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Theory of Irreversible Processes in a Plasma—Derivation of a Convergent Kinetic Equation from the Generalized Master Equation

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The theory of irreversible processes in plasmas is rigorously developed from the cluster formulation of the exact generalized Master equation. This equation is a time-dependent analog of the equilibrium virial expansion, and the development of the theory of nonequilibrium plasmas is found to parallel that of the well-known theory of equilibrium plasmas in a direct and simple way. A *convergent* kinetic equation for homogeneous plasmas is then derived which includes the effects of close collisions as well as long-range collisions and is exact to first order in the density—in the asymptotic limit of long times. The distinguishing feature of this kinetic equation is that it converges for the Coulomb potential. No arbitrary cutoffs or screened potentials are required to "make" it converge. The divergence of previous kinetic equations is directly attributed to the neglect of significant close collision terms (such equations are not exact to first order in the density). The method of derivation is quite general and the extension of the convergent kinetic equation to short times (non-Markoffian kinetic equation) as well as to general order in the density is indicated.

I. INTRODUCTION

In recent years there have appeared several derivations of kinetic equations for plasmas.[1,2] These equations, as is well known, are characterized by a divergence at what corresponds to small impact parameters (close collisions)—even for homogeneous plasmas with a uniform positive background. The divergence of these kinetic equations is directly related to the inherent assumption that close collisions may, in some sense, be neglected. The fact that these equations do diverge refutes this assumption in so far as the relaxation or evolution of the momentum distribution function is concerned. Strictly speaking, these equations—although useful—are not correct to within the stated order of the relevant expansion parameter.

The question then arises as to whether or not a convergent kinetic equation for a homogeneous plasma can be rigorously obtained when the equations of motion are described by classical mechanics. This question is of more than academic interest. It is pertinent, for example, to the dc conductivity of plasmas.

An equation which accounts for close collisions appears in an article by Hubbard.[3] The method used in this work, however, requires further justification. More recently, a partial answer to the convergence question has been provided by the derivation of a kinetic equation[4] in which close binary collisions are explicitly accounted for via the BBGKY approach. We believe, however, that the convergence question has not been entirely answered there. This is because it was necessary to arbitrarily introduce a cutoff or screened potential to achieve formal convergence of the kinetic equation.

It is our purpose to derive a convergent kinetic equation without ever taking recourse to a cutoff or screened potential. We shall deal only with the Coulomb potential. We shall also indicate how one can obtain a non-Markoffian convergent kinetic equation which is valid for all time (such an equation is relevant to the transient evolution of the momentum distribution function as well as the frequency dependence of transport coefficients). The generalization of this derivation to higher order in the density is clear.

The derivation, which will be found in Secs. II to V, can be outlined as follows. It begins with the cluster formulation of the generalized master equation.[5] This equation is an exact but simple expression for the evolution of the N-particle *momentum* distribution function $\varphi(t)$ in terms of time-dependent irreducible cluster integrals $\beta_s(t)$.

It is given by

$$\frac{\partial\phi(t)}{\partial t}=\int_0^t dy\left\{\sum_{s=1}^{\infty}\frac{\partial^2\beta_s(y)}{\partial y^2}\right\}\phi(t-y).$$

These cluster integrals correspond to collisions between $(s+1)$ particles and are a time-dependent analog of the Mayer irreducible clusters well known in equilibrium statistical mechanics. This master equation is to be viewed here as a time-dependent virial expansion for the evolution of the momentum distribution function in which $\beta_s(t)$ plays the role of the $(s+1)$th virial coefficient. The attractive feature of this formulation is that it casts the time-dependent problem into a form which closely resembles an analogous equilibrium problem. We shall hence be able to make use of notions which are well known in equilibrium theory to guide us in our understanding of nonequilibrium theory.

For example, the time-dependent cluster integral $\beta_s(y)$ diverges when the interaction potential is Coulombic—just as the equilibrium cluster integral does. [This, of course, is the essential feature of plasma theory, in the absence of magnetic fields, which distinguishes it from neutral gas theory—both equilibrium and nonequilib-

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rium.] To circumvent this difficulty we extract *ring integrals* of $(s+1)$ particles from $\beta_s(y)$ just as is done in equilibrium theory. These ring integrals $R_s(y)$ correspond to *weak* (long-range) $(s+1)$-particle collisions, and the remainder $[\beta_s(y)-R_s(y)]$ corresponds to *strong* (close) $(s+1)$-particle collisions. The time-dependent virial expansion for $\phi(t)$ may thus be divided into a weak (distant) collision part,

$$\sum_{s=1}^{\infty} R_s(y) \equiv \text{ring sum}\,,$$

plus a strong interaction part,

$$\sum_s [\beta_s(y)-R_s(y)]\,,$$

as follows

$$\frac{\partial\phi(t)}{\partial t}=\int_0^t dy\left\{\sum_{s=1}^{\infty} R_s''(y) + \sum_{s=1}^{\infty}[\beta_s''(y)-R_s''(y)]\right\}\phi(t-y)\,.$$

The ring sum accounts for long-range collisions (shielding) and leads to the Lenard-Balescu, Fokker-Planck equation. The cluster remainder $\beta_s(y)-R_s(y)$ leads to corrections to the Fokker-Planck equation which are due to close $(s+1)$-particle collisions and which are of the sth order in the density. The correction for close (Coulombic) binary collisions exactly cancel out the divergence which appears in the Fokker-Planck equation and leads to a convergent kinetic equation.

We shall present the details of this calculation for a homogeneous plasma with a uniform neutralizing background.

