W(E'+M)} & \dfrac{(W-W')(E+M)}{2W(E'-M)}\\[2ex] \dfrac{(W-W')(E-M)}{2W(E'+M)} & \dfrac{(W+W')(E-M)}{2W(E'-M)}\end{bmatrix}.\quad (3.4)$$

Instead of writing out the full expression for $D(s,s')$ we find it convenient to separate D into two parts:

$$D(s,s')=D^1(s,s')+D^2(s,s')\,,\quad (3.5)$$

$$D^1(s,s')=C(s,s')\sigma_3\,,\quad (3.5a)$$

$$D^2(s,s')=\alpha(s)[\sigma_3,\,\alpha^{-1}(s')]\,;\quad (3.5b)$$

except for the 12 element, the elements of D^2 are substantially smaller than those of D^1. D^2 is given by

$$D^2(s,s')=\begin{bmatrix}-\dfrac{(W-M)(E+M)}{W(E'+M)} & -\dfrac{(W'-M)(E+M)}{W(E'-M)}\\[2ex] -\dfrac{(W+M)(E-M)}{W(E'+M)} & \dfrac{(W'-M)(E-M)}{W(E'-M)}\end{bmatrix}.\quad (3.6)$$

We are especially interested in the effect of resonances in the crossed channel, i.e., what forces are generated by the exchange of various π-N "isobars?" (We shall find such forces to be essential in generating the resonances themselves.) Thus we represent the scattering amplitude under the integrals by the usual truncated Legendre series, Eqs. (2.14)–(2.15). The cosine x' of the scattering angle at energy s' and momentum transfer t is related to that (x) at energy s and momentum transfer t by

$$x'=\xi x+a;\quad \xi=k^2/(k')^2,\quad a=1-\xi.\quad (3.7)$$

For brevity, we label an f amplitude with a prime whenever it refers to the (s', t) variables. A given Legendre function $P_l(x')$ under the integral can be expressed as a sum of $P_n(x)$ where n runs from 0 to l using identities given in Appendix A. This procedure simplifies and systematizes the task of working out the partial-wave projection of the right-hand side of Eq. (3.1).

We now discuss in detail the first ("direct") term in (3.1). From (3.1) and (2.16) we find, suppressing the diagonal isospin index,

$$f_{l\pm}{}^d(s)=\frac{1}{\pi}\int\frac{ds'}{s'-s}\,\mathrm{Im}\Big[\sum_{j=1}^{2}C_{1j}(f_j')^l+C_{2j}(f_j')^{l\pm1}\Big];\quad (3.8)$$

$$(f_j'(s'))^l\equiv\frac{1}{2}\int_{-1}^{1}dxP_l(x)f_j'(s',x')\,.\quad (3.9)$$

Throughout the following it is assumed that the influence of waves of orbital momentum $l'>l$ can be neglected where l is the orbital momentum of the state of interest. (This assumption is valid for F, G, H states but not D. See Sec. IV.) It follows immediately that C_{12} does not contribute to (3.8), since $f_2'\propto P_{l'-1}(x)+\cdots$, where $+\cdots$ denotes P_n's of order lower than $l'-1$. Similarly no partial wave with $l'<l$ can contribute to $f_l{}^d$ in the C_{11} term. For $l'=l-1$ there is a C_{21} contribution to $(f_1')^{l-1}$ (for $j=l-\frac{1}{2}$) from the $j'=l'+\frac{1}{2}$ amplitude. However C_{21} is extremely small ($C_{21}<k^2/8M^2$) even at a pion lab energy of a BeV. We ignore C_{21} henceforth. [For $l'<l-1$ there is no C_{21} contribution to $f_{l\pm}{}^d(s)$.] Thus, with the neglect of the very small coefficient C_{21}, we have found that terms in f' with $l'<l$ do not contribute to $f_{l\pm}$. Using the definition (3.9)

and the results of Appendix A, we find

$$f_{l+}{}^{d}(s)=\frac{1}{\pi}\int\frac{ds'}{s'-s}\left(\frac{k}{k'}\right)^{2l}C_{11}(s,s')\,\mathrm{Im}f_{l+}(s')\,,\tag{3.10}$$

$$f_{l-}{}^{d}(s)=\frac{1}{\pi}\int\frac{ds'}{s'-s}\left(\frac{k}{k'}\right)^{2l-2}C_{22}(s,s')\,\mathrm{Im}f_{l-}(s')$$
$$+\frac{1}{\pi}\int\frac{ds'}{s'-s}\left(\frac{k}{k'}\right)^{2l}C_{11}(s,s')$$
$$\times(1-C_{22}/C_{11}\xi)\,\mathrm{Im}f_{l+}(s')\,.\tag{3.11}$$

From (3.4) it is observed that C_{11} is practically unity and $C_{22}=(E'+M)^2(C_{11}\xi)/(E+M)^2$ is nearly ξC_{11} (for $s=s'$, $C_{11}=1$ and $C_{22}=C_{11}$). Even for s' rather different than s these approximations are fairly good. Moreover, in applications the contributions of s' nearly equal to s are especially significant for $l'=l$. Although it is not difficult to carry along the exact factors in (3.10)–(3.11) we shall write these equations in the compact approximate form

$$f_{l\pm}{}^{d}(s)=\frac{1}{\pi}\int\frac{ds'}{s'-s}\left(\frac{k}{k'}\right)^{2l}\mathrm{Im}f_{l\pm}(s')\,.\tag{3.12}$$

[The relation to the static nucleon theory is direct, since $ds'/(s'-s)=d\omega_L'/(\omega_L'-\omega_L)$.] In Fig. 1, C_{11}, C_{22}/ξ, and $C_{11}-C_{22}/\xi$ are plotted for representative s and s'. It is clear that if $\mathrm{Im}f_{l-}$ is very small but f_{l+} resonant then (3.12) will be a poor approximation to (3.11). But in this case the contributions of the other terms in (3.1) will be more important than the direct term.

We now turn to the more complicated analysis of the "crossed" contribution [the second term on the right-hand side of Eq. (3.11)] to the lth partial waves ($f_l{}^c$). Again we ignore contributions to f' having $l'>l$. The essentially new feature of this term as compared to the direct term [apart from the substitution of $D(s,s')$ for $C(s,s')$] is the occurrence of the angular dependence in the denominator $(u'-u)^{-1}$. In our work this quantity has the physical significance of the propagator of the exchanged baryonic resonances. The angular dependence permits states of arbitrarily small l' to contribute to $f_l{}^c$. To find these contributions we expand the propagator in a "multipole expansion," Eq. (2.34). Frequent use is made of the following essential property of the Q functions: For arguments y relevant to the kinematical variables in the region of the higher resonances the Q functions decrease rapidly with increasing index. Physically the Q functions represent a centrifugal barrier effect. For example if $l=3$, $l'=1$, the P_2 term of (2.34) is the lowest that contributes to the F state ($l=3$). The factor $P_2(x)$, which represents the minimum orbital momentum (2 units) needed to supplement the "spin" of the exchange p state to give an F-state contribution, is accompanied by a centrifugal barrier (Q_2) appropriate to $l'=2$. The p-state angular factor occurs in the numerator and is not inhibited by a centrifugal barrier. The height of this barrier is determined by the average mass $(s')^{1/2}$ of the exchanged state. The factors governing the relative importance of the exchange of small l' states are thus (a) the mass of the exchanged object and (b) the order of the Q function. However it is not true that one can terminate the series as soon as the first (lowest index) significant Q contribution is found; because of the structure of $D(s,s')$ it is essential to keep the next higher index Q as well to obtain all contributions of the same order of magnitude. All these remarks are clarified by the explicit calculations carried out below.

We first discuss the contribution of the $l'=l$ component of f' to f^c. The non-negligible contribution comes from the term Q_0-3xQ_1 in the expansion (2.34). The analysis of the angle-independent Q_0 term parallels the discussion of f^d. Neglecting D_{21} as we did C_{21}, the Q_0 contribution is

$$\begin{pmatrix}f_{l-}{}^{c}(s)\\ f_{l+}{}^{c}(s)\end{pmatrix}=\frac{1}{2\pi k^2}\int ds'Q_0\left(1+\frac{s'-u_0}{2k^2}\right)$$
$$\times A^0(s,s')\left(\frac{k}{k'}\right)^{2l}\mathrm{Im}\begin{pmatrix}f_{l-}(s')\\ f_{l+}(s')\end{pmatrix}\tag{3.13}$$

leaving implicit the isospin sum implied by (3.1). A^0 is given by

$$A^0(s,s')=\begin{bmatrix}-\dfrac{(2M+W-W')(E'+M)}{2W(E+M)} & \dfrac{(W+W')}{2W}\left[\dfrac{E+M}{E'+M}+\dfrac{E'+M}{E+M}\right]+\Delta\\ 0 & \dfrac{(2M+W'-W)(E+M)}{2W(E'+M)}\end{bmatrix},\tag{3.14}$$

$$\Delta\equiv-\left(\frac{W-M}{W}\right)\left(\frac{E+M}{E'+M}\right)-\left(\frac{W'-M}{W}\right)\left(\frac{E'+M}{E+M}\right).\tag{3.15}$$

In Fig. 2 we have plotted the elements of A^0 as functions of W for an especially interesting case, $W'=W_{33}=8.86$. The general features are given by the special case $W=W'$:

$$A^0(s,s)=\frac{M}{W}\begin{pmatrix}-1 & 2\\ 0 & 1\end{pmatrix}\tag{3.16}$$

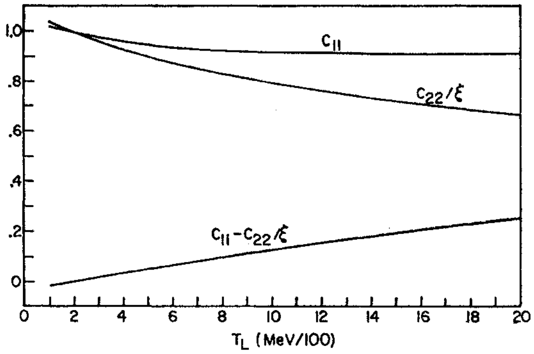


FIG. 1. Some relevant coefficients of the matrix C of Eq. (3.4) are plotted for W' equal to the value W_{33} appropriate to the 3–3 resonance.

which gives the appropriate "static" limit

$$A^0{}_{\text{static}}=\begin{pmatrix}-1 & 2\\ 0 & 1\end{pmatrix}. \tag{3.17}$$

It is noted that $A_{12}{}^0$, an especially important coefficient, is fairly close to the static value (2). Before discussing the remaining coefficients we consider the second term $-3xQ_1$ which gives rise to the l dependence of the crossing matrix.

