and

$$B=(1/\tau-1/\tau_a)L/c.$$

$I_T$ and $I_0$ are the source intensity at the time $t=T$ and 0, respectively, and $\tau_a$ is the life time of heavy nuclei. From Eq. (2) and the latitude effect shown in Figs. 1(f) and (h), we can estimate $\theta_c=60°\sim70°$, $I_T=10\sim20\%$ of air showers, and $B=-0.5\sim0.5$. Assuming the collision mean free path of the heavy nuclei, $m_a\approx5$ g/cm$^2$ and the matter density in the Galactic arm, $\rho\approx1$ proton/cm$^3$, we obtain $\tau_a=m_a/\rho c\approx10^{14}$ sec. Provided $L$ is smaller than the dimension of the Galaxy, the result $B<0.5$ gives the conditions $\tau\gtrsim2L/c\approx T$ and $I_0=10\sim60\%$ of air showers. This implies that a source at a distance $L$ light years has been emitting heavy nuclei since about $2L$ years ago, with an initial intensity $10\sim60\%$ of ordinary air shower and with a decay time longer than $2L$ years.

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# Complete Spin Tests for Fermions

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Complete spin tests for fermions of arbitrary spin, produced from a spin-zero boson on an unpolarized spin-$\frac{1}{2}$ fermion and decaying into a spin-zero boson and a spin-$\frac{1}{2}$ fermion, are derived. The tests constitute a set of necessary and sufficient conditions for a particular spin assignment, in the absence of more detailed dynamical information. Essential use is made of the $R$ invariance of the parity conserving production process. More general tests, applicable to arbitrary production processes, are also discussed.

## 1. INTRODUCTION

It is the main purpose of this paper to derive necessary and sufficient conditions for a spin assignment to a fermion that is produced from a spin-zero boson on an unpolarized spin-$\frac{1}{2}$ fermion and subsequently decays into a spin-$\frac{1}{2}$ fermion and a spin-zero boson. Our conclusions for this case are summarized in Sec. 4.2 in a form allowing their direct practical use.

Also, in this paper we discuss spin tests applicable to more general production processes. Our conclusions for the general case are summarized in Sec. 4.1.

The derivation of the necessary and sufficient conditions for the case of production from a spin-zero boson on an unpolarized spin-$\frac{1}{2}$ fermion is made possible by the application of the $R$ transformation to the production density matrix. Such a method was recently applied by Peshkin,[1] and is also related to work by A. Bohr[2] and by Eberhard and Good.[3] We obtain necessary and sufficient conditions as a consequence of a theorem that characterizes the most general density matrix for production of a fermion of spin $s$ from two incoherent helicity states related by an $R$ transformation.

For the general production process we obtain various tests, some of them involving the longitudinal and transverse polarizations. These results are closely related to results by Lee and Yang,[4] by Durand, Landovitz, and Leitner,[5] by Spitzer and Stapp,[6] by Gatto and Stapp,[7] by Capps,[8] by Ademollo and Gatto,[9] and by Byers and Fenster.[10]

## 2. THE DENSITY MATRIX FOR PRODUCTION FROM SPIN-ZERO BOSON ON UNPOLARIZED SPIN-1/2 FERMION

### 2.1

We consider the production process

$$a+f\to F+b\,, \tag{2.1}$$

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[10] N. Byers and S. Fenster, Phys. Rev. Letters **11**, 52 (1963).

where $a$ and $b$ are spin-zero bosons, $f$ is an unpolarized spin-$\frac{1}{2}$ fermion, and $F$ is a fermion of spin $s$. The production process (2.1) is assumed to be parity conserving.

As pointed out by Peshkin,[1] the two initial helicity states are related to each other by an $R$ transformation. The transformation $R$ is defined as a reflection through the production plane, or, equivalently, as the product of space inversion and a rotation of 180 deg around the normal to the production plane.

By virtue of the assumed reflection invariance and the fact that $b$ is spinless, the density matrix for the final fermion must be of the form

$$\rho^{(F)}=\tfrac{1}{2}|\psi\rangle\langle\psi|+\tfrac{1}{2}|R\psi\rangle\langle R\psi|. \tag{2.2}$$

The final amplitudes $\psi$ and $R\psi$ are not in general, orthogonal. Following Peshkin[1] we take the normal $\mathbf{n}$ to the production plane as the direction of spin quantization and denote by $\mu$ the component of the spin of $F$ along $\mathbf{n}$. We expand $|\psi\rangle$ in terms of the spin eigenfunctions $|\varphi_\mu\rangle$

$$|\psi\rangle=\textstyle\sum_\mu \beta_\mu|\varphi_\mu\rangle \tag{2.3}$$

with the normalization

$$\textstyle\sum_\mu|\beta_\mu|^2=1. \tag{2.4}$$

Apart from an unimportant phase factor, we have

$$|R\psi\rangle=\textstyle\sum_\mu(-1)^{s-\mu}\beta_\mu|\varphi_\mu\rangle, \tag{2.5}$$

as can easily be seen from the interpretation of $R$ as a space inversion followed by a rotation of $\pi$ around $\mathbf{n}$. From (2.2), (2.3), and (2.5) we obtain

$$\rho^{(F)}=\sum_{\mu\mu'}\tfrac{1}{2}[1+(-1)^{\mu-\mu'}]\beta_\mu\beta_{\mu'}^*|\varphi_\mu\rangle\langle\varphi_{\mu'}|. \tag{2.6}$$

According to (2.6) the density matrix of the final particle $F$ produced in reaction (2.1), though not a pure state density matrix, is obtained from a pure state density matrix by eliminating the elements corresponding to odd values of $\mu-\mu'$.

## 2.2

We shall now obtain the necessary and sufficient conditions that $\rho^{(F)}$ has to satisfy in order to be of such a form. These conditions will be used to formulate the complete set of conditions for the spin of $F$.

We first note that the conditions

$$\rho^{(F)\dagger}=\rho^{(F)},\quad \mathrm{Tr}[\rho^{(F)}]=1 \tag{2.7}$$

are identically satisfied by $\rho^{(F)}$, as can be seen from (2.6) and (2.4). We now split $\rho^{(F)}$ into the sum of two matrices $\rho'$ and $\rho''$ defined as follows: $\rho'$ has nonzero elements $\rho_{\mu\nu}'=\rho_{\mu\nu}^{(F)}$ only for $s-\mu$ and $s-\nu$ both odd; $\rho''$ has nonzero elements $\rho_{\mu\nu}''=\rho_{\mu\nu}^{(F)}$ only for $s-\mu$ and $s-\nu$ both even. We know that the matrix elements $\rho_{\mu\nu}^{(F)}$ with $s-\mu$ even (odd) and $s-\nu$ odd (even) are zero. Therefore,

$$\rho^{(F)}=\rho'+\rho'', \tag{2.8}$$

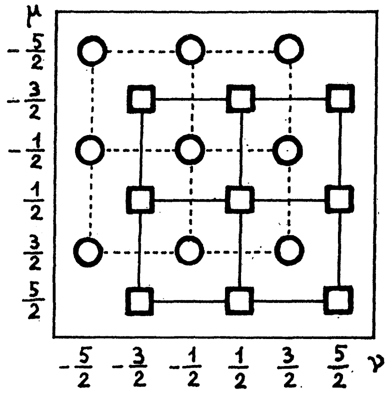


FIG. 1. The density matrix $\rho^{(F)}$ for spin $\frac{5}{2}$. The elements indicated by circles form the matrix $\rho'$, those indicated by squares form the matrix $\rho''$.

where $\rho'$ has only the odd rows and columns of $\rho^{(F)}$, and $\rho''$ only the even rows and columns.