II. GENERALIZED MASTER EQUATION

To begin the derivation of a convergent kinetic equation we consider the Hamiltonian of a system of N pair-interacting particles enclosed in a volume V:

$$H=\sum_{k=1}^{N}\frac{P_k^2}{2m}+\sum_{k<l}V_{kl}(\mathbf{R}_{kl})\,, \tag{1}$$

where $\mathbf{P}_k$ and $\mathbf{R}_k$ are the momentum and position of particle k, $\mathbf{R}_{kl}\equiv\mathbf{R}_k-\mathbf{R}_l$, and $V_{kl}(\mathbf{R}_{kl})$ is the interaction potential between particles k and s. The evolution of the N-particle distribution function $F(\{\mathbf{R}\};\{\mathbf{P}\};t)\equiv F(\mathbf{R}_1\cdots\mathbf{R}_N;\mathbf{P}_1\cdots\mathbf{P}_N;t)$, is determined by Liouville's equation:

$$\partial F/\partial t=-iLF\,, \tag{2}$$

where L is the Liouville operator defined by

$$L\equiv L_0+\sum_{k<l}L_{kl} \equiv -i\sum_k m^{-1}\mathbf{P}_k\cdot\frac{\partial}{\partial\mathbf{R}_k}+i\sum_{k<l}\frac{\partial V_{kl}}{\partial\mathbf{R}_{kl}}\cdot\left(\frac{\partial}{\partial\mathbf{P}_k}-\frac{\partial}{\partial\mathbf{P}_l}\right). \tag{3}$$

The generalized master equation is a closed equation for the evolution of the N-particle *momentum* distribution function $\phi(t)$, where $\phi(t)$ is defined by

$$\phi(t)\equiv V^{-N}\int d\{\mathbf{R}\}F(\{\mathbf{R}\};\{\mathbf{P}\};t)\equiv\phi(\{\mathbf{P}\};t)\,. \tag{4}$$

In Ref. 5 (hereafter referred to as I) we have derived a generalized master equation which is exact for all time in the limit of an infinite system (N, $V\to\infty$, $N/V=$constant) for distribution functions which are initially independent of positions. This equation is given by

$$\frac{\partial\phi(t)}{\partial t}=\int_0^t dy\left\{\sum_{s=1}^{\infty}\frac{\partial^2\beta_s(y)}{\partial y^2}\right\}\phi(t-y)\,, \tag{5}$$

where

$$\beta_s(y)\equiv\lim_{N\to\infty}\sum_{1\le i_1<\cdots<i_{s+1}\le N}\beta_s(i_1i_2\cdots i_{s+1};y) \tag{6}$$

and $\beta_s(i_1\cdots i_{s+1};y)$ is a time-dependent $(s+1)$-particle collision operator the explicit definition of which will be found in I. [Briefly, $\beta_s(i_1\cdots i_{s+1};y)$ involves the dynamics of the $(s+1)$ particles i_1, i_2, $\cdots i_{s+1}$. It is a propagator (Green function) which corresponds to a collision between particles $i_1\cdots i_{s+1}$ and is a time-dependent analog of the equilibrium irreducible cluster integral of $(s+1)$ particles. For example, the time-dependent, two-particle irreducible cluster integral $\beta_1(12;t)$ is defined in I by

$$\beta_1(12;t)\phi\equiv V^{-1}\int d\mathbf{R}_{12}[e^{it(L_0+L_{12})}-e^{itL_0}]\phi\,. \tag{7}$$

The integrand $[e^{it(L_0+L_{12})}-e^{itL_0}]$ of this cluster integral is nonzero for only that region of $\mathbf{R}_{12}$ space which leads to a collision between particles 1 and 2 within time t. The asymptotic two-particle operator

$$\lim_{t\to\infty}(\partial/\partial t)\beta_1(12;t)$$

is just the well-known Boltzmann collision integral.]

III. KINETIC EQUATION

The equation for the single-particle distribution function $\phi_1(\mathbf{P}_1)$,

$$\phi_1(\mathbf{P}_1)\equiv\phi_1(\mathbf{P}_1;t)\equiv\int d\mathbf{P}_2d\mathbf{P}_3\cdots d\mathbf{P}_N\phi(t)\,, \tag{8}$$

is referred to as a kinetic equation and is basic to the study of transport phenomena in fluids. Such an equation may be readily obtained from the master equation by integrating it over all momenta except $\mathbf{P}_1$—providing $\phi(t)$ satisfies the product condition. We thus integrate (5) over all momenta except $\mathbf{P}_1$ to obtain an equation for $\phi_1(\mathbf{P}_1)$:

$$\frac{\partial\phi_1(\mathbf{P}_1)}{\partial t}=\int d\mathbf{P}_2\cdots d\mathbf{P}_N\int_0^t dy\left\{\sum_{s=1}^{\infty}\frac{\partial^2\beta_s(y)}{\partial y^2}\right\}\phi(t-y). \quad (9)$$

There only remains to reduce the right side of (9) to a function of ϕ_1. This is a simple task if $\beta_s(y)$ converges—as it does for short-range forces.[6] The situation is more complicated for Coulomb interactions since then $\beta_s(y)$ does not converge [although $\sum \beta_s(y)$ does converge].

To circumvent this difficulty we shall make use of the similarities which exist between (9) and the partition function. That is, Eq. (9) and the master equation may be viewed as a time-dependent virial expansion in which $(N/V)^{-s}\beta_s(y)$ plays the role of an $(s+1)$th virial coefficient [$(s+1)$-particle irreducible cluster integral]. Like the equilibrium cluster integrals, however, $\beta_s(y)$ diverges for a Coulomb interaction potential,

$$V_{kl}=e^2/R_{kl}$$

because V_{kl} approaches zero very slowly at large R_{kl}. This means that (9), like the equilibrium virial expansion, is not directly applicable to Coulomb interactions in its present form and must be rearranged to yield a convergent result. In equilibrium theory this is accomplished by first expanding each cluster integral as a power series in the parameter e^2. The term which is of lowest order in e^2 in such an expansion of a cluster integral is called a *ring integral,* and the sum of all possible ring integrals is found to converge to the Debye-Huckel approximation. These ring integrals account for long-range correlations and lead to the phenomena of shielding [corrections from the remainder of the expansion can be obtained by various methods].

The same procedure can be applied to (9). We can extract time-dependent ring integrals of $(s+1)$ particles from the expansion of the time-dependent cluster integral $\beta_s(y)$. The sum of all such time-dependent ring integrals provides long-range shielding of the interaction (Balescu-Lenard, Fokker-Planck equation). The remainder of the time-dependent clusters (clusters minus rings) yield corrections which correspond to close collisions.