The contribution of the Q_1 term to $f_{l\pm}{}^c$ is

$$f_{l\pm}{}^c(s)=-\frac{3}{2\pi k^2}\int ds' Q_1\left(1+\frac{s'-u_0}{2k^2}\right)\times \operatorname{Im}\sum_{j=1}^{2}\left[D_{1j}(xf_j')^l+D_{2j}(xf_j')^{l\pm 1}\right]. \tag{3.18}$$

Neglecting D_{21} we find that (3.18) reduces to

$$f_{l-}{}^c(s)=\frac{-3}{2\pi k^2}\int ds' Q_1\left(1+\frac{s'-u_0}{2k^2}\right)\xi^l \operatorname{Im}\left\{\left[\frac{lD_{12}}{(2l+1)\xi}+\frac{(l-1)(1-\xi)D_{22}}{\xi^2}\right]\left[f_{l-}(s')-f_{l+}(s')\right]+lD_{11}(1-\xi)\xi^{-1}f_{l+}(s')\right\}. \tag{3.19}$$

$$f_{l+}{}^c(s)=\frac{-3}{2\pi k^2}\int ds' Q_1\left(1+\frac{s'-u_0}{2k^2}\right)\xi^l \times \operatorname{Im}\left\{\frac{l(1-\xi)}{\xi^2}D_{11}f_{l+}(s')+\frac{lD_{12}}{(2l+1)\xi}\times\left[f_{l-}(s')-f_{l+}(s')\right]\right\}. \tag{3.20}$$

The D_{12} terms in Eqs. (3.19)–(3.20) are the largest for $\xi\lesssim 1$. First of all for the substantial energies of interest D_{12} is about an order of magnitude bigger than D_{11} or D_{22}. Secondly, one is concerned here with waves of the same l, so that $\xi=k^2/(k')^2$ is of order unity in the relevant energy range. For $\xi>1$, $(\xi-1)/\xi^2$ has a maximum value of 0.25. Thus to better than, say 10%, one has for the Q_1 term

$$\begin{pmatrix}f_{l-}{}^c\\ f_{l+}{}^c\end{pmatrix}\cong\frac{-1}{2\pi k^2}\int ds' Q_0\left(1+\frac{s'-u_0}{2k^2}\right)\times A_l{}^1(s,s')\left(\frac{k}{k'}\right)^{2l}\operatorname{Im}\begin{pmatrix}f_{l-}(s')\\ f_{l+}(s')\end{pmatrix}, \tag{3.21}$$

$$A_l{}^1(s,s')\equiv\left(\frac{3yQ_1(y)}{Q_0(y)}\right)\left(\frac{-D_{12}(s,s')}{2\xi y}\right)\times\left(\frac{2l}{2l+1}\right)\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}. \tag{3.22}$$

The grouping of terms in (3.17) is motivated as follows: To a very good approximation one may replace Q_1 by its asymptotic form $Q_1\cong Q_0/3y$. For s' appropriate to the P_{33} resonance, $3Q_1y/Q_0$ is 1.05 at $T_L=1$ BeV and 1.1 at 2 BeV. For s' appropriate to the F_{15} resonance the deviation from unity is less than 0.01 for $T_L=1$ BeV and about 0.04 for $T_L=1$ BeV. (This approximation is not necessary, but makes the form of the resulting equation more transparent.) The factor $-D_{12}/2\xi y$ is just unity in the static limit as can be seen by writing y in the form $2M(\omega_L+\omega_L')/2k^2-1$ (ω_L is the pion lab energy) and neglecting the -1:

$$\frac{-D_{12}}{2\xi y}\cong\left(\frac{W+W'-2M}{\omega_L+\omega_L'}\right)\left(\frac{E+M}{2W}\right)\left(\frac{E'+M}{2M}\right). \tag{3.23}$$

Thus the "static" crossing matrix given by Eq. (2.25) is an approximation to the sum A^0+A^1 given in Eqs. (3.15) and (3.17); in the heavy mass limit we have

$$A^0+A^1\to\begin{pmatrix}-1 & 2\\ 0 & 1\end{pmatrix}+\frac{2l}{2l+1}\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix} \tag{3.24}$$

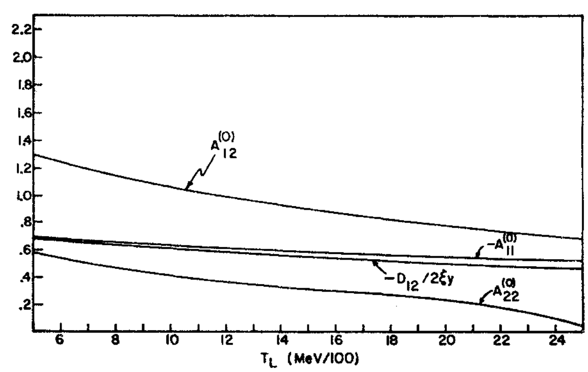


FIG. 2. The energy dependence of various components of the crossing matrix is depicted for $W'=W_{33}$. The corresponding "static" values are $A_{12}{}^0=2$, $A_{22}{}^0=-A_{11}{}^0=-D_{12}/2\xi y=1$.

which yields just Eq. (2.25). The qualitative features of the static crossing matrix persist in the energy-dependent crossing matrix $A=A^0+A^1$.

Figure 2 shows the behavior of $-D_{12}/(2\xi y)$ as a function of W for W' fixed at the P_{33} resonance value. For other values of W' [e.g., $W'=W$, $W'=W(F_{15})$] the curve may be lowered by as much as 10% at low values of W. When $-D_{12}/(2\xi y)$ is multiplied by the correction $3Q_1y/Q_0$, the curve almost coincides with that for $A_{11}{}^0$. Thus the crossing matrix elements [Eq. (3.15) plus Eq. (3.21)] are affected by kinematical corrections as follows. A_{11} is essentially the same as the static value but scaled down by the factor $|A_{11}{}^0|\approx|D_{12}|/2\xi y$. This element is rather small in any case due to the near cancellation between $A_{11}{}^0$ and $A_{11}{}^1$ [see Eq. (3.21) for the static value]. In the static A, A_{22} is very small due to cancellation; this cancellation is even more complete in the present case. For large $W(T_L>1$ BeV) A_{22} may become slightly negative. Since $A_{21}{}^0\cong 0$, A_{21} is given by the static value reduced by the factor $(3yQ_1/Q_0)(-D_{12}/2\xi y)$ shown in Eq. (3.21). This amounts to about 35% reduction at 900 MeV lab energy (Fig. 2) for P_{33} exchange. $A_{12}{}^0$ is rather smaller than the static value, as is $A_{12}{}^1$, which has a smaller *negative* value and therefore compensates somewhat the decrease of $A_{12}{}^0$. If we consider moderately large ξ, then the D_{11} term in (3.19) tends to cancel the D_{12} term. Thus for large ξ $(A^1)_{12}$ is smaller than given by (3.22) so that A_{12} is roughly equal to $A_{12}{}^0$, which is not very different from the static value of A_{12}. In summary, the unimportant diagonal terms are reduced to values even smaller than their "static" values. The effect of exchanging a $j=l+\frac{1}{2}$ object on a $j=l-\frac{1}{2}$ state and the effect of exchanging a $j=l-\frac{1}{2}$ object on a $j=l+\frac{1}{2}$ state is decreased characteristically by 20–30% for typical choices of kinematical parameters.

It is appropriate to summarize the results obtained thus far before deriving the contributions due to the exchange of low partial-wave resonances. By neglecting contributions with $l'>l$ and neglecting certain small kinematical factors we have found the approximate partial-wave dispersion relations

$$f_{l\mu}(s)=\frac{1}{\pi}\int\frac{ds'}{s'-s}\left(\frac{k}{k'}\right)^{2l}\operatorname{Im}f_{l\mu}(s')+\frac{1}{2\pi k^2}\sum_{\nu}\int ds'Q_0\left(1+\frac{s'-u_0}{2k^2}\right)\left(\frac{k}{k'}\right)^{2l}\times A_{\mu\nu}(s,s')\operatorname{Im}f_{l\nu}(s'). \quad (3.24)$$

One still has to add to (3.24) the contribution of the pole terms [Eq. (2.35)], ρ exchange (Appendix B) and the yet-to-be-found contributions of partial waves with $l'<l$. Equation (3.24) shows that coupling of states of the same l is very similar to that obtained using the static crossing matrix, Eq. (2.28). In particular the large components A_{14} and A_{41} are especially relevant for the operation of the bootstrap mechanism for arbitrary l, as discussed in a previous note.[9] We defer a discussion of the physical content of Eq. (3.24) to Sec. IV.