For instance, for spin $s=\frac{5}{2}$, the matrix $\rho^{(F)}$ has the form shown in Fig. 1. The matrix elements indicated by circles form the matrix $\rho'$; those indicated by squares form the matrix $\rho''$; the other matrix elements are zero.

Turning to the general discussion, we note that the matrices $\rho'$ and $\rho''$ are related by the condition

$$\mathrm{Tr}[\rho']+\mathrm{Tr}[\rho'']=\mathrm{Tr}[\rho^{(F)}]=1. \tag{2.9}$$

To $\rho'$ and $\rho''$ applies the following theorem: Let $\rho$ be a Hermitian matrix with trace $a$.

$$\rho^\dagger=\rho,\quad \mathrm{Tr}[\rho]=a. \tag{2.10}$$

The necessary and sufficient conditions for $\rho$ to be of the form

$$\rho=\beta\beta^\dagger, \tag{2.11}$$

where $\beta$ is a one-column matrix, are

(i) $a\geq 0$

(ii) $\rho$ is of rank one.

The theorem can easily be proved by reducing $\rho$ to diagonal form. The condition (ii) can be expressed as

$$\rho_{\mu\nu}\rho_{\sigma\tau}-\rho_{\mu\tau}\rho_{\sigma\nu}=0 \tag{2.12}$$

or equivalently

$$\rho^2=a\rho. \tag{2.13}$$

Taking Eq. (2.12) with $\mu=\nu$ and $\tau=\sigma$ we find

$$\rho_{\mu\mu}\rho_{\sigma\sigma}=\rho_{\mu\sigma}\rho_{\sigma\mu}=|\rho_{\mu\sigma}|^2, \tag{2.14}$$

the last step following from the hermiticity condition on $\rho$. Equation (2.14) tells us that all nonzero diagonal elements of $\rho$ have the same sign. The condition that the trace be positive, can thus be replaced by the condition that one nonzero diagonal matrix element of $\rho$ be positive.

If a matrix $\rho$ is of the form (2.11) it must clearly be true that:

$$\rho_{\mu_1\nu_1}\rho_{\mu_2\nu_2}\cdots\rho_{\mu_n\nu_n}=\rho_{\mu_1\nu_{i_1}}\rho_{\mu_2\nu_{i_2}}\cdots\rho_{\mu_n\nu_{i_n}}, \tag{2.15}$$

where $(i_1,i_2\cdots i_n)$ is any one of the $n!$ permutations on

1, $2\cdots n$, and $n$ is arbitrary. From the necessary and sufficient condition that we have derived, all the relations of the form (2.15) with $n>2$ must follow from those with $n=2$. This consequence can directly be shown to be true by induction. Spin tests using the relations (2.15) can also be formulated, but we shall not consider them here.

Not all the conditions (2.12) are independent. It is sufficient to impose (2.12) on the independent minors of the matrix $\rho$. Each of the matrices $\rho'$ and $\rho''$, into which we have split $\rho^{(F)}$, has $[(2s-1)/2]^2$ independent minors. There are many ways to choose $[(2s-1)/2]^2$ independent relations of the form (2.12) for each of the two matrices $\rho'$ and $\rho''$. One possible choice, convenient from the point of view of spin tests, is the following: Take $\mu=\nu=\alpha$ as a fixed index and let $\sigma$ and $\tau$ take on all possible values different from $\alpha$. The independent relations are then

$$\rho_{\alpha\alpha}'\rho_{\sigma\tau}'=\rho_{\alpha\tau}'\rho_{\sigma\alpha}', \tag{2.16}$$

and similar relations

$$\rho_{\beta\beta}''\rho_{\sigma\tau}''=\rho_{\beta\tau}''\rho_{\sigma\beta}'' \tag{2.16$'$}$$

for $\rho''$. There are $[(2s-1)/2]^2$ relations of the form (2.16) for $\rho'$ and also $[(2s-1)/2]^2$ relations of the form (2.16′) for $\rho''$.

### 2.3

In conclusion, for spin $s$ a total of $\frac{1}{2}(2s-1)^2$ independent relations must be satisfied. These relations are independent relations of the form (2.12), applied to both $\rho'$ and $\rho''$, that express the condition that both $\rho'$ and $\rho''$ must have rank one (that is, all minors of order larger than one must vanish). Equations (2.16) and (2.16′) are particular convenient choices of the independent relations. Furthermore, the two inequalities $\mathrm{Tr}[\rho']\geq 0$ and $\mathrm{Tr}[\rho'']\geq 0$ must be satisfied.

For applying the conditions given above to test the value of the spin of $F$ one needs the values of the matrix elements of $\rho^{(F)}$. These matrix elements can be determined from the angular distribution in the decay of $F$. In the next section we discuss the construction of $\rho^{(F)}$ from experimental data for the case that $F$ decays into a spin-$\frac{1}{2}$ fermion and a spin-0 boson.

The generalization of our results to more general production processes will be discussed elsewhere.

## 3. THE DECAY DENSITY MATRIX. CONSTRUCTION OF THE PRODUCTION DENSITY MATRIX

### 3.1

In this section we assume that the fermion $F$ produced in a general production process

$$A+B\rightarrow F+\cdots \tag{3.1}$$

decays according to

$$F\rightarrow f'+c. \tag{3.2}$$

The $A$ and $B$ are arbitrary, $f'$ is a spin-$\frac{1}{2}$ fermion and $c$ is a spin-zero boson. The decay process (3.2) is not assumed to be parity conserving and, hence, the analysis can be applied either to a weakly decaying $F$ (for instance $\Xi$ production and subsequent decay) or to a strongly decaying $F$ (production and decay of baryon resonant states).

The most general density matrix of $F$ can be written as

$$\rho^{(F)}=\sum_{\mu\mu'}\rho_{\mu\mu'}|\varphi_\mu\rangle\langle\varphi_{\mu'}|\,.$$

By specializing $\rho_{\mu\mu'}$ to

$$\rho_{\mu\mu'}=\tfrac{1}{2}[1+(-1)^{\mu-\mu'}]\beta_\mu\beta_{\mu'}^*\,,$$

one has the matrix (2.6), typical of the special production process (2.1).

We call $\mathbf{u}$ and $\mathbf{u}'$, respectively, the unit vectors along the momenta of $B$ and $F$ in the center of mass frame of reaction (3.1); $\mathbf{n}=(\mathbf{u}\times\mathbf{u}')/|\mathbf{u}\times\mathbf{u}'|$ is the normal to the production plane, and we call $\mathbf{v}$ the unit vector along the momentum of $f'$ in the rest system of $F$.

After $F$ decays, the eigenfunction $|\varphi_\mu\rangle$, in the general expression for $\rho^{(F)}$, becomes in its angular and spin dependence

$$|\varphi_\mu\rangle\rightarrow\sum_{lm}(-1)^l T_l(lm,\tfrac{1}{2}\nu|s\mu)Y_l^m(\mathbf{v})|\chi_\nu\rangle, \tag{3.3}$$

where $T_l$ are the decay matrix elements, $(lm,\frac{1}{2}\nu|s\mu)$ are the usual Clebsch-Gordan-Wigner coefficients, and $|\chi_\nu\rangle$ are the spin eigenstates of $f'$. We have explicitly introduced a factor $(-1)^l$ because our angular distribution refers to the final fermion $f'$, instead of the final boson $c$.