These time-dependent ring integrals, analogous to those in equilibrium theory, are the first nonvanishing terms in the expansion of the time-dependent clusters in powers of the coupling constant e^2. That is, if we set $V_{kl}=e^2/R_{kl}$ so that, in the definition of $\beta_s(y)$,

$$L_{kl}=e^2\frac{\partial R_{kl}^{-1}}{\partial \mathbf{R}_{kl}}\cdot\left(\frac{\partial}{\partial \mathbf{P}_k}-\frac{\partial}{\partial \mathbf{P}_l}\right),$$

then the expansion of $\beta_s(y)$ in powers of e^2 yields the ring integrals $R_s(y)$ plus terms which are of higher order in e^2. That is, we define $R_s(y)$ by

$$\int d\mathbf{P}_2\cdots d\mathbf{P}_N R_s(y)\equiv e^{2s+2}\lim_{e^2\to 0} e^{-(2s+2)}\int d\mathbf{P}_2\cdots d\mathbf{P}_N\beta_s(y)$$

and find, by expanding the expression for $\beta_s(y)$ (Ref. I),

$$\int d\mathbf{P}_2\cdots d\mathbf{P}_N R_s(y)=\int d\mathbf{P}_2\cdots d\mathbf{P}_N\sum_{i_1<\cdots<i_{s+1}}\sum_{\{\alpha_1\cdots\alpha_{s+1}\}}^{i_1\cdots i_{s+1}}\int d\mathbf{R}_{i_1i_2}\cdots d\mathbf{R}_{i_1i_{s+1}}\int_0^y dt_1\cdots\int_{t_s}^y dt_{s+1}$$
$$\times iL_{\alpha_1}e^{i(t_2-t_1)L_0}iL_{\alpha_2}\cdots e^{i(t_{s+1}-t_s)L_0}iL_{\alpha_{s+1}}=O(e^{2s+2}), \quad (10a)$$

where the binary indices $\alpha_1, \alpha_2, \cdots\alpha_{s+1}$ each denote some pair of particle indices from among particles $i_1, i_2, \cdots i_{s+1}$, $\mathbf{R}_\alpha$ denotes the relative vector distance between the pair of particles α, and

$$\sum_{\{\alpha_1\cdots\alpha_{s+1}\}}^{i_1\cdots i_{s+1}}$$

denotes the sum of each of the binary indices $\alpha_1, \cdots\alpha_{s+1}$ over all *permissable* pairs of particle indices from among particles $i_1, \cdots i_{s+1}$. [Each term in this sum is such that when a Mayer-type diagram (as described in I) is drawn, by joining the pair indices together with lines, there results a topological ring.] This sum includes, for example, the ring term in which $\alpha_1=12$, $\alpha_2=23$, $\cdots\alpha_s=s(s+1)$, $\alpha_{s+1}=1(s+1)$.

The sum in (10) for $R_s(y)$ can be identified as the sum of all $(s+1)$-particle ring integrals which can be formed from the N particles of the system. These ring integrals are, aside from notation, the same as Balescu's ring terms.

The order of magnitude of the remaining terms in the expansion of $\int d\mathbf{P}_2\cdots d\mathbf{P}\beta_s(y)$ in powers of e^2 is equal to $O(e^{2s+1})$. That is

$$\int d\mathbf{P}_2\cdots d\mathbf{P}_N\beta_s(y)=\int d\mathbf{P}_2\cdots d\mathbf{P}_N R_s(y)+O(e^{2s+4}). \quad (10b)$$

[6] If $\phi(t)$ satisfies the product condition

$$[\phi(t)=\prod_{j=1}^{N}\phi_1(\mathbf{P}_j;t)]$$

then, from (9), the kinetic equation for $\phi_1(\mathbf{P}_1,t)$ to all orders in the density and for all times t is given by

$$\frac{\partial\phi_1(\mathbf{P}_1)}{\partial t}=\sum_{s=1}^{\infty}(N/V)^s\int_0^t dy\int d\mathbf{P}_2\cdots d\mathbf{P}_{s+1}\beta_s(1,2\cdots s+1;y)\times\prod_{j=1}^{s+1}\phi_1(\mathbf{P}_j;t-y)$$

providing β_s converges.

[There are other terms in the expansion of $\beta_s(y)$ which are of the same or lower order in e^2 than $R_s(y)$. These terms vanish,[7] however, when they are integrated over all momenta but one and, consequently, will not appear in the kinetic equation.]

Let us now write $\beta_s(y)$ in the form

$$\beta_s(y)\equiv R_s(y)+[\beta_s(y)-R_s(y)]$$

and substitute this form into (9) to obtain

$$\frac{\partial\phi_1(\mathbf{P}_1)}{\partial t}=\int d\mathbf{P}_2\cdots d\mathbf{P}_N\int_0^t dy\{R''(y)+\sum_{s=1}^{\infty}[\beta_s''(y)-R_s''(y)]\}\phi(t-y), \quad (11)$$

where primes denote differentiation with respect to y, and

$$R(y)\equiv\sum_{s=1}^{\infty}R_s(y)$$

is the *sum* of *all* ring integrals.

We thus see, in (11), that the ring integrals have been extracted and separated from the cluster integrals. The ring sum $R(y)$ accounts for distant Coulomb collisions (long-range correlations) and the term $[\beta_s(y)-R_s(y)]$ provides a correction due to close $(s+1)$-particle collisions.[8] Equation (11) then explicitly divides the exact equation for the evolution of $\phi_1\mathbf{P}_1$ into a weak but highly correlated collision part, $R(y)$, plus strong $(s+1)$-particle collision parts, $[(\beta_s(y)-R_s(y)]$. (This equation is of great interest for plasmas, and we shall find that it directly leads to a convergent kinetic equation.)

The ring term has already been evaluated by Balescu in the asymptotic limit of large t under the condition that $\phi(t)$ satisfies the product condition

$$[\phi(t)=\prod_{j=1}^{N}\phi_1(\mathbf{P}_j)].$$

We shall merely quote Balescu's result later on, and shall henceforth consider the ring term as known.

IV. CONVERGENT EQUATION TO LOWEST ORDER IN THE DENSITY

Equation (11) is exact to all orders. We wish to obtain from it a convergent equation for ϕ_1 which is exact to lowest order in the density—in the limit of large t.