We next consider the effect of states in the crossed channel having $l'<l$. As in the preceding case, one cannot terminate the series at the first nonzero contribution Q_n $(n=l-l')$. Here too, part of the Q_{n+1} term gives a contribution of the same order of magnitude. In fact, by a slight modification we can take over the results found above for $l'=l$. Consider the sum

$$\sum_n(-1)^n(2n+1)P_n(x)\frac{1}{2\pi k^2}\int ds'Q_n\left(1+\frac{s'-u_0}{2k^2}\right)D(s,s')\operatorname{Im}\begin{pmatrix}f_1'\\f_2'\end{pmatrix}. \quad (3.25)$$

Consider the n and $(n+1)$ terms; using identity (A1),

$$\begin{aligned}(2n+1)P_n(x)Q_n-(2n+3)P_{n+1}(x)Q_{n+1}\\=(2n+1)P_n(x)\{Q_n-[(2n+3)/(n+1)]xQ_{n+1}\}\\+\text{const}P_{n-1}(x),\quad (3.26)\end{aligned}$$

where the term constP_{n-1} does not contribute for $n=l-l'$. Dropping P_{n-1}, we note further that because of the asymptotic form appropriate here, $Q_{n+1}(y)\cong(n+1)Q_n(y)/(2n+3)y$ Eq. (3.26) is effectively

$$(2n+1)P_n(x)Q_n(y)(1-x/y), \quad (3.27)$$

in exact analogy to the case $l'=l$. As there we need not commit ourselves to the approximation (3.27) but can insert the ratio Q_{n+1}/Q_n explicitly in the crossing matrix. In the case $l'=l$ we learned that the angular momentum composition of $(1-x/y)f_i'$ is given by the second term of (3.24) (for $l\to l'$). Thus the n and $(n+1)$ terms of (3.25) contribute to $f_1+\boldsymbol{\sigma}\cdot\hat{q}_2\boldsymbol{\sigma}\cdot\hat{q}_1f_2$ the amount (recall that we are taking one value of l' in the f_i' amplitudes and systematically dropping angular factors that do not contribute to f_l):

$$(-1)^n(2n+1)P_n(x)\frac{1}{2\pi k^2}\int ds'Q_n\left(1+\frac{s'-u_0}{2k^2}\right)\times\sum_\mu\mathcal{G}_\mu(\hat{q}_2,\hat{q}_1)A_{\mu\nu}{}^{l'}(s,s')\left(\frac{k}{k'}\right)^{2l'}\operatorname{Im}f_\nu(s'), \quad (3.28)$$

where the indices μ, ν and the crossing matrix $A_{\mu\nu}{}^{l'}$ refer to orbital momentum l' [more precisely $A_{\mu\nu}$ will differ slightly from that used before in that $3Q_1/Q_0$ will be replaced by a different ratio, discussed above Eq. (3.27)]. It is then straightforward to compute the coefficient of $\mathcal{G}_{l\pm}(\hat{q}_2,\hat{q}_1)$ occurring in the product $P_{l-l'}(x)\mathcal{G}_{l\pm}(\hat{q}_2,\hat{q}_1)$. The general crossing matrix will then involve in a crucial way the coefficients $C_n(l,l';\mu\beta)$

defined by

$$P_n(x)\mathcal{J}_{l'\mu}(\hat{q}_2,\hat{q}_1)=\sum_{l\beta} C_n(l,l';\mu\beta)J_{l\beta}(\hat{q}_2,\hat{q}_1)\,. \quad (3.29)$$

To illustrate the simplicity of the procedure, we work out in detail the contribution of P_{33} exchange to the F waves. First note that the exchange of isospin $\frac{3}{2}$ leads mostly to isospin $\frac{1}{2}$ since the second column of the isospin crossing matrix, Eq. (2.23) gives $\frac{4}{3}$ for $T=\frac{1}{2}$ and $\frac{1}{3}$ for $T=\frac{3}{2}$. In the following, $A_{\mu\nu}$ is the angular momentum part of the crossing matrix, given by Eqs. (3.14) and (3.17). Thus (3.28) gives

$$\binom{4}{1}\left(\frac{5}{6\pi k^2}\right)P_2(x)\int ds' Q_2\left(1+\frac{s'-u_0}{2k^2}\right)\left(\frac{k}{k'}\right)^2 \times\{\mathcal{J}_{1-}(\hat{q}_2,\hat{q}_1)A_{12}{}^1(s,s')+\mathcal{J}_{1+}(\hat{q}_2,\hat{q}_1)A_{22}{}^1(s,s')\} \times\mathrm{Im} f_{P_{33}}(s')\,, \quad (3.30)$$

where the 4, 1 standing in front refer to isospin $\frac{1}{2}$ and $\frac{3}{2}$, respectively. The largest coefficient is A_{12}, so we expect the $P_2(x)\mathcal{J}_{1-}$ angular factor to be the decisive one. (The static values for $l'=1$ are $\frac{4}{3}$ and $\frac{1}{3}$; the ratio A_{12}/A_{22} is substantially larger in the exact expression.) Using the definitions (2.19) and (2.20), Appendix A, one finds

$$P_2(x)\mathcal{J}_{1-}(\hat{q}_2,\hat{q}_1)=(7/35)\mathcal{J}_{3-}(\hat{q}_2,\hat{q}_1)+\cdots, \quad (3.31)$$

$$P_2(x)\mathcal{J}_{1+}(\hat{q}_2,\hat{q}_1)=(2/35)\mathcal{J}_{3}(\hat{q}_2,\hat{q}_1)+(9/35)\mathcal{J}_{3+}(\hat{q}_2,\hat{q}_1)+\cdots, \quad (3.32)$$

where $+\cdots$ indicates contributions to p waves. Thus for $F_{5/2}$ we obtain the energy-dependent crossing coefficient

$$\Gamma_{3-}=(7A_{12}+2A_{22})/35\,, \quad (F_{5/2}) \quad (3.33)$$

and for $F_{7/2}$

$$\Gamma_{3+}=9A_{22}/35\,, \quad (\Gamma_{7/2})\,. \quad (3.34)$$

The "static" values $A_{12}{}^s=\frac{4}{3}$, $A_{22}{}^s=\frac{1}{3}$ give $\Gamma_{3-}=10/35$ and $\Gamma_{3+}=3/35$ for $F_{5/2}$ and $F_{7/2}$, respectively. Thus we obtain the contribution of P_{33} exchange to the F states:

$$\binom{4}{1}\left(\frac{5}{6\pi k^2}\right)\int ds' Q_2\left(1+\frac{s'-u_0}{2k^2}\right)\left(\frac{k}{k'}\right)^2 \times\Gamma_{3\pm}(s,s')\,\mathrm{Im} f_{P_{33}}(s')\,. \quad (3.35)$$

For the static values of $\Gamma_{3\pm}{}^s$ the ratio of (attractive) forces in the F states in $F_{15}{:}F_{17}{:}F_{35}{:}F_{37}=40{:}12{:}10{:}3$, so that the F_{15} force is by far the most significant one induced by P_{33} exchange.

To make clear the pattern of forces we consider another case in detail, the contribution of D_{13} exchange (the 600-MeV resonance) to the F states. Here the Q_1 term is relevant:

$$\binom{1}{-2}\left(\frac{P_1(x)}{2\pi k^2}\right)\int ds' Q_1\left(1+\frac{s'-u_0}{2k^2}\right)\left(\frac{k}{k'}\right)^4 \times\{A_{11}\mathcal{J}_{2-}(\hat{q}_2,\hat{q}_1)+A_{21}\mathcal{J}_{2+}(\hat{q}_2,\hat{q}_1)\}\,\mathrm{Im} f_{D_{13}}(s')\,. \quad (3.36)$$

(The static values of A for $l'=2$ are $A_{11}{}^s=-\frac{1}{5}$ and $A_{21}{}^s=\frac{4}{5}$.) We can use the general formulas

$$P_1(x)\mathcal{J}_{(l-1)-}(\hat{q}_2,\hat{q}_1)=[(l-1)/(2l-1)]\mathcal{J}_{l-}(\hat{q}_2,\hat{q}_1)+\cdots, \quad (3.37)$$

$$P_1(x)\mathcal{J}_{(l-1)+}(\hat{q}_2,\hat{q}_1)=[1/(4l^2-1)][\mathcal{J}_{l-}(\hat{q}_2,\hat{q}_1)+l(2l-1)\mathcal{J}_{l+}(\hat{q}_2,\hat{q}_1)]+\cdots \quad (3.38)$$

(where $+\cdots$ indicates the noncontributing terms of low l) to obtain

$$P_1(x)\mathcal{J}_{2-}(\hat{q}_2,\hat{q}_1)=(14/35)\mathcal{J}_{3-}(\hat{q}_2,\hat{q}_1)+\cdots, \quad (3.39)$$

$$P_1(x)\mathcal{J}_{2+}(\hat{q}_2,\hat{q}_1)=(1/35)\mathcal{J}_{3-}(\hat{q}_2,\hat{q}_1)+(15/35)\mathcal{J}_{3+}(\hat{q}_2,\hat{q}_1)+\cdots, \quad (3.40)$$

which yield the crossing coefficients

$$\Gamma_{3-}{}^d=(14A_{11}+A_{21})/35 \quad (F_{5/2})\,, \quad (3.41)$$

$$\Gamma_{3+}{}^d=15A_{21}/35 \quad (F_{7/2})\,. \quad (3.42)$$

The "static" values of $\Gamma_{3\pm}{}^d$ are $\Gamma_{3-}{}^d=-2/35$, $\Gamma_{3+}{}^d=15/35$. In this approximation the ratio of forces $F_{15}{:}F_{17}{:}F_{35}{:}F_{37})$ is $-2{:}15{:}4{:}-30$. The result for D_{13} exchange in the F states is thus summarized by

$$\binom{1}{-2}\frac{1}{2\pi k^2}\int ds' Q_1\left(1+\frac{s'-u_0}{2k^2}\right)\left(\frac{k}{k'}\right)^4 \times\Gamma_{3\pm}{}^d\,\mathrm{Im} f_{D_{13}}(s')\,. \quad (3.43)$$