From (3.3) we obtain, for the final state density matrix $\rho^{(f)}$,

$$\rho^{(f)}=\sum_{\mu\mu'}\rho_{\mu\mu'}\sum_{ll'}(-1)^{l-l'}T_lT_{l'}^*\sum_{mm'\nu\nu'}(lm,\tfrac{1}{2}\nu|s\mu)\times(l'm',\tfrac{1}{2}\nu'|s\mu')Y_l^m(\mathbf{v})Y_{l'}^{m'*}(\mathbf{v})|\chi_\nu\rangle\langle\chi_{\nu'}|\,. \tag{3.4}$$

We can now calculate the angular distribution of $f'$, $I(\mathbf{v})$, and its polarization $\mathbf{P}$, using the formulas

$$I(\mathbf{v})=\mathrm{Tr}[\rho^{(f)}], \tag{3.5}$$

$$I(\mathbf{v})\mathbf{P}=\mathrm{Tr}[\rho^{(f)}\boldsymbol{\sigma}]. \tag{3.6}$$

The explicit calculations are briefly outlined in the Appendix. The results are listed below. We shall use the following normalization for the matrix elements $T_l$:

$$|T_{s-\frac{1}{2}}|^2+|T_{s+\frac{1}{2}}|^2=1\,, \tag{3.7}$$

so that the angular distribution $I(\mathbf{v})$ becomes normalized to unity over the whole solid angle. We also shall use the following notation:

$$\alpha=2\,\mathrm{Re}[T_{s-\frac{1}{2}}T_{s+\frac{1}{2}}^*] \tag{3.8}$$

$$\beta=2\,\mathrm{Im}[T_{s-\frac{1}{2}}T_{s+\frac{1}{2}}^*] \tag{3.8$'$}$$

$$\epsilon=(-1)^{s-\frac{1}{2}}[|T_{s-\frac{1}{2}}|^2-|T_{s+\frac{1}{2}}|^2]. \tag{3.8$''$}$$

These coefficients satisfy

$$\alpha^2+\beta^2+\epsilon^2=1. \tag{3.8'''}$$

The angular distribution is given by

$$I(\mathbf{v})=\sum_{LM} a(L,M)Y_L{}^M(\mathbf{v}), \tag{3.9}$$

where for even $L$

$$a(L,M)=(-1)^{s-\frac{1}{2}}\left(\frac{2s+1}{4\pi}\right)^{1/2}(s\tfrac{1}{2},\, s-\tfrac{1}{2}|L0) \times\sum_\mu \rho_{\mu,\mu-M}(s\mu-M,\, LM|s\mu), \tag{3.9'}$$

and for odd $L$

$$a(L,M)=(-1)^{s-\frac{1}{2}}\alpha\left(\frac{2s+1}{4\pi}\right)^{1/2}(s\tfrac{1}{2},\, s-\tfrac{1}{2}|L0) \times\sum_\mu \rho_{\mu,\mu-M}(s\mu-M,\, LM|s\mu). \tag{3.9''}$$

For the longitudinal polarization we have

$$I(\mathbf{v})\mathbf{P}\cdot\mathbf{v}=\sum_{LM} b_1(L,M)Y_L{}^M(\mathbf{v}), \tag{3.10}$$

where for even $L$

$$b_1(L,M)=(-1)^{s-\frac{1}{2}}\alpha\left(\frac{2s+1}{4\pi}\right)^{1/2}(s\tfrac{1}{2},\, s-\tfrac{1}{2}|L0) \times\sum_\mu \rho_{\mu,\mu-M}(s\mu-M,\, LM|s\mu), \tag{3.10'}$$

and for odd $L$

$$b_1(L,M)=(-1)^{s-\frac{1}{2}}\left(\frac{2s+1}{4\pi}\right)^{1/2}(s\tfrac{1}{2},\, s-\tfrac{1}{2}|L0) \times\sum_\mu \rho_{\mu,\mu-M}(s\mu-M,\, LM|s\mu). \tag{3.10''}$$

The polarization along $\mathbf{n}\times\mathbf{v}$ is given by

$$I(\mathbf{v})\mathbf{P}\cdot\mathbf{n}\times\mathbf{v}=\sum_{LM} b_2(L,M)Y_L{}^M(\mathbf{v}), \tag{3.11}$$

where for odd $L$

$$b_2(L,M)=i\epsilon\left(\frac{2s+1}{4\pi}\right)^{1/2}\frac{2s+1}{[L(L+1)]^{1/2}}(s\tfrac{1}{2},\, s-\tfrac{1}{2}|L0) \times(LM,10|LM)\sum_\mu \rho_{\mu,\mu-M}(s\mu-M,LM|s\mu), \tag{3.11'}$$

and for even $L$

$$b_2(L,M)=(-1)^{s-\frac{1}{2}}\beta\left(\frac{2s+1}{4\pi}\right)^{1/2}\frac{2s+1}{(2L+1)^{1/2}} \times\sum_{\text{odd } f} C_f(s\tfrac{1}{2},\, s-\tfrac{1}{2}|f0)(fM,10|LM) \times\sum_\mu \rho_{\mu,\mu-M}(s\mu-M,\, fM|s\mu). \tag{3.11''}$$

Here

$$\begin{aligned} C_f&=(f+1)^{-1/2} && \text{for } f=L-1 \\ &=f^{-1/2} && \text{for } f=L+1. \end{aligned} \tag{3.11'''}$$

Finally, for the polarization along $\mathbf{v}\times(\mathbf{n}\times\mathbf{v})$ we have

$$I(\mathbf{v})\mathbf{P}\cdot\mathbf{v}\times(\mathbf{n}\times\mathbf{v})=\sum_{LM} b_3(L,M)Y_L{}^M(\mathbf{v}), \tag{3.12}$$

where for odd $L$

$$b_3(L,M)=(-1)^{s-\frac{1}{2}}i\beta\left(\frac{2s+1}{4\pi}\right)^{1/2}(s\tfrac{1}{2},\, s-\tfrac{1}{2}|L0) \times(LM,10|LM)\sum_\mu \rho_{\mu,\mu-M}(s\mu-M,\, LM|s\mu), \tag{3.12'}$$

and for even $L$

$$b_3(L,M)=-\epsilon\left(\frac{2s+1}{4\pi}\right)^{1/2}\frac{2s+1}{(2L+1)^{1/2}} \times\sum_{\text{odd } f} C_f(s\tfrac{1}{2},\, s-\tfrac{1}{2}|f0)(fM,10|LM) \times\sum_\mu \rho_{\mu,\mu-M}(s\mu-M,\, fM|s\mu). \tag{3.12''}$$

Again $C_f$ is given by (3.11‴).

If parity is conserved in reaction (3.2) then $\alpha$ and $\beta$ are zero and $\epsilon$ is $+1$ for even final $l$ waves and $-1$ for odd final $l$ waves.