The ring term $R(y)$, as previously mentioned, diverges at zero impact parameter. This implies that the ring term does not adequately account for close collisions. To remove, or cancel, this divergence we must include the close collision term, $\sum_s[\beta_s(y)-R_s(y)]$, in the kinetic equation. We need not, however, include all the close collision terms. This is because

$$\int d\mathbf{P}_2\cdots d\mathbf{P}_N[\beta_s''(y)-R_s''(y)]\phi=O([N/V]^s),$$

so that to lowest order in the density we need only include the binary collision term $[\beta_1(y)-R_1(y)]$. This term accounts for close two-particle collisions. The remaining close collision terms $(s\geqslant 2)$ are of higher order in the density and correspond to close collisions between three or more particles. Hence, to lowest order in the density or, equivalently, neglecting close collisions between three or more particles, Eq. (11) becomes:

$$\frac{\partial\phi_1(\mathbf{P}_1)}{\partial t}=\int d\mathbf{P}_2\cdots d\mathbf{P}_N\int_0^t dy\{R''(y)+\beta_1''(y)-R_1''(y)\}\phi(t-y). \quad (12)$$

Equation (12) is exact for all t to lowest order in the density and, it will be noted, has a non-Markoffian memory which is significant for small t and for high-frequency phenomena. This memory is interesting in itself and shall be the subject of future investigations. For our present purpose, however, we shall confine our discussion to the behavior of (12) in the asymptotic limit of long times (Markoffian limit). In this limit we have

$$\int_0^t dy\{R''(y)+\beta_1''(y)-R_1''(y)\}\phi(t-y)\sim\int_0^{\infty}dy\{R''(y)+\beta_1''(y)-R_1''(y)\}\phi(t)=\{R'(\infty)+\beta_1'(\infty)-R_1'(\infty)\}\phi(t) \quad (13)$$

since, as is shown in I (Appendix D), $\beta_s(0)=0$ for all s. The asymptotic time limit in (13) has been shown to be exact to lowest order in the density (see I, Sec. IIIB).

Inserting (13) into (12) and assuming that $\phi(t)$ satisfies the product condition we have

$$\frac{\partial\phi_1(\mathbf{P}_1)}{\partial t}=\int d\mathbf{P}_2\cdots d\mathbf{P}_N\{R'(\infty)+\beta_1'(\infty)-R_1'(\infty)\}\times\prod_{j=1}^{N}\phi_1(\mathbf{P}_j), \quad (14)$$

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[7] For example, consider the four-particle term

$$\int d\mathbf{R}_{12}\int d\mathbf{R}_{34}\int_0^y dt_1\int_{t_1}^y dt_2\int_{t_2}^y dt_3 iL_{12}e^{i(t_2-t_1)L_0}iL_{34}e^{i(t_3-t_2)L_0}iL_{12}\phi.$$

This is a term in the expansion of $\beta_3(y)$ that is of lower order in e^2 than the four-particle ring term. The integral of this term over $\mathbf{P}_3$ and $\mathbf{P}_4$ becomes

$$\int d\mathbf{R}_{12}\int d\mathbf{R}_{34}\int_0^y dt_1\int_{t_1}^y dt_2\int_{t_2}^y dt_3 iL_{12}e^{i(t_2-t_1)L_0}i\times\int d\mathbf{P}_3 d\mathbf{P}_4 L_{34}e^{i(t_3-t_2)L_0}iL_{12}\phi,$$

since $\int\mathbf{P}_3$ and $\int\mathbf{P}_4$ commute with the left side of this term. But

$$L_{34}\equiv e^2\frac{\mathbf{R}_{34}}{R_{34}^3}\cdot\left(\frac{\partial}{\partial\mathbf{P}_3}-\frac{\partial}{\partial\mathbf{P}_4}\right)$$

so that the integral of this term over $\mathbf{P}_3$ and $\mathbf{P}_4$ yields only surface integrals which vanish providing ϕ vanishes at P_3, $P_4=\infty$.

[8] $\beta_s(y)$ corresponds to all possible $(s+1)$-particle collisions—weak and strong. The ring term $R_s(y)$ corresponds to weak $(s+1)$-particle collisions so that the difference, $[\beta_s(y)-R_s(y)]$, corresponds to strong collisions.

which, in the limit of large t, is still exact to lowest order in the density providing $\phi(t)$ satisfies the product condition.

As the final step in the derivation of a kinetic equation we must evaluate the momentum integrations in (14) for all three terms: the ring sum $R'(\infty)$ the two-particle cluster $\beta_1'(\infty)$, and the two-particle ring $R_1'(\infty)$.

This is a relatively easy task. The ring sum has already been evaluated in Ref. 3 and is given there by[9] (except for a factor of $8\Pi^3$).

$$\int d\mathbf{P}_2\cdots d\mathbf{P}_N R'(\infty)\prod_j \phi_1(\mathbf{P}_j) = 2e^4C\int d\mathbf{P}_2\int d\mathbf{l}\,\mathbf{R}(\mathbf{l}), \quad (15)$$

where we have defined $\mathbf{R}(\mathbf{l})$ by

$$\mathbf{R}(\mathbf{l})\equiv\left(\mathbf{l}\cdot\frac{\partial}{\partial\mathbf{P}_1}\right)\times\frac{\delta(\mathbf{l}\cdot\mathbf{g}_{12})}{\left|l^2+4\pi ie^2C\int d\mathbf{P}_3\delta_-(\mathbf{l}\cdot\mathbf{g}_{13})\mathbf{l}\cdot(\partial\phi_1(\mathbf{P}_3)/\partial\mathbf{P}_3)\right|^2}\times\mathbf{l}\cdot\left(\frac{\partial}{\partial\mathbf{P}_1}-\frac{\partial}{\partial\mathbf{P}_2}\right)\phi_1(\mathbf{P}_1)\phi_1(\mathbf{P}_2). \quad (16)$$

Here $\mathbf{g}_{kl}\equiv m^{-1}(\mathbf{P}_k-\mathbf{P}_l)$, $C\equiv N/V$ and

$$\delta_-(a)\equiv\pi\delta(a)-iP(a^{-1}).$$

It will be noted that the $\mathbf{l}$ integration in (15) diverges logarithmically at large l (large l corresponds to small impact parameter).