From the preceding examples one can see some simple rules. The isospin crossing matrix shows that exchange of a $T=\frac{3}{2}$ object gives primarily a $T=\frac{1}{2}$ force; similarly $T=\frac{3}{2}$ is favored by exchange of $T=\frac{1}{2}$. A similar situation holds within the set of states of the same l: the off-diagonal elements (A_{21} and A_{12}) dominate so that exchange of an object with spin and orbital momentum "parallel" induces the strongest force in the "antiparallel" configuration and vice versa. Putting together the above considerations, one arrives at the bootstrap situation where the exchange of $T=\frac{1}{2}$, $j=l-\frac{1}{2}$ helps create a $T=\frac{3}{2}$, $J=l+\frac{1}{2}$ state, the exchange of which enhances the $T=\frac{1}{2}$, $J=l-\frac{1}{2}$ state. The case of $l'<l$ is slightly more complicated. First there is a factor $(-1)^{l-l'}$ from the expansion of the denominator $(u'-u)$, which factor causes an alteration between repulsion and attraction. If a state of $J=l+\frac{1}{2}$ is exchanged, the behavior is determined mainly by $P_{l-l'}(x)\mathcal{J}_{l'-}$ because of the dominance of A_{12} relative to A_{22}. From Eqs. (3.31) and (3.39) one sees examples of the way $P_{l-l'}\mathcal{J}_{l'-}$ contains mainly $\mathcal{J}_{l-}$. Similarly when $j=l-\frac{1}{2}$ is exchanged the resulting amplitude is principally determined by $P_{l-l'}\mathcal{J}_{l'+}$, as A_{21} dominates A_{11}. As in the special cases of Eqs. (3.32) and (3.40), $P_{l-l'}\mathcal{J}_{l'+}$ gives substantially more $\mathcal{J}_{l+}$ than $\mathcal{J}_{l-}$. Thus the large forces due to exchange occur in the states of oppositely directed isospin vectors and angular momentum vectors; the sign of the force is determined by the

coefficient $(-1)^{l-l'}$ in the "multipole expansion" of the "propagator" $(u'-u)^{-1}$, except when $T=\frac{1}{2}$ is exchanged, in which case the sign is opposite in the $T=\frac{1}{2}$ is exchanged, in which case the sign is opposite in the $T=\frac{1}{2}$ state. The following formulas are useful:

$$P_2(x)\mathcal{g}_{l-}(\hat{q}_2,\hat{q}_1)=\frac{3l(l+1)}{2(2l+1)(2l+3)}\mathcal{g}_{(l+2)-}+\cdots, \quad (3.44)$$

$$P_2(x)\mathcal{g}_{l+}(\hat{q}_2,\hat{q}_1)=\frac{3(l+1)}{2(2l+1)(2l+3)(2l+5)}\{(2l+1)(l+2)\times\mathcal{g}_{(l+2)-}+2\mathcal{g}_{(l+2)+}\}+\cdots, \quad (3.45)$$

dropping additional terms containing $\mathcal{g}_{l\pm}$ and $\mathcal{g}_{(l-2)\pm}$ contributions.

To obtain a compact notation, we observe that the exchange contributions can all be written in the form

$$\sum_\nu \frac{1}{2\pi k^2}\int ds' Q_{l-l'}\left(1+\frac{s'-u_0}{2k^2}\right)\left(\frac{k}{k'}\right)^{2l'} \times\Gamma_{\mu\nu}(s,s')\,\mathrm{Im}f_\nu(s'), \quad (3.46)$$

where the index μ signifies $l\pm\frac{1}{2}$ and the isospin of the state of interest [cf. Eq. (3.24)] but the ν sum runs over the various l' terms contributing, and includes the information of $l'\pm\frac{1}{2}$ and isospin. The $\Gamma_{\mu\nu}(s,s')$ includes the Γ's defined in Eqs. (3.33) and (3.34), (3.41) and (3.42), e.g., and also contains the factor $(-1)^{l-l'}\times[2(l-l')+1]$ as well as the appropriate component of the isospin crossing matrix. Explicitly $\Gamma_{\mu\nu}$ is given by

$$\Gamma_{\mu\nu}=(-)^{l-l'}(2l-2l'+1)M_{\mu\nu}\textstyle\sum_\alpha A^{l'}{}_{\alpha\nu}(s,s') \times C_{l-l'}(l,l';\mu\alpha). \quad (3.47)$$

In all numerical calculations we have taken into account the energy dependence of the $\Gamma_{\mu\nu}$ coefficients. For illustration we write out the result for the F_{15} state, including the Born terms, P_{33} and D_{13} exchange and the F-wave "bootstrap":

$$\begin{aligned}f_{F_{15}}(s) &= f_{F_{15}}{}^B(s)+\frac{1}{\pi}\int\frac{ds'}{s'-s}\left(\frac{k}{k'}\right)^6 \mathrm{Im}f_{F_{15}}(s') \\ &+\frac{1}{2\pi k^2}\sum_\nu\int ds' Q_0\left(1+\frac{s'-u_0}{2k^2}\right)A_{1\nu}(s,s')\left(\frac{k}{k'}\right)^6 \\ &\times\mathrm{Im}f_\nu(s')+\frac{1}{2\pi k^2}\int ds' Q_2\left(1+\frac{s'-u_0}{2k^2}\right)\Gamma(F_{15};P_{33}) \\ &\times\left(\frac{k}{k'}\right)^2\mathrm{Im}f_{P_{33}}(s')+\frac{1}{2\pi k^2}\int ds' Q_1\left(1+\frac{s'-u_0}{2k^2}\right) \\ &\times\Gamma(F_{15};D_{13})\left(\frac{k}{k'}\right)^4\mathrm{Im}f_{D_{13}}(s'). \quad (3.48)\end{aligned}$$

The sum over ν in Eq. (3.48) runs over the four F states. The "static" values are $A_{14}=32/21$, $A_{13}=-4/21$, $A_{12}=-8/21$, $A_{11}=1/21$, $\Gamma(F_{15},P_{33})=40/21$, $\Gamma(F_{15},D_{13})=-2/35$.

IV. FORCES DUE TO EXCHANGE OF NUCLEONIC STATES

In the preceding section we obtained an approximate system of coupled partial-wave dispersion relations. With an assist from experiment these equations can be decoupled by inspection, i.e., the qualitative pattern of forces coincides so completely with the experimental results that one has confidence that representing the crossed channel by empirical results (the energy, width, and quantum numbers of each resonance) will be sensible. By virtue of its small mass, the nucleon occupies a critical place in the sequence of nucleonic states. As discussed in Sec. III, the strongest attractive forces due to nucleon exchange occur in P_{33}, F_{37}, $H_{3,11}\cdots$, i.e., in the states of "stretched" isospin, angular momentum configurations of $\Delta l=2$. Likewise strong relatively long-range repulsive forces are induced in D_{35}, G_{39}, $\cdots$. The well-known theory of the P_{33} resonance is especially simple in that the contributing force is primarily due to one process, nucleon exchange, though in the usual approach[21] the effects of short-range forces are lumped into one adjustable constant, the high-energy cutoff. The "reciprocal bootstrap" relation advocated by Chew,[11] whereby P_{33} exchange generates nucleon-sustaining forces in the P_{11} state is the crucial notion which we here generalize to the entire set of (j,l,T) states. The first step was the recognition that the qualitative character of the static p-wave bootstrap generalizes to arbitrary l, i.e., the exchange of $T=\frac{1}{2}$, $j=l-\frac{1}{2}$ induces $T=\frac{3}{2}$, $J=l+\frac{1}{2}$ and vice versa.[9] This was done within the heavy mass approximation [cf. Eq. (2.28)] but as shown in Sec. III the kinematical corrections do not change the qualitative conclusions regarding the significant couplings among states of the same l.

Thus for alternate l the bootstrap mechanism (we reserve this appellation for the mutual influence of states having the same l) *cooperates* with the nucleon pole terms. The known existence of an F_{15} resonance at 900 MeV thus leads one to suspect that the 1.3-BeV π^+-p maximum must be largely due to an F_{37} resonance. The dynamical similarity of P_{11} and P_{33} then suggest the (verified) observation that P_{33} exchange gives its strongest attractive forces to the F_{15} state. It will be noticed that the "trivial details" of spin and isospin so often abused as "inessential complications" are in fact directly responsible for nearly everything that is significant and interesting in the observed systematic pattern of quantum numbers. Nucleon exchange gives a negligible force in F_{15} (the contrary was incorrectly claimed in a previous paper[10]) and P_{33} exchange has a negligible effect on F_{37}. Qualitatively (except for the energy differences) the major features of

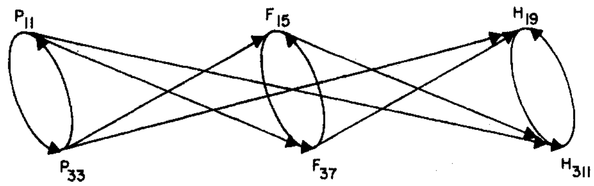


FIG. 3. The dynamical relations among the members of the nucleon (P_{11}) and isobar (P_{33}) trajectories are shown. Each arrow indicates that the exchange of the state at the base of the arrow induces a strong attractive force in the state at the tip of the arrow.

the above discussed resonances are invariant under the reflection (cf. Fig. 3) $P_{11} \leftrightarrow P_{33}$, $F_{15} \leftrightarrow F_{37}$ (and also $H_{19} \leftrightarrow H_{311}$ as discussed below). This "symmetry" corresponds to a reflection (or rotation of 180°) of Fig. 3 about its horizontal axis, or equivalently to an interchange of the nucleon and P_{33} trajectories. Whether this symmetry has any deep significance is a question deserving further attention.

It is to be observed that the significant nucleonic exchange (we use the expression "nucleonic" in the

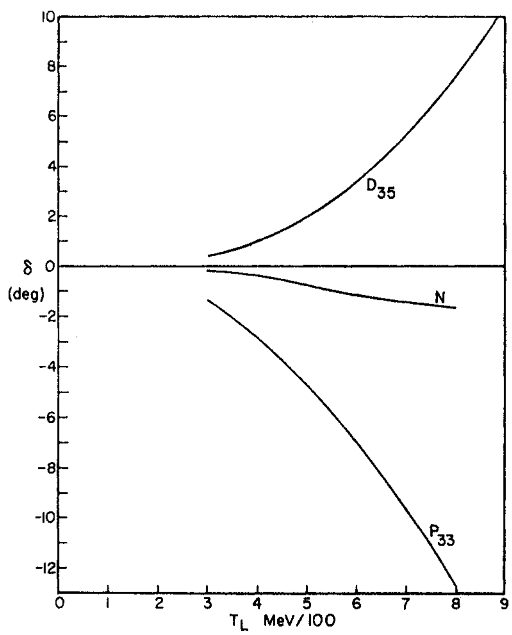


FIG. 4. The "forces" in D_{13} due to the exchange of the states labeling the various curves are given in terms of an effective phase shift, equal to the amplitude f_l divided by k. The curve labeled D_{35} shows the effect of an assumed D_{35} state at 850 MeV with $\Gamma=1$ ($\Gamma=1$ may over estimate this effect due to the large inelasticity at 850 MeV). N stands for nucleon.