We observe that in the Adair's[11] limit of 0° or 180° production angle, we must have symmetry about the direction of the incident beam, and hence $M\equiv\mu-\mu'=0$ and $\rho_{\mu\mu}=\rho_{-\mu-\mu}$. In this case the angular distribution contains only the even $L$ terms

$$I(\mathbf{v})=\sum_{\text{even } L} a(L,0)Y_L{}^0(\mathbf{v}), \tag{3.13}$$

and the polarization $\mathbf{P}$ is completely longitudinal and constant in magnitude.

$$\mathbf{P}=\alpha\mathbf{v}. \tag{3.14}$$

### 3.2

The coefficients $a(L,M)$ and $b_i(L,M)$ can be obtained from experiment by certain weighted averages of the angular distribution and polarization of the final fermion. In particular, we have

$$a(L,-M)=\langle Y_M{}^L\rangle, \tag{3.15}$$

$$b_1(L,-M)=\langle\mathbf{P}\cdot\mathbf{v}Y_L{}^M\rangle, \tag{3.16}$$

[11] R. K. Adair, Phys. Rev. **100**, 1540 (1955).

$$b_2(L,-M)=\langle \mathbf{P}\cdot(\mathbf{n}\times\mathbf{v})Y_L{}^M\rangle, \quad (3.17)$$

$$b_3(L,-M)=\langle \mathbf{P}\cdot\mathbf{v}\times(\mathbf{n}\times\mathbf{v})Y_L{}^M\rangle, \quad (3.18)$$

where

$$\langle A\rangle=\int I(\mathbf{v})A(\mathbf{v})d\Omega_{\mathbf{v}}.$$

Owing to the reality of $I(\mathbf{v})$ we have

$$a(L,-M)=(-1)^M a(L,M)^*,$$

and also analogous relations for the $b_i(L,M)$.

From (3.9′)–(3.12″) we obtain the following general relations among the coefficients $a(L,M)$ and $b_i(L,M)$:

$$b_1(L,M)=\alpha a(L,M) \quad \text{for even } L, \quad (3.19)$$

$$b_1(L,M)=(1/\alpha)a(L,M) \quad \text{for odd } L, \quad (3.19')$$

$$b_2(L,M)=(-1)^{s-\frac{1}{2}}\frac{i\epsilon}{\alpha}\frac{2s+1}{L(L+1)}Ma(L,M) \quad \text{for odd } L, \quad (3.20)$$

$$b_2(L,M)=\frac{\beta}{\alpha}(2s+1)\left\{\frac{1}{L}\left[\frac{(L-M)(L+M)}{(2L-1)(2L+1)}\right]^{1/2}a(L-1,M)-\frac{1}{L+1}\left[\frac{(L-M+1)(L+M+1)}{(2L+1)(2L+3)}\right]^{1/2}a(L+1,M)\right\} \quad \text{for even } L, \quad (3.20')$$

$$b_3(L,M)=-\frac{i\beta}{\alpha}\frac{2s+1}{L(L+1)}Ma(L,M) \quad \text{for odd } L, \quad (3.21)$$

$$b_3(L,M)=(-1)^{s+\frac{1}{2}}\frac{\epsilon}{\alpha}(2s+1)\left\{\frac{1}{L}\left[\frac{(L-M)(L+M)}{(2L-1)(2L+1)}\right]^{1/2}a(L-1,M)-\frac{1}{L+1}\left[\frac{(L-M+1)(L+M+1)}{(2L+1)(2L+3)}\right]^{1/2}a(L+1,M)\right\} \quad \text{for even } L, \quad (3.21')$$

and also

$$b_3(L,M)=(-1)^{s-\frac{1}{2}}(\beta/\epsilon)b_2(L,M) \quad \text{for odd } L, \quad (3.22)$$

$$b_3(L,M)=(-1)^{s+\frac{1}{2}}(\epsilon/\beta)b_2(L,M) \quad \text{for even } L, \quad (3.22')$$

which are equivalent to (3.21) and (3.21′), respectively.

For $L=0$ in the right-hand side of (3.20′) and (3.21′) only the second term in the curly brackets exists.

All the above relations are valid for any production process of $F$.

These relations can, by themselves, furnish general tests for determining the spin and the decay parameters of $F$, provided only that the transverse polarization is appreciable. In fact, fitting these relations for various values of $L$ and $M$, we can obtain the parameters

$$\alpha; \quad \epsilon'=(-1)^{s-\frac{1}{2}}(2s+1)\epsilon; \quad \text{and} \quad \beta'=(2s+1)\beta$$

directly from experiment. Using (3.8‴) we obtain

$$\beta'^2+\epsilon'^2=(2s+1)^2(1-\alpha^2). \quad (3.23)$$

This gives unambiguously the spin $s$ which, in turn, permits also the determination of $\beta$ and $\epsilon$ provided $\beta$, $\epsilon\neq0$, and $\alpha\neq\pm1$. Other tests given below are also possible for this particular situation.

Let us consider a particular case of the above procedure. Using (3.20′) and (3.21′) with $L=M=0$ and also (3.19′) and the definitions (3.15)–(3.18) we have, for any value of $\alpha$:

$$\beta'=-\frac{\langle \mathbf{P}\cdot\mathbf{n}\times\mathbf{v}\rangle}{\langle(\mathbf{P}\cdot\mathbf{v})(\mathbf{n}\cdot\mathbf{v})\rangle},$$

$$\epsilon'=\frac{\langle \mathbf{P}\cdot\mathbf{v}\times(\mathbf{n}\times\mathbf{v})\rangle}{\langle(\mathbf{P}\cdot\mathbf{v})(\mathbf{n}\cdot\mathbf{v})\rangle}.$$

Thus (3.23) can be written

$$(2s+1)^2=\frac{\langle \mathbf{P}\cdot\mathbf{n}\times\mathbf{v}\rangle^2+\langle \mathbf{P}\cdot\mathbf{v}\times(\mathbf{n}\times\mathbf{v})\rangle^2}{\langle(\mathbf{P}\cdot\mathbf{v})(\mathbf{n}\cdot\mathbf{v})\rangle^2-\langle\mathbf{n}\cdot\mathbf{v}\rangle^2}. \quad (3.24)$$

This relation provides a very direct general test for the spin.

### 3.3

Let us consider now the case in which the particle $F$ is produced in reaction (2.1). Equations (3.15), (3.7′), and (3.7″) can be written as

$$\langle Y_L{}^M\rangle=(-1)^{s-\frac{1}{2}}\left(\frac{2s+1}{4\pi}\right)^{1/2}g_L(s\tfrac{1}{2},s-\tfrac{1}{2}|L0)\times\sum_\mu \rho_{\mu,\mu+M}(s\mu,LM|s\mu+M), \quad (3.25)$$

where $M$ is even and

$$g_L=\tfrac{1}{2}[(1+\alpha)+(-1)^L(1-\alpha)]=1 \text{ for even } L$$
$$=\alpha \text{ for odd } L.$$

Equation (3.25) can be inverted. This gives

$$\rho_{\mu,\mu+M}=(-1)^{\mu-\frac{1}{2}}\frac{(4\pi)^{1/2}}{2s+1}\sum_L \frac{1}{g_L}(2L+1)^{1/2} \times\frac{(s\mu,\, s-\mu-M\,|\,L-M)}{(s\tfrac{1}{2},\, s-\tfrac{1}{2}\,|\,L0)}\langle Y_L{}^M\rangle. \quad (3.26)$$

We easily see that the conditions

$$\rho^{(F)\dagger}=\rho^{(F)};\quad \mathrm{Tr}[\rho^{(F)}]=1$$

are identically satisfied by (3.26).

We now discuss a possible way of using (3.26) to test the spin of $F$.

From the experimental averages $\langle Y_L{}^M\rangle$ and for an assumed value of $s$, we can construct the matrix $\rho^{(F)}$ whose elements $\rho_{\mu\nu}{}^{(F)}$ are given by the right-hand side of (3.26).