The two-particle cluster integral, $\beta_1'(\infty)$, in (14) is quite familiar in nonequilibrium statistical mechanics. It leads directly to the Boltzmann collision integral for binary collisions. This is proven in Appendix A where it is found [Eq. (A8)] that

$$\int d\mathbf{P}_2\cdots d\mathbf{P}_N\beta_1'(\infty)\prod_{j=1}^{N}\phi_1(\mathbf{P}_j) = C\int d\mathbf{P}_2 m^{-1}P_{12}\times\int b\,db\,d\theta\{\phi_1(\mathbf{P}_1')\phi_1(\mathbf{P}_2')-\phi_1(\mathbf{P}_1)\phi_1(\mathbf{P}_2)\}, \quad (17)$$

where b and θ are the impact parameter and azimuthal angle for a collision between particles 1 and 2, and $\mathbf{P}_{12}\equiv\mathbf{P}_1-\mathbf{P}_2$. The asymptotic momenta $\mathbf{P}_1'$ and $\mathbf{P}_2'$ are the momenta of 1 and 2 after they have completed a collision with each other (completed over an infinite time interval) and are explicitly given, for a Coulomb potential, in Eq. (A11) as functions of b and θ (this equation for $\mathbf{P}_1'$ and $\mathbf{P}_2'$ is simply the expression for Rutherford scattering).

Equation (17) is simply the Boltzmann collision integral for a Coulomb interaction potential. This integral diverges logarithmically at large b. We shall find, however, that this divergence is exactly cancelled by the $R_1'(\infty)$ term in (14).

In order to examine the cancellation of the divergent terms in (14) it will be convenient to transform the integration over b and θ in (17) into an integration over the same three-dimensional vector $\mathbf{l}$ as in (15). This is done in Appendix B where it is found, (B4) and (B5), that (17) becomes

$$\int d\mathbf{P}_2\cdots d\mathbf{P}_N\beta_1'(\infty)\prod_j\phi_1(\mathbf{P}_j) = 2e^4C\int d\mathbf{P}_2\int d\mathbf{l}\,\mathbf{B}_1(\mathbf{l}), \quad (18)$$

where $\mathbf{B}_1(\mathbf{l})$ is defined by

$$\begin{aligned}\mathbf{B}_1(\mathbf{l})&\equiv 2^{-1}m^{-2}e^{-4}P_{12}{}^2\delta(\mathbf{l}\cdot\mathbf{g}_{12})l^{-4}\{\phi_1(\mathbf{P}_1')\phi_1(\mathbf{P}_2')-\phi_1(\mathbf{P}_1)\phi_1(\mathbf{P}_2)\}\\ &\equiv 2^{-1}m^{-2}e^{-4}P_{12}{}^2\delta(\mathbf{l}\cdot\mathbf{g}_{12})l^{-4}\left\{\phi_1\left(\mathbf{P}_1+\left[\frac{m^2e^4l^2}{m^2e^4l^2+P_{12}{}^4}\right]\mathbf{P}_{12}-\left[\frac{mP_{12}{}^3e^2}{m^2e^4l^2+P_{12}{}^4}\right]\mathbf{l}\right)\right.\\ &\qquad\left.\times\phi_1\left(\mathbf{P}_2-\left[\frac{m^2e^4l^2}{m^2e^4l^2+P_{12}{}^4}\right]\mathbf{P}_{12}+\left[\frac{mP_{12}{}^3e^2}{m^2e^4l^2+P_{12}{}^4}\right]\mathbf{l}\right)-\phi_1(\mathbf{P}_1)\phi_1(\mathbf{P}_2)\right\}. \quad (19)\end{aligned}$$

We see, from (18) and (19), that the logarithmic divergence occurs at small l. This is because l corresponds to the reciprocal of b (Appendix B).

Thus far we have considered the $R'(\infty)$ and $\beta_1'(\infty)$ terms in (14). There only remains the two-particle ring integral, $R_1'(\infty)$, which is easily evaluated from either $R'(\infty)$ or $\beta_1'(\infty)$. That is, we see from (10) that $R_1'(\infty)$ is simply the first term in the expansion of $\beta_1'(\infty)$ about e^2. In addition, we see from (12) that $R_1'(\infty)$ is also the first term in the expansion of $R'(\infty)$ about e^2. Consequently, we may obtain $R_1'(\infty)$ from either $R'(\infty)$ or $\beta_1'(\infty)$. It is more convenient to obtain $R_1'(\infty)$ from

[9] The asymptotic ring sum $R'(\infty)$ in Eq. (14) is—aside from notation—equal to Balescu's ring sum.

$R'(\infty)$. Thus, from (10B), (12), and (15) we find that

$$\int d\mathbf{P}_2\cdots d\mathbf{P}_N R_1'(\infty)\prod_j \phi_1(\mathbf{P}_j)$$

$$=e^4\lim_{e^2\to 0} e^{-4}\int d\mathbf{P}_2\cdots d\mathbf{P}_N R'(\infty)\prod_j \phi_1(\mathbf{P}_j)$$

$$=2e^4C\int d\mathbf{P}_2\int d\mathbf{l}\,\mathbf{R}_1(\mathbf{l}), \tag{20}$$

where

$$\mathbf{R}_1(\mathbf{l})\equiv \mathbf{l}\cdot\frac{\partial}{\partial \mathbf{P}_1}\frac{\delta(\mathbf{l}\cdot\mathbf{g}_{12})}{l^4}\mathbf{l}\cdot\left(\frac{\partial}{\partial \mathbf{P}_1}-\frac{\partial}{\partial \mathbf{P}_2}\right)\phi_1(\mathbf{P}_1)\phi_1(\mathbf{P}_2). \tag{21}$$

Substituting (15), (18), and (20) into (14) we finally obtain the desired kinetic equation:

$$\frac{\partial\phi_1(\mathbf{P}_1)}{\partial t}=2e^4C\int d\mathbf{P}_2\int d\mathbf{l}[\mathbf{R}(\mathbf{l})+\mathbf{B}_1(\mathbf{l})-\mathbf{R}_1(\mathbf{l})], \tag{22}$$

where $\mathbf{R}(\mathbf{l})$ is the ring sum defined by (16), $\mathbf{B}_1(\mathbf{l})$ is the Boltzmann collision term (for a Coulomb potential) defined by (19), and $\mathbf{R}_1(\mathbf{l})$ is the two-particle ring term defined by (21).