generic sense, to describe all the nucleonic objects: nucleon and its excited states) forces giving rise to the P_{33} trajectory are due to the exchange of the states belonging to the nucleon trajectory and vice versa. The object-image relationship between exchange state and induced state suggests the name *image resonances.* Thus P_{33} is the image of P_{11}, which in turn is the image of P_{33}. F_{37} is the image of P_{11} and F_{15}; F_{15} the image of P_{33} and F_{37}. Similarly we find H_{19} to be the image of P_{33}, F_{37} and H_{311}, and $H_{3,11}$ to be the image of P_{11}, F_{15} (very weak) and H_{19}. We call the operation of the mutually dependent pattern of forces a *super-bootstrap* to distinguish it from the simple bootstrap within states of the same l. We also think it is fitting to call the mutually dependent grouping of resonances on the nucleon and isobar trajectories a *constellation* of

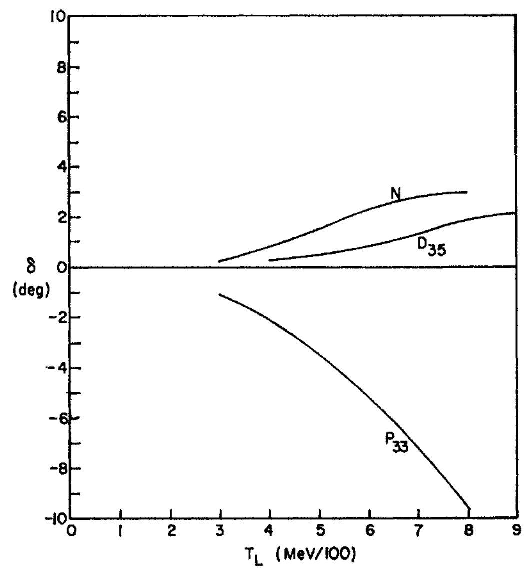


FIG. 5. Contributions to D_{33} are shown.

resonances. Physically it seems appropriate, in view of the results of the proposed theory, to speak of the resonances on the P_{11} and P_{33} trajectories as legitimate "excited states" of the nucleon.

In the present section we give the results for the "forces" due to the exchange of various resonances. To facilitate numerical work we use the sharp resonance approximation

$$\mathrm{Im} f_l(s) = (\pi\Gamma_r W_r/k_r)\delta(s-s_r)\,, \tag{4.1}$$

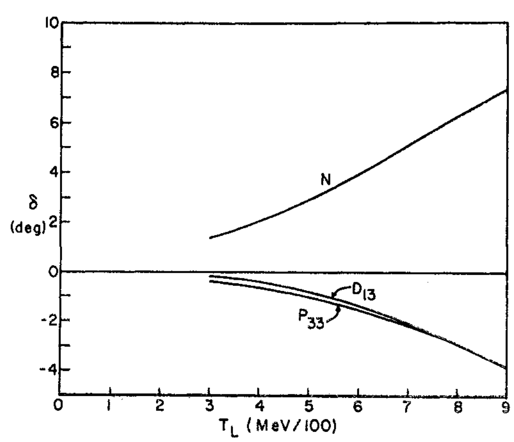


FIG. 6. Various contributions to D_{15} are shown. D_{13} was assumed to have a Γ of unity.

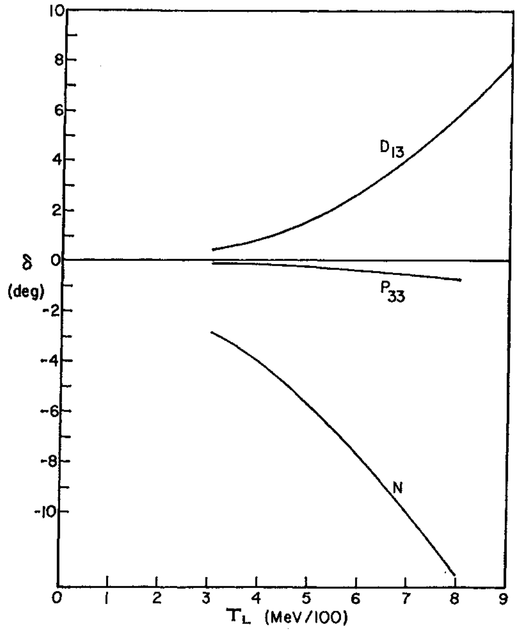


FIG. 7. Forces in D_{35} are shown. Γ was taken to be unity for D_{13}.

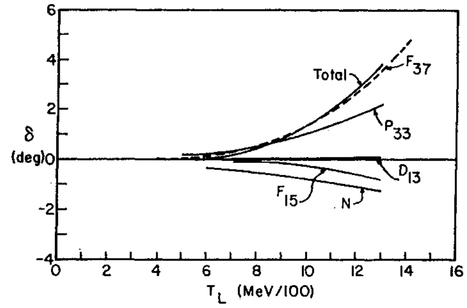


FIG. 9. The F_{35} nucleonic exchange forces are shown.

where Γ_r is the full width, W_r the total c.m. resonance energy, and k_r the c.m. resonance momentum. These "Born" terms are then essential ingredients for any complete calculation and much can be learned from them. For example, the effective phase shift by which we classify the various contributions, rises rapidly at about the proper energy for the F_{15} and F_{37} states. F_{35} is (in the same approximation) weakly attractive while F_{17} is quite repulsive. Thus it is quite reasonable to suppose, in analogy to the 3–3 resonance, that repeated resonance exchange will enhance F_{15} and leave F_{17} small, unless inelasticity drastically modifies the dynamics. In these calculations we have taken into account the exchange of the nucleon, P_{33}, D_{13}, F_{15}, F_{37}, H_{19}, and H_{311}. The parameters used are, in the same order, $W_r=6.72$, 7.79, 10.80, 12.08, 13.58, 15.70, 17.03 corresponding to lab energies, $\cdots$, 190, 600, 900, 1300, 1950, and 2400 MeV. The widths were chosen as follows, noting that inelasticity lowers the effective Γ: P_{33}, $\Gamma=1.1$; D_{13}, $\Gamma=\frac{1}{2}$; F_{15}, $\Gamma=1$; F_{37}, $\Gamma=2$; H_{19}, $\Gamma=0.78$; H_{311}, $\Gamma=0.48$. We have not included the 850-MeV shoulder because we are not sure what the assignment of quantum numbers is. This phenomenon, and the second resonance (D_{13}) are so inelastic that we hesitate to make any firm commitments on the basis of the present theory. The omission of the 850-MeV object is not serious (because of its high mass and large inelasticity) in that few results are substantially altered by it. A more complete discussion is given in Sec. V. (Also the chosen values of Γ for the H states may be too large; we have taken values greater than suggested by a simple subtraction from a smooth background.) The resultant G phases are very small when positive, or else are negative, consistent with the probable lack of resonant phenomena in the $l=4$ states. The pattern of forces in the H states is qualitatively very similar to that in the F states, though the H forces seem to be somewhat weaker than the F forces at the corresponding energies of the observed resonances, largely because of the weakness of the H bootstrap mechanism.

The D states are complicated by the presence of both attractive and repulsive forces of differing range but comparable absolute value. For D_{13}, nucleon exchange

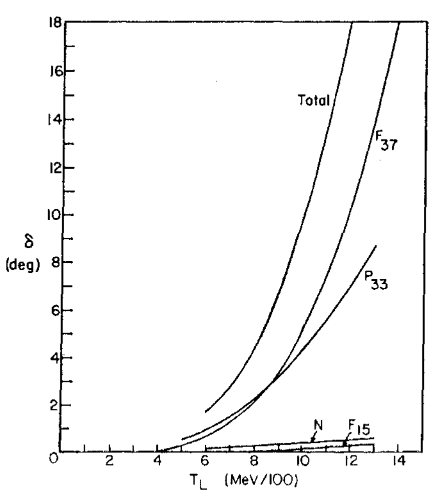


FIG. 8. The F_{15} nucleonic exchange forces are shown.

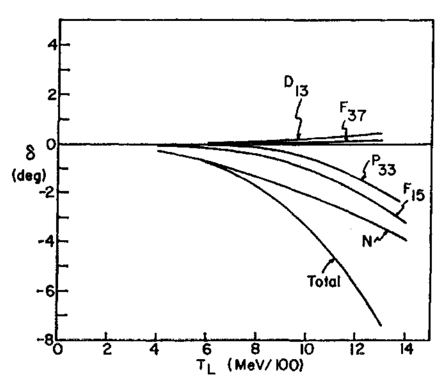


FIG. 10. The F_{17} nucleonic exchange forces are shown.

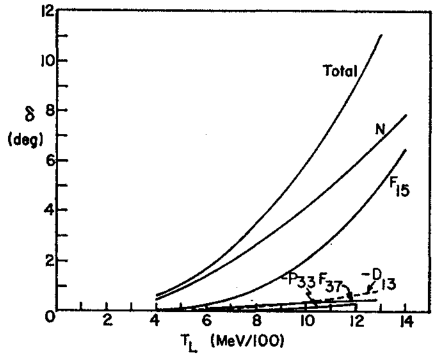
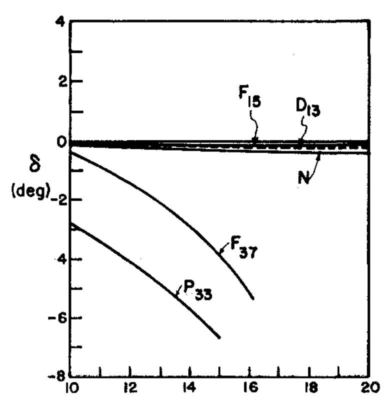


FIG. 11. The F_{37} nucleonic exchange forces are shown.

gives an unimportant contribution while P_{33} exchange gives a strong repulsion. If the shoulder at 850 MeV is D_{35} then the bootstrap contribution gives a short-range D_{13} attractive force that is rapidly increasing at 600 MeV. Similarly, for D_{35}, P_{33} exchange is very small while N exchange gives a strong, rather long-range repulsion. However D_{13} exchange induces a short-range attractive force. Whether these "long-range" repulsions actually sharpen the resonances (the 600-MeV maximum is rather narrow) or are in fact deleterious to the existence of resonances is not clear. There seems to be no escape from accounting for production amplitudes in a detailed way in order to understand the 600- and 850-MeV phenomena. For D_{15}, N exchange gives a non-negligible attraction while D_{13} and P_{33} exchange give small repulsions. P_{33} exchanges makes D_{33} repulsive while N and D_{35} give small positive contributions. It is entirely possible that the (longest range) repulsions can be essentially neutralized by the ease with which the $2\pi N$ channel can be reached: the pion and nucleon can undergo a transition to $2\pi N$ rather than exchange a state obnoxious to it. In Figs. 4–7 we summarize the above results for the D states. It should be said that the exchange of F_{15} and F_{37} has a non-negligible influence on the D states. These results will be described elsewhere in an attempt to describe the 600- and 850-MeV phenomena.

