Mc/sec obtained by the Leiden group[18] from calorimetric measurements at 78 Mc/sec. On the basis of an extrapolation to $\nu=0$ of their χ''/ν data, Locher and Gorter[7] confirm the calorimetric value, but it is not clear on what basis they drew their extrapolating curve. The value of τ_S^{-1} obtained by Broer on the basis of a Gaussian shape function of rms width ν_0, is 7100 Mc/sec.

In the $S=\frac{1}{2}$ case, the width of the zero-frequency band is approximately ν_0, a frequency which also corresponds to the absorption of a quantum by a single-ion flipping in a field H_i. In reality, of course, the single-ion view is not applicable. For small fields ($H_0 \ll H_i$) pure states exist only for the crystal as a whole. There are no individual spin flips, but a sort of "cooperative flipping." Yet if we adhere to the notion that considering a spin flip in a field H_i provides an estimate of the width of the band, the band which we observe in chrome alum ($S=\frac{3}{2}$, $H_i=310$ G) corresponds to the complete flipping of a dipole (i.e., the transition $-\frac{3}{2} \rightarrow +\frac{3}{2}$ absorbs quanta of frequency $3\nu_0 \approx 2700$ Mc/sec). This rule of thumb might be verified by measurements on suitable systems with various values of S. Magnetic field dependence studies would also be helpful. For instance, some preliminary studies we have made show the manner in which the flat-topped curve in Fig. 7 evolves continuously into a resonance spectrum as increasingly larger static magnetic fields are applied perpendicular to the rf fields.

Figure 6 indicates that the shape function in chromic potassium alum is independent of temperature at liquid-helium temperatures.

V. CONCLUSIONS

The method which has been developed for measuring spin absorption as a continuous function of frequency enhances the understanding of the zero-frequency band. In cupric salts in zero field the theories of shape function moments and relaxation times are generally confirmed, though there remain theoretical and experimental difficulties for cases of very large exchange interaction. The results for the chromic alum are entirely unexpected, and suggest several avenues of approach for further study.

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Spin-Wave Contribution to Low-Temperature Specific Heat of Yttrium Iron Garnet in Zero Applied Fields*

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Recent results of specific heat measurements of yttrium iron garnet in zero applied field have been analyzed previously by using the approximate spin-wave dispersion relation $\hbar\omega=Dk^2$, neglecting the effects of the dipolar interaction and anisotropy. It is shown that this approximation is valid for temperatures above $\sim$0.2°K.

I. INTRODUCTION

THE low-temperature specific heat of yttrium iron garnet (YIG) has been measured and analyzed recently in terms of the spin-wave and lattice contributions in order to determine the exchange constant D.[1–4] In analyzing the data, the spin-wave dispersion relation was approximated by

$$\hbar\omega_k=Dk^2, \tag{1}$$

where the exchange constant D is related to the Landau exchange constant A by the relation

$$A=DM_s/2g\mu_B.$$

In this paper we examine the effect of this approximation and show that in the temperature range above 1°K in which the experiments were performed the approximation is quite good. At lower temperatures, of the order of 0.2°K in YIG, the approximation is no longer valid. The experimental results of references one through four are summarized in Shinozaki's paper.

II. SPECIFIC HEAT CALCULATION

The spin-wave contribution to the specific heat is

$$C_0=\frac{d}{dT}\frac{1}{(2\pi)^3}\int d\mathbf{k}\frac{\hbar\omega_k}{\exp(\hbar\omega_k/k_BT)-1}. \tag{2}$$

In the approximation (1) the integral is easily evalu-

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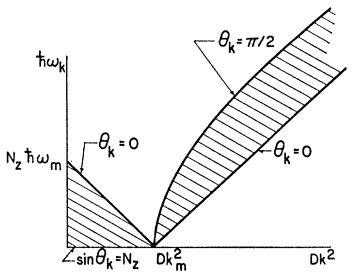

FIG. 1. Dispersion relation (4) for no applied field.