Equation (22) is a kinetic equation for a plasma in which binary collisions have been rigorously accounted for. It is exact to first order in the density in the limit of large t providing $\phi(t)$ satisfies the product condition. We must now prove that it converges.

V. CONVERGENCE OF THE KINETIC EQUATION

We wish to show that despite the fact that each of the three terms in (22) $[\mathbf{R}(\mathbf{l}),\mathbf{B}_1(\mathbf{l}),\mathbf{R}_1(\mathbf{l})]$ diverge the sum of all three terms converge. Thus, we see from (16), (19), and (21) that $\int d\mathbf{l}\mathbf{R}(\mathbf{l})$ diverges logarithmically at large l, $\int d\mathbf{l}\mathbf{B}_1(\mathbf{l})$ diverges logarithmically at small l, and $\int d\mathbf{l}\mathbf{R}_1(\mathbf{l})$ diverges logarithmically at both small l and large l.

But it is easily seen from (16) and (21) that the large l divergencies due to $\mathbf{R}(\mathbf{l})$ and $-\mathbf{R}_1(\mathbf{l})$ cancel each other out exactly.

There only remains the divergence of (22) due to $\mathbf{B}_1(\mathbf{l})$ and $-\mathbf{R}_1(\mathbf{l})$ at small l. To understand how these divergencies at small l cancel each other out we must expand $\mathbf{B}_1(\mathbf{l})$ about $l=0$. It is not necessary, however, to go through all the details of such an expansion. That is, we see from the expression for $\mathbf{B}_1(\mathbf{l})$ in (19) that the term $[\phi_1(\mathbf{P}_1')\phi_1(\mathbf{P}_2')-\phi_1(\mathbf{P}_1)\phi_1(\mathbf{P}_2)]$, which must be expanded in $\mathbf{B}_1(\mathbf{l})$, contains e^2 and $\mathbf{l}$ in the combination $e^2\mathbf{l}$. An expansion of $[\phi_1(\mathbf{P}_1')\phi_1(\mathbf{P}_2')-\phi_1(\mathbf{P}_1)\phi_1(\mathbf{P}_1)]$ about small e^2 is, hence, the same as its expansion about small l. But, the expansion of $\beta_1(y)$ about small e^2 and, hence, about small l is given by (10B). Consequently, we may substitute (10B) into (18) to obtain

$$2e^4C\int d\mathbf{P}_2\int d\mathbf{l}\mathbf{B}_1(\mathbf{l})$$

$$=\int d\mathbf{P}_2\cdots d\mathbf{P}_N R_1'(\infty)[1+O(e^2l)]\prod\phi_1. \tag{23}$$

Substituting (20) into (23) we then have

$$2e^4C\int d\mathbf{P}_2\int d\mathbf{l}\mathbf{B}_1(\mathbf{l})$$

$$=2e^4C\int d\mathbf{P}_2\int d\mathbf{l}\,\mathbf{R}_1(\mathbf{l})[1+O(e^2l)]. \tag{24}$$

[The reader can verify that (24) is correct by expanding $\mathbf{B}_1(\mathbf{l})$ about small l and comparing with $\mathbf{R}_1(\mathbf{l})$.]

Substituting (24) into (22) we see that the singularity of $\int d(\mathbf{l})\mathbf{B}_1(\mathbf{l})$ is exactly cancelled by the singularity of $\int d\mathbf{l}\mathbf{R}_1(\mathbf{l})$ at small l.

We have thus proven that the singularities of the kinetic equation, Eq. (22), cancel each other out leaving a convergent result. The distinctive feature of this convergent kinetic equation is that no cutoff or screened potential is required to make it converge. It only involves the Coulomb potential.

VI. SUMMARY AND DISCUSSION

A convergent kinetic equation, Eq. (22), has been derived for a homogeneous plasma without introducing cutoff or screened potentials. This equation was rigorously derived from the cluster formulation of the generalized Master equation by retaining terms which correspond to close binary collisions $[B_1(\infty)-R_1(\infty)]$ in addition to the familiar terms which correspond to long-range correlations [ring sum, $R(\infty)$]. Neglected, were terms which correspond to close collisions between three or more particles,

$$\sum_{s=2}^{\infty}[\beta_s(y)-R_s(y)].$$

Such terms are of second or higher order in the density. The kinetic equation, in the limit of long time, is thus exact to first order in the density although it contains terms to all orders in the density from the ring sum $\mathbf{R}(\mathbf{l})$. The only assumption made was the product condition. Further (density) corrections can be obtained by including the terms for close collisions between three or more particles.

Equation (22) will lead to different values of transport coefficients than the Lenard-Balescu equation when the latter is appropriately cut off at large l. The order of magnitude of this difference will, approximately, vary with the logarithm of the arbitrary cutoff in the Lenard-Balescu equation.

As a final remark we note that the kinetic equation has been derived from the generalized master equation in the asymptotic limit of large time t. This limit is not necessary, however. The master equation is valid for all time so that one can derive from it a kinetic equation which is also valid for all time. To accomplish this one must simply evaluate the ring sum, $R'(y)$, and the binary collision integral $\beta_1'(y)$ as a function of the time y instead of in the limit of large y (such a calculation may not be difficult for Coulomb interactions since then the particle trajectories are simple and well-known functions of the time). This would lead to a non-Markoffian kinetic equation of the Volterra type with which one may study the frequency dependence of transport coefficients. This equation would, of course, reduce to (22) when t is large.

Note added in proof. It has come to my attention that Dr. D. E. Bladwin is aware that a convergent equation need not have a cutoff and has published such a result in a technical report.