FIG. 12. The G_{17} nucleonic exchange forces are shown.



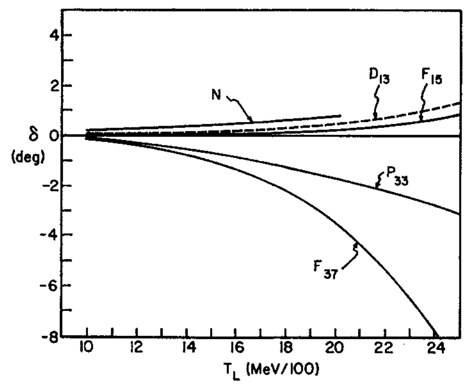


FIG. 13. The G_{37} nucleonic exchange forces are shown.

The nucleonic exchange contributions to the F states are shown in Figs. 8–11. Especially interesting is the predominance of N, F_{15} exchange in the F_{37} state and P_{33}, F_{37} exchange in the F_{15} state (cf. Fig. 3). In F_{35} the nucleonic exchange forces are of mixed character, being weakly attractive on the whole. The non-negligible nucleonic exchange forces in F_{17} are all repulsive.

The results for the G states are shown in Figs. 12–15. The G_{17} state is very repulsive; G_{37} is also repulsive. In G_{19} weak "long-range" attractive forces due to N and P_{33} exchange are overcome by shorter range repulsive forces (F_{15} and F_{37} exchange) that increases rapidly with energy. In G_{39} there is a short-range attraction via F_{15} exchange that is compensated for by a longer range repulsion due to nucleon exchange. Clearly any maxima occurring in G states must have a dynamical mechanism different from that considered here.

The results for the H states (Figs. 16–19) are qualitatively similar to those obtained for the F states, although there are more contributions to the former. However the $H_{19}-H_{311}$ bootstrap is quite weak, because of the rather weak development (as compared to what unitarity permits) of these maxima. Thus at the positions of the highest energy observed maxima the H_{19} and H_{311} forces are somewhat weaker than those in F_{15} and F_{37} at their respective resonance energies. Of course this circumstance may be responsible for the underdeveloped character of the newest maxima.

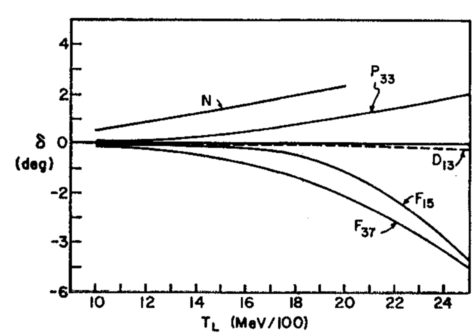


FIG. 14. The G_{19} nucleonic exchange forces are shown.

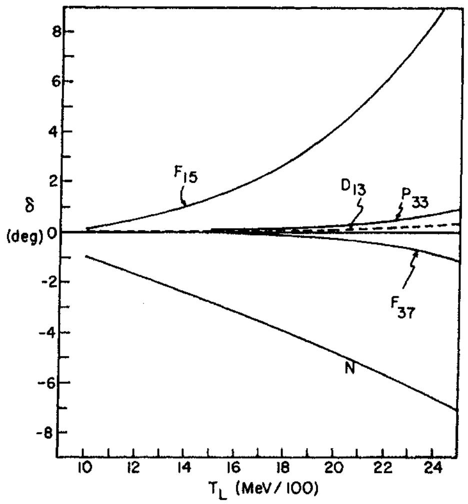


FIG. 15. The G_{39} nucleonic exchange forces are shown.

We have not performed any calculations for states with $l>5$. Although the same pattern will persist at even higher energy the exchange forces become less effective and inelasticity more complete. It is not clear that any further variations in the total cross section should be observable in experiments of plausible accuracy. However the effects of the nucleonic exchange terms might be observable in large angle $\pi-p$ scattering at high energies.

In Fig. 20 we have summarized the nucleonic exchange forces for D through H waves. In this graph no account has been taken of the effect of $l'>l$ states on l waves.

Next we examine in more detail the analytic properties of the nucleonic exchange terms, using the delta-function approximation of Eq. (4.1). (We discuss the amplitudes f_l/k^{2l} as a function of s, though $W=(s)^{1/2}$ is more convenient for actual computation, as emphasized by Frautschi and Walecka. Our results are easily transcribed to the W variable if the MacDowell symmetry[25] is invoked to define the amplitudes in the left-hand W

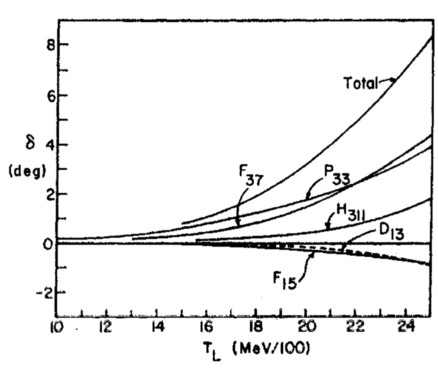


FIG. 16. The H_{19} nucleonic exchange forces are shown.

[25] S. W. MacDowell, Phys. Rev. **116**, 774 (1960).

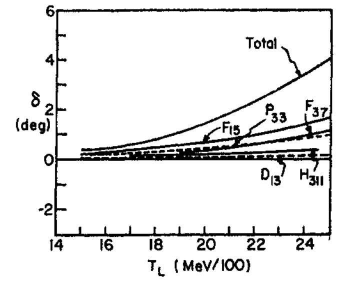


FIG. 17. The H_{39} nucleonic exchange forces are shown.

plane.) In this approximation the forces due to nucleonic exchange are represented by cuts below the threshold $(M+\mu)^2$ on the real axis, due to the Q functions. From the integral representation[24]

$$Q_n(y)=\frac{1}{2}\int_{-1}^{1}\frac{P_n(x)dx}{x-y}, \qquad (4.2)$$

where $y=1+(s_r-u_0)/2k^2$, one sees that the amplitude is discontinuous for real s such that y lies between -1 and 1. The branch points $(y=\pm1)$ occurs for $s=0$, ∞ and $S_\pm$, where $S_+=(M^2-\mu^2)^2/S_r$ and $S_-=2(M^2+\mu^2)-S_r$ (see Table I). A qualitative difference distinguishes

TABLE I. The positions of the cuts arising from the exchange of resonances of mass $(S_r)^{1/2}$ is determined by $S_\pm$ [see discussion following Eq. (4.2)].

T_L	190	600	850	900	1300	1950	2400
S_r	77.9	117.2	141.5	146.0	184.4	247.4	290.0
S_+	25.0	16.6	13.8	13.4	10.6	7.9	6.7
S_-	14.4	−24.9	−49.1	−53.7	−92.1	−155.1	−197.7

the cases $s_r \gtrless 2(M^2+\mu^2)$. For $s_r<2(M^2+\mu^2)$ one has cut from S_- to S_+ and a second cut from $-\infty$ to 0, as in the case of the well-known nucleon exchange contribution. For $s_r>2(M^2+\mu^2)$ the two cuts run from $-\infty$ to s_- and from 0 to s_+ (cf. Fig. 21). In Fig. 22 we have plotted $k^2(s)$, and $y(s)$ for P_{33} and F_{15} exchange to illustrate both situations. In constructing such figures it is useful to use $y(\pm\infty)=y(s_+)=-y(s_-)=1$, $y[(M+\mu)^2+]=y[(M-\mu)^2-]=+\infty$, $y[(M+\mu)^2-]=y[(M-\mu)^2+]=-\infty$ and

$$y'(0)=2[s_r-2(M^2+\mu^2)]/(M^2-\mu^2)^2. \qquad (4.3)$$

As a function of y, $\mathrm{Im}Q_n(y)=(\pi/2)P_n(y)$, as is seen

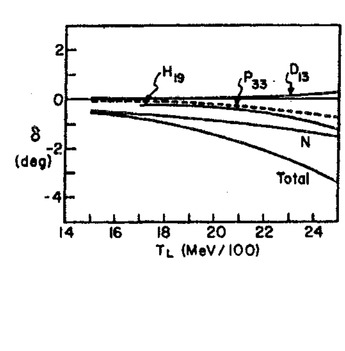


FIG. 18. The $H_{1,11}$ nucleonic exchange forces are shown.

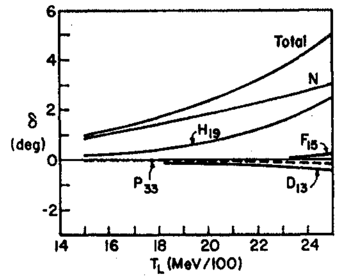


FIG. 19. The $H_{3,11}$ nucleonic exchange forces are shown.

from Eq. (4.2). From this point the calculation is of the standard N/D type, easily soluble in the pole approximation.

Because of the extensive numerical work involved we have decided to defer the discussion of the calculations now in progress to a subsequent paper. Thus the results shown in Figs. 4 to 20 are "raw theoretical data," not yet in shape to be confronted with experiment. [That resonances occur in the proper F states (F_{15} and F_{37}) at *some* energy is almost beyond doubt in view of the rapidity of increase of the exchange terms at high energy.] However the qualitative features of our results should be useful in selecting a preferred set among competing phase shift solutions.