If $\alpha=0$, we can for the odd-$L$ terms use (3.19′) to replace the indeterminate form $(1/\alpha)\langle Y_L{}^M\rangle$ by $\langle \mathbf{P}\cdot\mathbf{v}Y_L{}^M\rangle$.

The matrix so constructed should, apart from experimental error, satisfy the conditions discussed in Sec. 2.2, provided the assumption regarding the spin is correct. On the contrary, it is possible to show that a matrix $\rho^{(F)}$ constructed by assuming a wrong value of the spin cannot satisfy the same conditions. For this purpose let us suppose $s$ to be the true value of the spin and take $s'>s$ as the test value for the computation of the matrix, which we denote by $\rho^{(s')}$. The experimental averages must satisfy

$$\langle Y_L{}^M\rangle=0 \quad \text{for} \quad L>2s,$$

so that all the matrix elements $\rho_{\mu\nu}{}^{(s')}$ with $|\nu-\mu| =|M|>2s-1$ will vanish. If now we suppose that $\rho'^{(s')}$ and $\rho''^{(s')}$ both satisfy (2.12) it is easy to see that for each of these submatrices all the elements must be zero, except those of one minor across the diagonal, of the order $\frac{1}{2}(2s+1)$. For each value of $|\nu-\mu|\leq 2s-1$ the vanishing of the matrix elements gives, by (3.26), $2(s'-s)$ linear homogeneous equations between the averages, which can be seen to be incompatible.

## 4. CONCLUSIONS

In this section we summarize the possible tests for the determination of the spin of the particle $F$.

### 4.1

General tests, valid for any production process, are those considered by Lee and Yang[4] and by Durand, Landovitz, and Leitner,[5] which follow from the conditions

$$\rho_{\mu\mu}{}^{(F)}\geq 0,\quad \textstyle\sum_\mu \rho_{\mu\mu}{}^{(F)}=1. \quad (4.1)$$

$\rho_{\mu\mu}{}^{(F)}$ is, in fact, the probability $P_\mu$ to find $F$ with spin component $\mu$ along any direction $\mathbf{n}$.

There are two kinds of such inequalities. First we have limitations on the experimental averages (3.15)–(3.18) for $M=0$. In fact, all the expressions of $a(L,0)$ and $b_i(L,0)$ contain the factor

$$\textstyle\sum_\mu \rho_{\mu\mu}(s\mu,L0\,|\,s\mu)$$

which is the expectation value of the Clebsch-Gordan coefficient $(s\mu,L0\,|\,s\mu)$ in the initial state. Thus, we have

$$|\textstyle\sum_\mu \rho_{\mu\mu}(s\mu,L0\,|\,s\mu)|\leq \max|(s\mu,L0\,|\,s\mu)|, \quad (4.2)$$

where the notation max on the right-hand side of (4.2) means the maximum value of the argument, with respect to $\mu$.

For the angular distribution we have

$$|\langle P_L(\mathbf{n}\cdot\mathbf{v})\rangle|\leq|(s\tfrac{1}{2},L0\,|\,s\tfrac{1}{2})|\max|(s\mu,L0\,|\,s\mu)|, \quad (4.3)$$

where we have used the Legendre polynomials $P_L$ instead of the spherical harmonics. For odd $L$ we have also made use of the condition $|\alpha|\leq 1$. For $L=1$, we have $\max|(s\mu,10\,|\,s\mu)|=(s/s+1)^{1/2}$ from which follows the well-known limitaton of Lee and Yang:

$$|\langle\mathbf{n}\cdot\mathbf{v}\rangle|\leq 1/(2s+2). \quad (4.4)$$

For the longitudinal polarization we have limitations analogous to (4.3) and (4.4):

$$|\langle\mathbf{P}\cdot\mathbf{v}P_L(\mathbf{n}\cdot\mathbf{v})\rangle| \leq|(s\tfrac{1}{2},L0\,|\,s\tfrac{1}{2})|\max|(s\mu,L0\,|\,s\mu)|. \quad (4.5)$$

For $L=1$, this becomes

$$|\langle(\mathbf{P}\cdot\mathbf{v})(\mathbf{n}\cdot\mathbf{v})\rangle|\leq 1/(2s+2). \quad (4.6)$$

Analogous limitations can also be found for transverse polarization, but they are less useful than the preceding inequalities. For example, from (3.9″) and for $L=0$, we obtain

$$|\langle\mathbf{P}\cdot\mathbf{n}\times\mathbf{v}\rangle|\leq(2s+1)/(2s+2). \quad (4.7)$$

More general tests that can be derived from (4.1) are limitations on some test functions that can be constructed from the experimental averages.

We have, independent of the value of $\alpha$,

$$T_\mu=(-1)^{\mu-\frac{1}{2}}\frac{1}{2s+1} \times\sum_L(2L+1)\frac{(s\mu,\, s-\mu\,|\,L0)}{(s\tfrac{1}{2},\, s-\tfrac{1}{2}\,|\,L0)}\langle P_L\rangle\geq 0. \quad (4.8)$$

In fact, the sum over the even-$L$ terms gives $\frac{1}{2}(P_\mu+P_{-\mu})$ whereas the sum over the odd-$L$ terms gives $\frac{1}{2}(P_\mu-P_{-\mu})$. Since $|\alpha|\leq 1$ the entire sum is positive.

We notice that limitations entirely analogous to (4.8)

TABLE I. Numerical values of the coefficients $C_{\mu,\mu+M}^{(s)}(L)$, $s=\frac{1}{2}$.

| | | $\mu$ | |
|---|---|---|---|
| $L$ | $M$ | $+\frac{1}{2}$ | $-\frac{1}{2}$ |
| 0 | 0 | 1.7725 | 1.7725 |
| 1 | 0 | 3.0700 | −3.0700 |

TABLE II. Numerical values of the coefficients $C_{\mu,\mu+M}^{(s)}(L)$, $s=\frac{3}{2}$.

| | | $\mu$ | | | |
|---|---|---|---|---|---|
| $L$ | $M$ | $+\frac{3}{2}$ | $+\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{3}{2}$ |
| 0 | 0 | 0.8862 | 0.8862 | 0.8862 | 0.8862 |
| 1 | 0 | 4.6050 | 1.5350 | −1.5350 | −4.6050 |
| 2 | 0 | −1.9817 | 1.9817 | 1.9817 | −1.9817 |
| 2 | 2 | 0.0000 | 0.0000 | −2.8025 | −2.8025 |
| 3 | 0 | −0.7816 | 2.3447 | −2.3447 | 0.7816 |
| 3 | 2 | 0.0000 | 0.0000 | −2.4716 | 2.4716 |

hold for the test functions obtained by the replacement in (4.8) of $\langle P_L\rangle$ by $\langle \mathbf{P}\cdot\mathbf{v}P_L\rangle$.

Analogous but a little more complicated test functions can be defined for the transverse polarization (see, e.g., Ref. 5).

We emphasize that limitations (4.3)–(4.7), or alternatively (4.8) and analogous ones must be satisfied for any production process, but they are, in general, not sufficient for the spin determination. We also remark that such limitations do not involve any knowledge of the decay parameters $\alpha$, $\beta$, and $\epsilon$.

More precise tests, also valid in the general case, are those discussed in subsection 3.2. They consist in verifying the relations (3.19)–(3.21) among the averages $a(L,M)$ and $b_i(L,M)$ defined as in (3.15)–(3.18). This would give the possibility of a direct determination of the spin and decay parameters of $F$ [see e.g., Eq. (3.24)].

### 4.2

We now discuss the particular case of production of $F$ from a spin-zero boson on an unpolarized spin-$\frac{1}{2}$ fermion.