APPENDIX A

We shall derive here the result expressed by (17). For convenience we let $I(y)$ denote

$$\int d\mathbf{P}_2\cdots d\mathbf{P}_N\beta_1'(y)\prod_{j=1}^{N}\phi_1(\mathbf{P}_j)$$

so that, with (6) and (7),

$$I(y)\equiv\int d\mathbf{P}_2\cdots d\mathbf{P}_N\sum_{k<s}V^{-1}\int d\mathbf{R}_{ks}\frac{d}{dy}[e^{-iy(L_0+L_{ks})}-1]\prod_{j=1}^{N}\phi_1$$

$$=\sum_{k=2}^{N}V^{-1}\int d\mathbf{P}_k\int d\mathbf{R}_{1k}\frac{d}{dy}[e^{-iy(L_0+L_{1k})}-1]\int_{(\mathbf{P}_k)}d\mathbf{P}_2\cdots d\mathbf{P}_N\prod_{j=1}^{N}\phi_1,\qquad\text{(A1)}$$

since

$$\int d\mathbf{P}_s d\mathbf{P}_k\int d\mathbf{R}_{ks}\frac{d}{dy}[e^{-iy(L_0+L_{ks})}-1]\phi_1(\mathbf{P}_k)\phi_1(\mathbf{P}_s)=0.$$

But

$$\int_{(\mathbf{P}_k)}d\mathbf{P}_2\cdots d\mathbf{P}_N\prod_{j=1}^{N}\phi_1\equiv\phi_2(\mathbf{P}_1,\mathbf{P}_k)=\phi_1(\mathbf{P}_1)\phi_1(\mathbf{P}_k)$$

so that (A1) becomes

$$I(y)=\sum_{k=2}^{N}V^{-1}\int d\mathbf{P}_k\int d\mathbf{R}_{1k}\frac{d}{dy}[e^{-iy(L_0+L_{1k})}-1]\phi_1(\mathbf{P}_1)\phi_1(\mathbf{P}_k)$$

$$=(N/V)\int d\mathbf{P}_2\int d\mathbf{R}_{12}\frac{d}{dy}[e^{-iy(L_0+L_{12})}-1]\phi_1(\mathbf{P}_1)\phi_1(\mathbf{P}_2)N.\qquad\text{(A2)}$$

But $e^{-iy(L_0+L_{12})}$ is the formal Green function solution of Liouville's equation for a two-body system; i.e., if $f[\mathbf{P}_1(t),\mathbf{P}_2(t)]$ is any function of the momenta of particles 1 and 2 at time t, $\mathbf{P}_1(t)$ and $\mathbf{P}_2(t)$, then

$$f[\mathbf{P}_1(-t),\,\mathbf{P}_2(-t)]=e^{-it(L_0+L_{12})}f[\mathbf{P}_1,\mathbf{P}_2]\qquad\text{(A3)}$$

is the solution of

$$\frac{\partial f[\mathbf{P}_1(-t),\,\mathbf{P}_2(-t)]}{\partial t}=-i(L_0+L_{12})f[\mathbf{P}_1(-t),\,\mathbf{P}_2(-t)],$$

where $\mathbf{P}_1(-t)$ and $\mathbf{P}_2(-t)$ are the momenta that 1 and 2 (considered isolated) must have had at time $(-t)$ in order that they will have momenta and relative position $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{R}_{12}$ at time zero. That is, $\mathbf{P}_1(-t)$ and $\mathbf{P}_2(-t)$ are solutions of the two-body problem and are functions of $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{R}_{12}$.

Substituting (A3) into (A2) we have, with $N/V\equiv C$,

$$I(y)=C\int d\mathbf{P}_2\frac{d}{dy}\int d\mathbf{R}_{12}\{\phi_1[\mathbf{P}_1(-y)]\times\phi_1[\mathbf{P}_2(-y)]-\phi_1(\mathbf{P}_1)\phi_1(\mathbf{P}_2)\}.\qquad\text{(A4)}$$

The integrand of (A4) is nonzero only if $\mathbf{R}_{12}$ is such that 1 and 2 are aimed to collide within time y in which event

$$\mathbf{P}_1(-y)\neq\mathbf{P}_1,$$
$$\mathbf{P}_2(-y)\neq\mathbf{P}_2.$$

We may, therefore, restrict the integration over $\mathbf{R}_{12}$ in (A4) to that region, $\Omega_{12}(y)$, which leads to a collision between 1 and 2 as follows:

$$\int_{\text{all space}}d\mathbf{R}_{12}\{\phi_1[\mathbf{P}_1(-y)]\phi_1[\mathbf{P}_2(-y)]-\phi_1(\mathbf{P}_1)\phi_1(\mathbf{P}_2)\}$$

$$=\int_{\Omega_{12}(y)}dR_{12}\{\phi_1[\mathbf{P}_1(-y)]\phi_1[\mathbf{P}_2(-y)]-\phi_1(\mathbf{P}_1)\phi_1(\mathbf{P}_2)\},\qquad\text{(A5)}$$

where $\Omega_{12}(y)$ is a *collision cylinder* whose length is $m^{-1}P_{12}y$ and whose cross section is the total scattering cross section. If y is large enough then the momenta

$\mathbf{P}_1(-y)$ and $\mathbf{P}_2(-y)$ will approach the asymptotic momenta $\mathbf{P}_1'$ and $\mathbf{P}_2'$, and (A5) may be integrated to yield[10]

$$m^{-1}P_{12}y\int bdbd\theta\{\phi_1(\mathbf{P}_1')\phi_1(\mathbf{P}_2')-\phi_1(\mathbf{P}_1)\phi_1(\mathbf{P}_2)\} +\text{smaller terms in } y, \quad \text{(A6)}$$

where

$$\mathbf{P}_1'\equiv\mathbf{P}_1-\mathbf{P}_{12}\cdot\mathbf{kk}, \quad \mathbf{P}_2'\equiv\mathbf{P}_2+\mathbf{P}_{12}\cdot\mathbf{kk}, \quad \text{(A7)}$$

and $\mathbf{k}$ is the unit vector in the perihelion direction for a collision with impact parameter b and azimuthal angle θ. In other words, $\mathbf{P}_1'$ and $\mathbf{P}_2'$ are the asymptotic momenta that 1 and 2 will have after completing their interaction with each other given that they were, initially, infinitely separated and approaching each other with impact parameter b and azimuthal angle θ.