As an example, we discuss recent Berkeley results[26] near the second resonance. It was hoped that polarization measurements would provide a definitive determination of the quantum numbers of this maximum. However, equally good fits were obtained with either

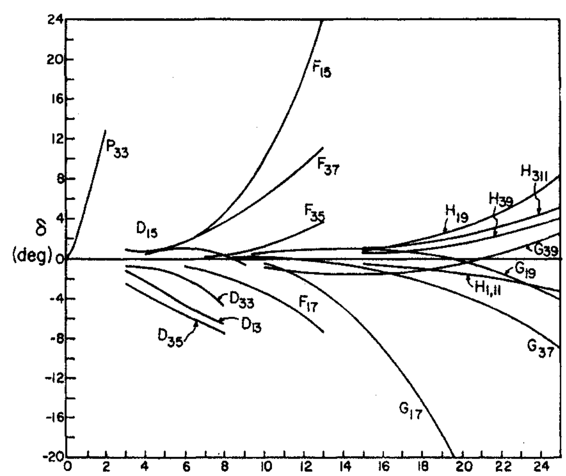


FIG. 20. The total "forces" due to exchange of nucleonic states are shown for D through H states. Although this figure obscures the fact that each curve receives contributions of differing range, the general pattern is significant in showing the emergence of the resonance states of even parity. For comparison the P_{33} "force" due to nucleon exchange is shown. The P_{11} (attractive) force arising from P_{33} exchange is not shown, but would lie above the P_{33} curve.

[26] R. D. Eandi, Lawrence Radiation Laboratory Report, UCRL-10629 (unpublished).

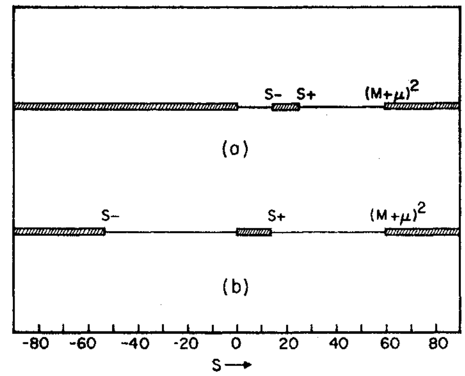


FIG. 21. The left-hand cuts in the s plane due to (a) P_{33} and (b) F_{15} exchange are shown [the sharp resonance approximation, Eq. (4.1), has been used].

D_{13} or P_{13} for the 600-MeV resonance. For both solutions the change of D_{35} from strong repulsions at 523 and 572 MeV to a weak attraction at 689 MeV is consistent with the effect of D_{13} exchange described above. (In fact, the $T=\frac{3}{2}$ D and F waves are nearly the same for both solutions.) Besides the theoretical reasons for preferring D_{13} to P_{13} (and also considering the photoproduction analysis[27] for the same purpose) the P_{13} solution has D_{15} rather repulsive at 523 MeV, surprising in view of the substantial (theoretical) attraction due to nucleon exchange. The $T=\frac{1}{2}$ F waves provide important clues. The D_{13} solution has a large positive F_{15} phase and a small F_{17} phase shift, in agreement with the baryonic exchange forces giving strongest attractive forces in F_{15} and moderate repulsions in F_{17}. On the other hand the P_{13} solution gives large positive F_{17} phase shifts at all three energies and smaller F_{15} phase shifts. In making these comparisons one should bear in mind the qualitative influence of the neglected but not insignificant contributions due to ρ exchange, and $T=J=0$ $\pi\pi$ exchange. The effect of the latter is to give

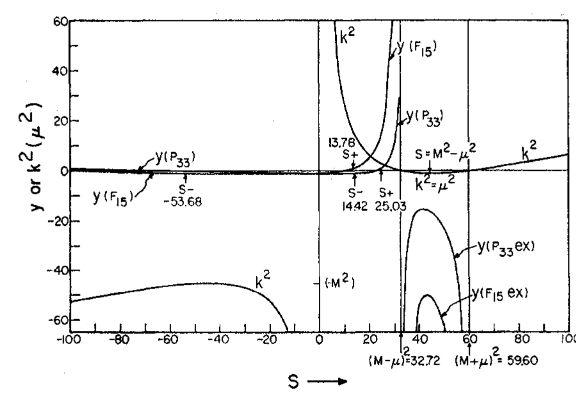


FIG. 22. The c.m. momentum squared k^2, and the variable y [for P_{33} and F_{15} exchange, of Eq. (4.2)] are given as a function of s. $S_\pm$ locate branch points occurring for $y=\pm 1$. $y=0$ and ∞ are also branch points.

[27] R. F. Peierls, Phys. Rev. **118**, 325 (1960).

an attraction in all states, while exchange enhances $T=\frac{1}{2}$ twice as much as it diminishes $T=\frac{3}{2}$ (Appendix B). Another virtue of the D_{13} solution is the substantial P_{11} phase shift, possibly associated with the maximum in $\pi^-+p\rightarrow\Lambda+K^0$ at 900 MeV.[13] Previously it was suggested[10] that the ρ exchange force, which is fairly substantial in this state, might be important in generating this maximum. Numerical evaluation[19] indicates that P_{33} exchange is substantially larger than both the ρ exchange and the repulsive N exchange term, at high energy. Although more work is in order concerning the origin (and even the assignment of quantum numbers) of this maximum it is very appealing that this object be P_{11}, induced by P_{33} exchange. In this case one has the beginning of a second trajectory with the nucleon quantum numbers. Is it possible that another P_{33} resonance lies obscured by the broad maximum we have previously associated solely with the F_{37} state?

V. DISCUSSION

In the previous sections the forces due to the exchange of nucleonic states have been analyzed. It was found that a self-sustaining dynamical entity, a *constellation* of resonances, composed of two Regge trajectories originating in the nucleon and P_{33} resonances was suggested by the structure of the crossing matrix. Moreover the members of one trajectory are "images" of the resonances comprising the other trajectory, as discussed in Sec. IV. Although the involved numerical calculation of the resonance energies has not yet been completed, it was found that the states in which resonances are expected surmount the centrifugal barrier at roughly the proper energies and in the right order (e.g., F_{15} precedes F_{37} as the energy is increased). Thus the energy spacing between the members of the trajectories, or equivalently the slope of the trajectories, is qualitatively correlated with the range of forces due to the exchange of baryonic objects. As we do not expect to achieve especially accurate results in the calculation of resonance energies because of divergences in the N/D solutions we cannot understand why the slopes should be so remarkably constant. For instance $\Delta s=100\mu^2$ for the three members of the nucleon trajectory and about $106\mu^2$ for the P_{33} trajectory. The principal inadequacy of the analysis presented above resides in the assumed unimportance of inelasticity as a generator of resonances. As already remarked such a position cannot be maintained for the 2nd resonance and the 850-MeV shoulder. Similarly the influence of inelasticity on the constellation will probably be significant though we believe it to be a secondary consideration. Thus it may be significant that the F_{15} resonance lies near the threshold for ρ production and the H_{19} maximum near the f^0 threshold. Although the precise resonance energies may thus be influenced by inelastic thresholds we do not agree with the position maintained by many people to the effect that the even parity higher resonances are "cusps" or sundry threshold effects. Rather we claim that the constellation exhibits and interprets in a systematic way the compellingly beautiful empirical regularity of the excited states of that most fundamental object, the nucleon.

While the attractive isospin-$\frac{1}{2}$ forces caused by ρ exchange and production no doubt cause the $T=\frac{1}{2}$ resonances to develop at a lower energy than would be the case with purely nucleonic forces we view the ρ admixture in much the same way that proponents of the eightfold way regard the "symmetry-breaking" terms. (It is possible, though, that there is a subtle conspiracy wherein all these forces cooperate in a simple way as yet undiscovered.)

As already mentioned it appears that one dare not neglect the influence of the exchange of nucleonic states on the production process. We now discuss qualitatively the D-state forces due to N, P_{33} and D exchange. The situation is considerably more intricate than might be gathered from the discussion of a previous paper,[10] in which we perhaps overemphasized the "filtering" action of the D-wave crossing matrix. P_{33} exchange creates a moderately strong repulsion in D_{13}. If the shoulder is D_{35} then there is a substantial short-range attraction as well. The attraction in D_{15} due to nucleon exchange is helpful in understanding the positive value of the phase shift in this state. Both D_{35} and D_{33} receive strong repulsions due to N and P_{33} exchange. This is most likely the reason why the π^+p cross section is so small in the natural domain of the D waves (around 600-MeV pion lab energy). However at about 800 MeV the cross section suddenly increases giving rise to the shoulder predicted in an earlier paper.[28] In that work D_{33} was suggested on the basis of (a) the anomalously small charge exchange $\pi^-+p\rightarrow\pi^0+n$ near the second resonance, which indicated substantial interference with the D_{13} state on the basis of a simple resonance model, and (b) the small size of the shoulder which seemed to rule out $J=\frac{5}{2}$ unless the latter were almost completely absorbed. It now appears that the absorption is very strong indeed in the shoulder, so that $J=\frac{5}{2}$ cannot be so easily disposed of. Moreover the simple resonance model [point (a)] is probably not too reliable since so many states seem to be significant. It was subsequently pointed out[29] that D_{33} would be generated if ρ production (threshold: 890 MeV) had a significant effect. On the other hand, one can make a good case for D_{35}. As the lab energy is increased to 850 MeV the short-range attraction due to D_{13} exchange suddenly sets in. The longer range repulsion due to N exchange possibly confines the pion to shorter distances, enhancing the amount of inelasticity. (Experimentally[30] it appears that the most interesting feature of the π^--p angular distribution near the third resonance, the

[28] P. Carruthers, Phys. Rev. Letters 4, 303 (1960).
[29] P. Carruthers, Phys. Rev. Letters 6, 567 (1961).
[30] J. A. Hellard, T. J. Devlin, D. E. Hagge, M. J. Longo, B. J. Moyer, and C. D. Wood, Phys. Rev. Letters 10, 27 (1963).

TABLE II. The angular distribution for the reaction $\pi+N\rightarrow N^*+\pi$ is given for various channels. The notation L_J means that the incident πN channel of angular momentum J and orbital momentum L undergoes a transition to the $N^*-\pi$ configuration of orbital momentum l. X is the cosine of the production angle in the total c.m. system.