In this case, the density matrix $\rho^{(F)}$ must be of the form (2.6), so that the angular and polarization distributions (3.9)–(3.12) must be expressible in terms of the $2s+1$ complex numbers $\beta_\mu$ defined in (2.3). These are $4s$ real independent parameters because of the condition (2.4) and an indeterminate relative phase between $\beta_\mu$ and $\beta_\nu$ for odd values of $\mu-\nu$.

Without considering complications coming from experimental errors, a possible way to obtain definite tests for the spin of $F$ is to compute the matrix $\rho^{(F)}$ directly from the experimental data. Summarizing the results of subsections 3.3 and 2.2 one should proceed as follows:

(i) From the experimental angular distribution $I(\mathbf{v})$ of $f'$ in the decay of $F$ according to reaction (3.2), one computes the averages

$$\langle Y_L{}^M\rangle=\int Y_L{}^M(\mathbf{v})I(\mathbf{v})d\Omega_{\mathbf{v}}.$$

(ii) One then constructs the test matrix for spin $s$

$$\rho_{\mu\nu}{}^{(s)}=\sum\nolimits_L C_{\mu\nu}{}^{(s)}(L)(1/g_L)\langle Y_L{}^M\rangle\,, \qquad (4.9)$$

where $\nu=\mu+M$, $M$ even, and

$$C_{\mu\nu}{}^{(s)}(L)=(-1)^{\mu-\frac{1}{2}}\frac{(4\pi)^{1/2}}{2s+1}(2L+1)^{1/2} \times\frac{(s\mu,\, s-\nu\,|\,L-M)}{(s\tfrac{1}{2},\, s-\tfrac{1}{2}\,|\,L0)}. \qquad (4.10)$$

The coefficients $C_{\mu\nu}{}^{(s)}(L)$ have been numerically evaluated for $s=\frac{1}{2}$, $\frac{3}{2}$, and $\frac{5}{2}$. Their values are reported in Tables II and III. In (4.9), $g_L$ is given by

$$g_L=1 \text{ for even } L$$
$$=\alpha \text{ for odd } L.$$

The decay parameter $\alpha$ can be determined, for instance, from (3.19) as the average longitudinal polarization

$$\alpha=\langle\mathbf{P}\cdot\mathbf{v}\rangle. \qquad (4.11)$$

(iii) The test matrix $\rho_{\mu\nu}{}^{(s)}$ must be such that the two matrices $\rho_{\mu\nu}{}'^{(s)}$ [obtained from $\rho_{\mu\nu}{}^{(s)}$ by putting equal

TABLE III. Numerical values of the coefficients $C_{\mu,\mu+M}^{(s)}(L)$, $s=\frac{5}{2}$.

| | | $\mu$ | | | | | |
|---|---|---|---|---|---|---|---|
| $L$ | $M$ | $+\frac{5}{2}$ | $+\frac{3}{2}$ | $+\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{3}{2}$ | $-\frac{5}{2}$ |
| 0 | 0 | 0.5908 | 0.5908 | 0.5908 | 0.5908 | 0.5908 | 0.5908 |
| 1 | 0 | 5.1166 | 3.0700 | 1.0233 | −1.0233 | −3.0700 | −5.1166 |
| 2 | 0 | −1.6514 | 0.3303 | 1.3211 | 1.3211 | 0.3303 | −1.6514 |
| 2 | 2 | 0.0000 | 0.0000 | −1.2792 | −1.7162 | −1.7162 | −1.2792 |
| 3 | 0 | −1.9540 | 2.7355 | 1.5632 | −1.5632 | −2.7355 | 1.9540 |
| 3 | 2 | 0.0000 | 0.0000 | −3.3843 | −1.5135 | 1.5135 | 3.3843 |
| 4 | 0 | 0.8862 | −2.6587 | 1.7725 | 1.7725 | −2.6587 | 0.8862 |
| 4 | 2 | 0.0000 | 0.0000 | 2.6587 | −1.9817 | −1.9817 | 2.6587 |
| 4 | 4 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 3.3160 | 3.3160 |
| 5 | 0 | 0.1960 | −0.9798 | 1.9595 | −1.9595 | 0.9798 | −0.1960 |
| 5 | 2 | 0.0000 | 0.0000 | 0.8980 | −2.0079 | 2.0079 | −0.8980 |
| 5 | 4 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 2.1996 | −2.1996 |

to zero the matrix elements for which $s-\mu$ and $s-\nu$ are both even] and $\rho_{\mu\nu}''^{(s)}$ [obtained from $\rho_{\mu\nu}^{(s)}$ by putting equal to zero the elements with $s-\mu$ and $s-\nu$ both odd] both have rank one, and non-negative traces. These conditions are such that for given $\langle Y_L^M\rangle$ they can be satisfied for only one test matrix $\rho^{(s)}$, thus, defining uniquely the spin $s$.

Of course, the general tests discussed in Sec. 4.1 for a general production process also apply to the case discussed in this section.

## APPENDIX

We shall briefly derive the general expressions of the coefficients $a(L,M)$ and $b_i(L,M)$ of Eqs. (3.9)–(3.12) using the standard Racah algebra.

### A. Angular Distribution

From (3.5) and (3.4) and using the formula

$$Y_l^m(\mathbf{v})Y_{l'}^{m'*}(\mathbf{v})=(-1)^{m'}\frac{\hat{l}\hat{l}'}{(4\pi)^{1/2}}\sum_{LM}\frac{1}{\hat{L}}(l0,l'0|L0)(lm,l'm'|LM)Y_L^M(\mathbf{v}),\tag{A1}$$

where we have used the abreviation

$$\hat{l}=(2l+1)^{1/2},$$

we obtain

$$I(\mathbf{v})=\sum_{LM}a(L,M)Y_L^M(\mathbf{v}),\tag{A2}$$

where

$$a(L,M)=\frac{(-1)^L}{(4\pi)^{1/2}\hat{L}}\sum_{\mu\mu'}\rho_{\mu\mu'}\sum_{ll'}T_lT_{l'}^*\hat{l}\hat{l}'(l0,l'0|L0)\sum_{mm'\nu}(-1)^{m'}(lm,\tfrac{1}{2}\nu|s\mu)(l'm',\tfrac{1}{2}\nu|s\mu')(lm,\,l'-m'|LM).\tag{A2$'$}$$

By virtue of the formula,

$$\sum_{\alpha\beta\epsilon}(a\alpha,b\beta|e\epsilon)(e\epsilon,d\delta|c\gamma)(b\beta,d\delta|f\varphi)=\hat{e}\hat{f}W(abcd;ef)(a\alpha,f\varphi|c\gamma),\tag{A3}$$

where $W$ is the standard Racah coefficient, and the symmetry properties of the Clebsch–Gordan (C.G.) coefficients, the sum over the magnetic quantum numbers in (A2′) becomes

$$(-1)^{L+s-\frac{1}{2}}\hat{s}\hat{L}W(slsl';\tfrac{1}{2}L)(s\mu',LM|s\mu).$$

We now use the remarkable formula

$$\hat{b}\hat{d}(b0,d0|f0)W(abcd;\tfrac{1}{2}f)=(a\tfrac{1}{2},\,c-\tfrac{1}{2}|f0),\tag{A4}$$

so that any dependence on $l$ and $l'$ disappears in the coefficient of $T_lT_{l'}^*$ in (A2′), and we obtain

$$a(L,M)=(-1)^{s-\frac{1}{2}}\frac{\hat{s}}{(4\pi)^{1/2}}(s\tfrac{1}{2},\,s-\tfrac{1}{2}|L0)\sum_{ll'}T_lT_{l'}^*\sum_{\mu\mu'}\rho_{\mu\mu'}(s\mu',LM|s\mu).\tag{A5}$$

Observing, now, that $l$ and $l'$ take the values $s\pm\frac{1}{2}$ and that $l+l'+L$ must be even, because of the C.G. coefficient $(l0,l'0|L0)$ in (A2), and remembering the definitions (3.7) and (3.8), we obtain from (A5) the expressions (3.9′) and (3.9″), respectively, for even and odd $L$.