Substituting (A6) into (A4) and taking the limit of infinite y we have, exactly,

$$I(\infty)=C\int d\mathbf{P}_2 m^{-1}P_{12}\int bdbd\theta\{\phi_1(\mathbf{P}_1')\phi_1(\mathbf{P}_2') -\phi_1(\mathbf{P}_1)\phi_1(\mathbf{P}_2)\}. \quad \text{(A8)}$$

Equation (A8) is exact because the "smaller terms in y" vanish in the limit where y is infinite. This equation is simply the Boltzmann collision integral for Coulomb scattering (Rutherford scattering). [This integral actually diverges logarithmically at large b. This divergence, however, is cancelled by another term of the complete kinetic equation.]

Asymptotic Momenta P_1' and P_2'

The asymptotic momenta $\mathbf{P}_1'$ and $\mathbf{P}_2'$ in the Boltzmann collision integral, (A8), can be readily expressed in terms of b and θ as follows. We consider a coordinate system in which the $\mathbf{z}$ axis lies along $\mathbf{P}_{12}$, and we let $\mathbf{x}$ and $\mathbf{y}$ denote unit vectors perpendicular to each other and to the $\mathbf{z}$ axis. If we also let α denote the scattering angle then we have

$$\begin{aligned}\mathbf{P}_1'-\mathbf{P}_1&=\mathbf{P}_2-\mathbf{P}_2'=-\mathbf{P}_{12}\cdot\mathbf{kk}=P_{12}\sin(\tfrac{1}{2}\alpha)\mathbf{k}\\ &=P_{12}\sin(\tfrac{1}{2}\alpha)\{(\mathbf{k}\cdot\mathbf{z})\mathbf{z}+(\mathbf{k}\cdot\mathbf{x})\mathbf{x}+(\mathbf{k}\cdot\mathbf{y})\mathbf{y}\}\\ &=-P_{12}\sin^2(\tfrac{1}{2}\alpha)\mathbf{z}+\sin(\tfrac{1}{2}\alpha)\cos(\tfrac{1}{2}\alpha)\\ &\quad\times\{\mathbf{x}\cos\theta+\mathbf{y}\sin\theta\}. \quad \text{(A9)}\end{aligned}$$

For Coulomb scattering $\sin(\frac{1}{2}\alpha)$ is simply given by

$$\sin(\tfrac{1}{2}\alpha)=\left[1+\left(\frac{bP_{12}^2}{2me^2}\right)^2\right]^{-1/2}, \quad \text{(A10)}$$

which we substitute into (A9) to obtain

$$\begin{aligned}-\mathbf{P}_{12}\cdot\mathbf{kk}&=-P_{12}\left[1+\left(\frac{bP_{12}^2}{2me^2}\right)^2\right]^{-1}\mathbf{z}\\ &\quad+P_{12}\left[\left(\frac{bP_{12}^2}{2me^2}\right)^{-1}+\left(\frac{bP_{12}^2}{2me^2}\right)\right]^{-1}\\ &\quad\times\{\mathbf{x}\cos\theta+\mathbf{y}\sin\theta\}\\ &=\mathbf{P}_1'-\mathbf{P}_1=\mathbf{P}_2-\mathbf{P}_2'. \quad \text{(A11)}\end{aligned}$$

Equation (A11) serves to express $\mathbf{P}_1'$ and $\mathbf{P}_2'$ in terms of b and θ.

APPENDIX B

We wish to transform the integration over impact parameter b and azimuthal angle θ in (17) into an integration over the three-dimentional vector $\mathbf{l}$ as in (15). The integration over b and θ is a two-dimensional integral. That is, it is an integral over the plane perpendicular to the $\mathbf{P}_{12}$ direction. We may transform this two-dimensional integral into a three-dimensional integral by multiplying the integrand with a delta function. Thus, if we define the new variable of integration $\mathbf{l}$

$$\mathbf{l}\equiv l_x\mathbf{x}+l_y\mathbf{y}+l_z\mathbf{z} \quad \text{(B1)}$$

by

$$\begin{aligned}l&=b^{-1},\\ l_x&=b^{-1}\cos\theta,\\ l_y&=b^{-1}\sin\theta,\\ l_z&=0,\end{aligned} \quad \text{(B2)}$$

where $\mathbf{z}$ is a unit vector in the direction of $\mathbf{P}_{12}$, then

$$\begin{aligned}\int bdbd\theta&=\int dl_xdl_yl^{-4}=\int dl_xdl_ydl_zl^{-4}\delta(l_z)\\ &=m^{-1}P_{12}\int d\mathbf{l}l^{-4}\delta(\mathbf{l}\cdot\mathbf{g}_{12}). \quad \text{(B3)}\end{aligned}$$

Substituting (B3) into the right side of (17) we have

$$C\int d\mathbf{P}_2\int d\mathbf{l}m^{-2}P_{12}^2\delta(\mathbf{l}\cdot\mathbf{g}_{12}) \times l^{-4}\{\phi_1(\mathbf{P}_1')\phi_1(\mathbf{P}_2')-\phi_1(\mathbf{P}_1)\phi_1(\mathbf{P}_2)\}. \quad \text{(B4)}$$

Substituting (B2) into (A11) and (A11) into $\phi_1(\mathbf{P}_1')\phi_1(\mathbf{P}_2')$, we have

$$\begin{aligned}&\{\phi_1(\mathbf{P}_1')\phi_1(\mathbf{P}_2')-\phi_1(\mathbf{P}_1)\phi_1(\mathbf{P}_2)\}\\ &=\left\{\phi_1\left(\mathbf{P}_1+\left[\frac{m^2e^4l^2}{m^2e^4l^2+P_{12}^4}\right]\mathbf{P}_{12}-\left[\frac{mP_{12}^3e^2}{m^2e^4l^2+P_{12}^4}\right]\mathbf{l}\right)\right.\\ &\quad\times\phi_1\left(\mathbf{P}_2-\left[\frac{m^2e^4l^2}{m^2e^4l^2+P_{12}^4}\right]\mathbf{P}_{12}+\left[\frac{mP_{12}^3e^2}{m^2e^4l^2+P_{12}^4}\right]\mathbf{l}\right)\\ &\quad\left.-\phi_1(\mathbf{P}_1)\phi_1(\mathbf{P}_2)\right\}. \quad \text{(B5)}\end{aligned}$$

Combining (B4) and (B5) we have the right side of (17) given explicitly in terms of $\mathbf{l}$ instead of in terms of b and ϕ.

[10] J. Weinstock, Phys. Rev. **132**, 470 (1963).