$S_{1/2}\rightarrow d$	const.	$D_{3/2}\rightarrow d$	const.
$P_{1/2}\rightarrow p$	const.	$D_{5/2}\rightarrow d$	$1+10X^2-10X^4$
$P_{3/2}\rightarrow p$	$7-6X^2$	$D_{5/2}\rightarrow g$	$13-10X^2+45X^4$
$P_{3/2}\rightarrow f$	$1+2X^2$	$F_{5/2}\rightarrow p$	$1+2X^2$
$D_{3/2}\rightarrow s$	const.	$F_{5/2}\rightarrow f$	$7+34X^2-25X^4$

"backward bump," requires a substantial interference between resonant F_{15} and attractive $D_{5/2}$.) Moreover one can entertain the thought that the 850-MeV shoulder is associated with $\pi+N\rightarrow P_{33}+\pi$ through the F_{35} channel. The rapid rise occurs at a natural F-wave threshold energy. (It will be recalled that the nucleonic-exchange forces are weakly attractive in F_{35}.) The F_{35} assignment has been advocated by Peierls.[31] In this regard Peierls has emphasized[32] the importance of analyzing production data in terms of the angular distribution for $\pi+N\rightarrow N^*+\pi$. The one example known[33] for $\pi^++p\rightarrow\pi^++p+\pi^0$ at 820 MeV gives a distribution of the form $4.1-10.5\cos\theta+8.7\cos^2\theta$. This seems to favor $F_{35}\rightarrow P_{33}+p$-wave pion, which gives a distribution proportional to $1+2\cos^2\theta$. The point here is that all other states (leading to s- and p-wave recoil pions) give negative coefficients for the $\cos^2\theta$ term.[32] For d (or higher) wave recoil pions (e.g., $D_{5/2}\rightarrow P_{3/2}+d$ wave) large $\cos^4\theta$ terms appear. Of course complicated interference effects can occur but the analysis of the data in this manner might prove very informative. In Table II we give some of the relevant angular distributions. This table was computed for the author by L. M. Simmons and agrees with unpublished results of Peierls.[32]

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The author is indebted to Professor R. F. Peierls for many enlightening discussions of the problem of the higher resonances in pion-nucleon scattering.

APPENDIX A: USEFUL LEGENDRE IDENTITIES

In this appendix we summarize the identitites needed for the partial-wave reduction in Sec. III. Identities 1–4 are the common ones. The relations between $P_n(x')$ and $P_m(x)$ $(x'=\xi x+a)$ can be proved simply from Rodriguez' formula, e.g.,

$$xP_l(x)=[l/(2l+1)]P_{l-1}(x)+[(l+1)/(2l+1)]P_{l+1}(x)\,,\tag{A1}$$

$$xP_l'(x)=[l/(2l+1)]P_{l+1}'(x)+[(l+1)/(2l+1)]P_{l-1}'(x)\,,\tag{A2}$$

$$xP_l'(x)=lP_l(x)+P_{l-1}'(x)\,,\tag{A3}$$

$$P_l(x)=[1/(2l+1)][P_{l+1}'(x)-P_{l-1}'(x)]\,,\tag{A4}$$

$$P_l(x')=\xi^lP_l(x)+(2l-1)a\xi^{l-1}P_{l-1}(x)+(2l-3)\xi^{l-2}a(la-1)P_{l-2}(x)+\cdots\,,\tag{A5}$$

$$dP_l(x')/dx=\xi^{l+1}(2l+1)P_l(x)+(4l^2-1)\times a\xi^lP_{l-1}(x)+\cdots\,,\tag{A6}$$

$$P_2(x)P_l(x)=[3l(l-1)/2(4l^2-1)]P_{l-2}(x)+[3(l+1)(l+2)/2(2l+1)(2l+3)]P_{l+2}(x)+[l(l+1)/(2l-1)(2l+3)]P_{l+2}(x)\,,\tag{A7}$$

$$P_2(x)P_l'(x)=[3l(l+1)/2(2l+1)(2l+3)]P_{l+2}'(x)+[3l(l+1)/2(4l^2-1)]P_{l-2}'(x)+[(2l^3+3l^2-5l-3)/(4l^2-1)(2l+3)]\times P_l'(x)\,.\tag{A8}$$

APPENDIX B: ESTIMATE OF ϱ-EXCHANGE CONTRIBUTION

For a simple estimate we use perturbation theory with the interaction Hamiltonian

$$H'=f_{\rho NN}\varrho^\mu\cdot\bar{\psi}\gamma_\mu\tfrac{1}{2}\boldsymbol{\tau}\psi+f_{\rho\pi\pi}\varrho^\mu\cdot\boldsymbol{\phi}\times\partial_\mu\boldsymbol{\phi}\,,\tag{B1}$$

where ϱ^μ, $\boldsymbol{\phi}$, and ψ are, respectively, the ρ, pion, and nucleon fields. The vectorial symbols refer to isospace. The conventional transition matrix for scattering from k to k' is then

$$T=\tfrac{1}{2}f_{\rho NN}f_{\rho\pi\pi}\epsilon_{kk'j}\tau_j\times\frac{M}{(4E_pE_{p'}\omega_k\omega_{k'})^{1/2}}\frac{\bar{u}(p')(k+k')\cdot\gamma u(p)}{(k-k')^2-m_\rho^2}\,,\tag{B2}$$

where m_ρ is the ρ mass and p and p' label the initial and final nucleon four-momentum. The amplitude B in Eq. (2.4) for isospin $\frac{1}{2}$ is then $(B^{3/2}=-\frac{1}{2}B^{1/2})$

$$B^{1/2}=f_{\rho NN}f_{\rho\pi\pi}/(m_\rho^2-t)\,.\tag{B3}$$

The corresponding phase shifts for isospin $\frac{1}{2}$ are then $(y=1+m_\rho^2/2k^2)$

$$\delta_{l\pm}^{1/2}=(\gamma_\rho^2/4kW)[(E+M)(W-M)Q_l(y)+(E-M)(W+M)Q_{l\pm1}(y)]\,,\tag{B4}$$

where $\gamma_\rho^2=f_{\rho NN}f_{\rho\pi\pi}/4\pi$ as estimated from the width of the ρ, and also from low-energy pion nucleon scattering, is about 2.[34,35] For reference we also give the differential cross section in isospin $\frac{1}{2}$ following from (B2):

$$\frac{d\sigma^{1/2}}{d\Omega}=\gamma_\rho^4\frac{\{(s-M^2-\mu^2)^2+t(s-M^2)\}}{4s(t-m_\rho^2)^2}\,.\tag{B5}$$

[31] R. F. Peierls, Phys. Rev. Letters **5**, 166 (1960).
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[34] J. J. Sakurai, in the *International Conference on High-Energy Nuclear Physics, Geneva*, edited by J. Prentki (CERN Scientific Information Service, Geneva, Switzerland, 1962), p. 176.
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[Using (B4) and $\gamma_\rho^2=2$ we find for D_{13}, $\delta=0.38°$, 0.51°, 0.86°, 1.31° at 300, 400, 500, and 600 MeV. For D_{15} one finds 0.22°, 0.25°, 0.4°, 0.6°, 0.84°, 1.1°, 1.37° at 300, 400, 500, 600, 700, 800, 900 MeV. For F_{15} δ is 0.36°, 0.50° and 0.68° at 700, 800, and 900 MeV. For H_{19} δ is 0.37°, 0.56° and 1.05° at 1700, 2000, and 2500 MeV. The same parameters give for P_{11} $\delta=12°$ at 900 MeV.]

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Effect of the Baryon Excited States on the $N-\Lambda$ and $\Lambda-\Lambda$ Forces

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The N-Λ and Λ-Λ potentials caused by the exchange of two pions are calculated in the static theory, taking into account the resonance Y_1^* in the π-Λ system and the (3-3) resonance in the π-N system. The recoil of the baryons is included in an approximate way. It is shown that the presence of these resonances diminishes the spin-dependent part of the central potential and the tensor potential, and increases the spin-independent part of the central potential. The triplet potential turns out to be slightly stronger than the singlet potential at large distances, and slightly weaker than it closer in. If the resonances are omitted, the triplet potential is the stronger over the whole range. This last result is in mild disagreement with other work. Its relation to the choice of a one-channel or two-channel formalism is discussed.

1. INTRODUCTION

SOME experimental evidence on hypernuclei and on double-hypernuclei is now available and some phenomenological analyses of this evidence have been made with a view to determining the nature of the $N-\Lambda$[1] and $\Lambda-\Lambda$ forces.[2]

Various workers[3] have estimated the two-pion exchange contributions to these potentials using meson theory. However, no account seems yet to have been given of the effect upon these forces produced by the Y_1^* resonance in the $\pi-\Lambda$ system and the $3-3$ resonance in the $\pi-N$ system together.[4] The main purpose of this paper is to estimate this effect.

We shall take the $\Sigma-\Lambda$ parity to be even, as has now been almost conclusively established,[5] and we shall make the experimentally probable assumption that the Y_1^* resonance at 1385 MeV in the $\pi-\Lambda$ system is a $P_{3/2}$ state,[6] having the same mechanism as the $I=J=\frac{3}{2}$ resonance in the $\pi-N$ system. The Chew-Low theory for the pion-nucleon interaction can then be extended in a straightforward way to the pion-hyperon interaction, and the $N-\Lambda$ and $\Lambda-\Lambda$ potentials can be calculated by the method given by Miyazawa,[7] a method in which the resonances of the $\pi-N$ and $\pi-\Lambda$ systems can be treated.

It has been pointed out by Charap and Fubini and by Gupta[8] that the static limit of the two-pion exchange potential is not well defined. The difficulty comes from the fact that, when the two-pion exchange potential $V(x)$ is written in the form

$$V(x)=\int_{(2m_\pi)^2}^{\infty} dm^2\rho(m^2)\exp(-mx)/x, \qquad (1.1)$$

the inverse baryon mass expansion of the spectral function $\rho(m^2)$ does not converge at the lower mass end ($m\to 2m_\pi$). The relativistic effect is therefore important in the asymptotic region ($x\to\infty$), where the static limit would appear to be most justified. Akiba[9] has examined the accuracy of the inverse nucleon mass expansion, showing that this expansion provides us with a reasonable numerical approximation. Our calculation will be meaningful except for extremely large distances where $|V(x)|$ will be negligibly small, and of course for very short distances.

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