### B. Polarization

Let us define three orthogonal unit vectors as

$$\mathbf{v}_1=\mathbf{v};\quad \mathbf{v}_2=\frac{\mathbf{n}\times\mathbf{v}}{|\mathbf{n}\times\mathbf{v}|};\quad \mathbf{v}_3=\mathbf{v}_1\times\mathbf{v}_2.\tag{A6}$$

The polarization of the final fermion $f'$ along the direction $\mathbf{v}_i$ is, by (3.6) and (3.4),

$$I(\mathbf{v})\mathbf{P}\cdot\mathbf{v}_i=\sum_{\mu\mu'}\rho_{\mu\mu'}\sum_{ll'}(-1)^{l-l'}T_lT_{l'}^*\sum_{mm'\nu\nu'}(lm,\tfrac{1}{2}\nu|s\mu)(l'm',\tfrac{1}{2}\nu'|s\mu')Y_l^m(\mathbf{v})Y_{l'}^{m'*}(\mathbf{v})\langle\nu'|\boldsymbol{\sigma}\cdot\mathbf{v}_i|\nu\rangle.\tag{A7}$$

It is easy to see that

$$\langle\nu'|\boldsymbol{\sigma}\cdot\mathbf{v}_i|\nu\rangle=(4\pi)^{1/2}\sum_{\kappa}(\tfrac{1}{2}\nu',1\kappa|\tfrac{1}{2}\nu)Y_1^{\kappa}(\mathbf{v}_i).\tag{A8}$$

Furthermore, we can express the spherical harmonics of argument $\mathbf{v}_2$ and $\mathbf{v}_3$ by the one of argument $\mathbf{v}$

$$|\mathbf{n}\times\mathbf{v}|\,Y_1^{\kappa}(\mathbf{v}_2)=i\sqrt{2}(1\kappa,10|1\kappa)Y_1^{\kappa}(\mathbf{v}),\tag{A9}$$

$$|\mathbf{n}\times\mathbf{v}|\,Y_1^{\kappa}(\mathbf{v}_3)=-\left(\frac{4\pi}{3}\right)^{1/2}Y_1^{0}(\mathbf{v})Y_1^{\kappa}(\mathbf{v})+\left(\frac{3}{4\pi}\right)^{1/2}\delta_{\kappa 0}.\tag{A9$'$}$$

Substituting (A8), (A9), and (A9′) into (A7), and subsequently using (A1), we get

$$I(\mathbf{v})\mathbf{P}\cdot\mathbf{v}=\sum_{LM}b_1(L,M)Y_L^M(\mathbf{v}),\tag{A10}$$

where

$$b_1(L,M)=\left(\frac{3}{4\pi}\right)^{1/2}\frac{1}{\hat{L}}\sum_{\mu\mu'}\rho_{\mu\mu'}\sum_{ll'}(-1)^{l-l'}T_lT_{l'}{}^*\hat{l}\hat{l}'\sum_{l''}(l0,l'0|l''0)(l''0,10|L0)\sum_{mm'm''}\sum_{\nu\nu'\kappa}(-1)^{m'}(lm,\tfrac{1}{2}\nu|s\mu)$$
$$\times(l'm',\tfrac{1}{2}\nu'|s\mu')(\tfrac{1}{2}\nu',1\kappa|\tfrac{1}{2}\nu)(lm,l'-m'|l''m'')(l''m'',1\kappa|LM);\tag{A10$'$}$$

$$I(\mathbf{v})\mathbf{P}\cdot(\mathbf{n}\times\mathbf{v})=\sum_{LM}b_2(L,M)Y_L^M(\mathbf{v}),\tag{A11}$$

where

$$b_2(L,M)=i\left(\frac{3}{2\pi}\right)^{1/2}\frac{1}{\hat{L}}\sum_{\mu\mu'}\rho_{\mu\mu'}\sum_{ll'}(-1)^{l-l'}T_lT_{l'}{}^*\hat{l}\hat{l}'\sum_{l''}(l0,l'0|l''0)(l''0,10|L0)\sum_{mm'm''}\sum_{\nu\nu'\kappa\kappa'}(-1)^{m'}(lm,\tfrac{1}{2}\nu|s\mu)$$
$$\times(l'm',\tfrac{1}{2}\nu'|s\mu')(\tfrac{1}{2}\nu',1\kappa|\tfrac{1}{2}\nu)(lm,l'-m'|l''m'')(l''m'',1\kappa'|LM)(1\kappa,10|1\kappa');\tag{A11$'$}$$

and

$$I(\mathbf{v})\mathbf{P}\cdot\mathbf{v}\times(\mathbf{n}\times\mathbf{v})=\sum_{LM}b_3(L,M)Y_L^M(\mathbf{v});\tag{A12}$$

where

$$b_3(L,M)=-\left(\frac{3}{4\pi}\right)^{1/2}\frac{1}{\hat{L}}\sum_{\mu\mu'}\rho_{\mu\mu'}\sum_{ll'}(-1)^{l-l'}T_lT_{l'}{}^*\hat{l}\hat{l}'\Big[\sum_{l''l'''}(l0,l'0|l''0)(l''0,10l'''0)(l'''0,10|L0)$$
$$\times\sum_{mm'm''}\sum_{m'''\nu\nu'\kappa}(-1)^{m'}(lm,\tfrac{1}{2}\nu|s\mu)(l'm',\tfrac{1}{2}\nu'|s\mu')(\tfrac{1}{2}\nu',1\kappa|\tfrac{1}{2}\nu)(lm,l'-m'|l''m'')(l''m'',1\kappa|l'''m''')$$
$$\times(l'''m''',10|LM)-(l0,l'0|L0)\sum_{mm'\nu\nu'}(-1)^{m'}(lm,\tfrac{1}{2}\nu|s\mu)(l'm',\tfrac{1}{2}\nu'|s\mu')(\tfrac{1}{2}\nu',10|\tfrac{1}{2}\nu)(lm,l'-m'|LM)\Big].\tag{A12$'$}$$

### 1. *Calculation of* $b_1(L,M)$

To perform the summation on the magnetic quantum numbers we shall use the formula

$$\sum_{\alpha\beta\delta}\sum_{\epsilon\eta\theta}(a\alpha,b\beta|c\gamma)(d\delta,e\epsilon|f\varphi)(g\eta,h\theta|k\kappa)(a\alpha,d\delta|g\eta)(b\beta,e\epsilon|h\theta)=\hat{c}\hat{f}\hat{g}\hat{h}(c\gamma,f\varphi|k\kappa)X\begin{Bmatrix}a&b&c\\d&e&f\\g&h&k\end{Bmatrix},\tag{A13}$$

where $X$ is the Wigner $9-j$ coefficient.

The summation on the magnetic quantum numbers in (A10′) gives then

$$(-1)^l\sqrt{2}\hat{s}\hat{L}\hat{l}''X\begin{Bmatrix}l&s&\frac{1}{2}\\l'&s&\frac{1}{2}\\l''&L&1\end{Bmatrix}(s\mu',LM|s\mu).$$

Using now the identity

$$\hat{a}\hat{d}\hat{h}\hat{k}\sum_{g}(a0,d0|g0)(h0,k0|g0)X\begin{Bmatrix} a & b & \frac{1}{2} \\ d & e & \frac{1}{2} \\ g & h & k \end{Bmatrix}=(-1)^{d+e+k-\frac{1}{2}}(1/\sqrt{2})(b\tfrac{1}{2},e-\tfrac{1}{2}|h0)\,, \tag{A14}$$

which can be proved using the expression of the $X$ coefficient in terms of Racah $W$ coefficients and (A4), we get finally

$$b_1(L,M)=(-1)^{s-\frac{1}{2}}\frac{\hat{s}}{(4\pi)^{1/2}}(s\tfrac{1}{2},s-\tfrac{1}{2}|L0)\sum_{ll'}T_lT_{l'}^*\sum_{\mu\mu'}\rho_{\mu\mu'}(s\mu',LM|s\mu)\,. \tag{A15}$$

Observing now that in (A10′) $l+l'$ must be odd and remembering (3.7) and (3.8) we see that (A15) coincides with (3.10′) and (3.10″) for even $L$ and odd $L$, respectively.

### 2. *Calculation of* $b_2(L,M)$

In (A11′) we first perform the sum over $\kappa'$; using (A3) we obtain

$$\sum_{\kappa'}(1\kappa,10|1\kappa')(l''m'',1\kappa'|LM)=-\sqrt{3}\sum_{f\varphi}\hat{f}W(11Ll'';1f)(1\kappa,l''m''|f\varphi)(10,f\varphi|LM)\,.$$

We now perform the sum over the other magnetic quantum numbers by means of (A13) getting

$$b_2(L,M)=i\frac{6}{(4\pi)^{1/2}}\frac{\hat{s}}{\hat{L}}\sum_{\mu\mu'}\rho_{\mu\mu'}\sum_{ll'}(-1)^{l-l'}T_lT_{l'}^*\hat{l}\hat{l}'\sum_{l''}\hat{l}''(l0,l'0|l''0)(l''0,10|L0)$$

$$\times\sum_{f\varphi}\hat{f}^2W(11Ll'';1f)X\begin{Bmatrix} l & s & \frac{1}{2} \\ l' & s & \frac{1}{2} \\ l'' & f & 1 \end{Bmatrix}(s\mu',f\varphi|s\mu)(10,f\varphi|LM)\,. \tag{A16}$$

We must now distinguish between the case of odd $L$ and the case of even $L$. In the former case, the sum over $f$ contains the one term $f=L$. Furthermore, we can use the identity

$$\hat{l}\hat{l}'\sum_{l''}(l0,l'0|l''0)(l''0,10|L0)W(11Ll'';1L)X\begin{Bmatrix} l & s & \frac{1}{2} \\ l' & s & \frac{1}{2} \\ l'' & L & 1 \end{Bmatrix}=-\frac{1}{6}\frac{\hat{s}^2}{\hat{L}[L(L+1)]^{1/2}}(s\tfrac{1}{2},s-\tfrac{1}{2}|L0)\,, \tag{A17}$$

so that for odd $L$, using the fact $l'=l$ and introducing (3.8″) we easily obtain (3.11′).

For even $L$ we have $f=l''=L\pm1$. Using now the identity

$$\hat{l}\hat{l}'(l0,l'0|l''0)X\begin{Bmatrix} l & s & \frac{1}{2} \\ l' & s & \frac{1}{2} \\ l'' & l'' & 1 \end{Bmatrix}=\frac{1}{\sqrt{6}}\frac{\hat{s}^2}{\hat{l}''[l''(l''+1)]^{1/2}}(s\tfrac{1}{2},s-\tfrac{1}{2}|l''0) \tag{A18}$$

and the explicit expression

$$(l''0,10|L0)W(11Ll'';1l'')=-\frac{1}{\sqrt{6}}\frac{1}{2l''+1}F_{l''}\,, \tag{A19}$$

where

$$\begin{aligned} F_{l''}&=(l'')^{1/2} && \text{for}\quad l''=L-1 \\ &=(l''+1)^{1/2} && \text{for}\quad L''=L+1 \end{aligned} \tag{A19′}$$

and using the definition (3.8′) we immediately obtain (3.11″).

### 3. *Calculation of* $b_3(L,M)$

Performing the summation on the magnetic quantum numbers in (A12′) by making use of (A13), we obtain

$$b_3(L,M)=\left(\frac{6}{4\pi}\right)^{1/2}\frac{\hat{s}}{\hat{L}}\sum_{\mu\mu'}\rho_{\mu\mu'}\sum_{ll'}(-1)^{l-l'}T_lT_{l'}^{*}\hat{l}\hat{l}'$$
$$\times\left[\sum_{l''l'''}(-1)^{l'+l'''}\hat{l}''\hat{l}'''(l0,l'0|l''0)(l''0,10|l'''0)(l'''0,10|L0)(s\mu',l'''M|s\mu)(l'''M,10|LM)X\begin{Bmatrix}l & s & \frac{1}{2}\\ l' & s & \frac{1}{2}\\ l'' & l''' & 1\end{Bmatrix}\right.$$
$$\left.+(-1)^{l}\hat{L}(l0,l'0|L0)\sum_{f}\hat{f}(s\mu',fM|s\mu)(10,LM|fM)X\begin{Bmatrix}l & s & \frac{1}{2}\\ l' & s & \frac{1}{2}\\ L & f & 1\end{Bmatrix}\right].\qquad\text{(A20)}$$

In the first term in the square brackets we perform the summation over $l''$ using the identity (A14) and change $l''$ in $f$ so that $b_3(L,M)$ can be written

$$b_3(L,M)=\frac{\hat{s}}{(4\pi)^{1/2}}\sum_{\mu\mu'}\rho_{\mu\mu'}\sum_{ll'}(-1)^{l'}T_lT_{l'}^{*}\sum_{f}(10,LM|fM)(s\mu',fM|s\mu)D(f,L),\qquad\text{(A21)}$$

where

$$D(f,L)=\sqrt{6}\hat{l}\hat{l}'\hat{f}(l0,l'0|L0)X\begin{Bmatrix}l & s & \frac{1}{2}\\ l' & s & \frac{1}{2}\\ L & f & 1\end{Bmatrix}-(-1)^{L+l+s+\frac{1}{2}}(f0,10|L0)(s\tfrac{1}{2},s-\tfrac{1}{2}|f0).\qquad\text{(A21′)}$$

Let us consider first the odd-$L$ case. If $f=L\pm1$ then $D(f,L)=0$. There remains the term with $f=L$ which, by virtue of $l=l'\pm1$, (A18), and (3.8′), gives rise to (3.12′). For even $L$, $D(f,L)$ is zero if $f=L$. For $f=L\pm1$ is

$$D(f,f-1)=\frac{2s+1}{[f(2f+1)]^{1/2}}(s\tfrac{1}{2},s-\tfrac{1}{2}|f0)$$
$$D(f,f+1)=\frac{2s+1}{[(f+1)(2f+1)]^{1/2}}(s\tfrac{1}{2},s-\tfrac{1}{2}|f0).$$

Putting these expressions in (A21) and using (3.8″) we obtain (3.12″).