ated to give

$$C_0=\frac{15}{32}\frac{\zeta(\frac{5}{2})}{\pi^{3/2}}k_B\left(\frac{k_B}{D}\right)^{3/2}T^{3/2}=0.113k_B\left(\frac{k_B}{D}\right)^{3/2}T^{3/2}, \quad (3)$$

where ζ is the Riemann zeta function. In evaluating this integral it was assumed that Dk^2 at the edge of the Brillouin zone is much larger than k_BT so that this upper limit may be replaced by ∞. When the approximation (1) is not valid, Dk^2 must be replaced by the more general expression for the magnon energy $\hbar\omega_k$. The expression for $\hbar\omega_k$ including cubic anisotropy is[5]

$$\hbar\omega_k=[(Dk^2+\hbar\omega_i)(Dk^2+\hbar\omega_i+\hbar\omega_m\sin^2\theta_k)+\hbar\omega_A(\hbar\omega_m\sin^2\theta_k\cos^2\phi_k-\hbar\omega_A)]^{1/2},$$

where θ_k is the angle between the applied field and the wave vector $\mathbf{k}$, $\omega_m=4\pi\gamma M_s$, ω_A is related to the anisotropy constant, and the internal frequency is $\omega_i=\gamma(H_0-4\pi N_zM_s+A)$, where H_0 is the applied field, N_z is the z demagnetization factor, and A is an effective average anisotropy field. In the case of YIG,[5] $\omega_A=0.035\,\omega_m$, and hence can be dropped from the dispersion relation. For no applied field, $H_0=0$, this becomes

$$\hbar\omega_k=[(Dk^2-N_z\hbar\omega_m)(Dk^2-N_z\hbar\omega_m+\hbar\omega_m\sin^2\theta_k)]^{1/2}. \quad (4)$$

This relation is illustrated in Fig. 1 as a graph of $\hbar\omega_k$ as a function of Dk^2 with θ_k as a parameter drawn for the case $N_z=\frac{1}{3}$ for a single domain spherical sample. Since there is no applied field, the sample is not saturated in general. We assume that (4) is valid within a single domain. Since the sample is not saturated, N_z in (3) will not be the z demagnetization appropriate to the sample shape; but N_z will depend on the magnetization of the neighboring domains. The determination of N_z will be avoided by treating (4) in an approximate way. Since the integral in (2) is difficult to evaluate when $\hbar\omega_k$ is given by (4), we replace (4) by three models which contain the general features of (4). These models (A),

$$\hbar\omega_k=\begin{cases}Dk^2 & \text{for } Dk^2>\hbar\omega_M\\ 0 & \text{for } Dk^2<\hbar\omega_M\end{cases}, \quad (5a)$$

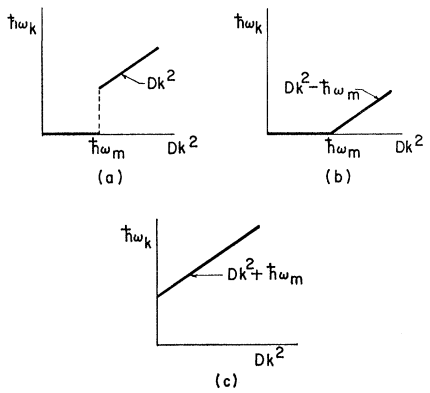

FIG. 2. Three approximate dispersion relations.

(B),

$$\hbar\omega_k=\begin{cases}Dk^2-\hbar\omega_M & \text{for } Dk^2>\hbar\omega_M\\ 0 & \text{for } Dk^2<\hbar\omega_M\end{cases}, \quad (5b)$$

and (C),

$$\hbar\omega_k=Dk^2+\hbar\omega_M, \quad (5c)$$

are illustrated in Figs. 2(a), 2(b), and 2(c). All three models give the same general result that the correction to (3) is small when $k_BT\gg\hbar\omega_M$.

In case (A) from (2) and (5a) the specific heat is given by

$$C=\frac{d}{dT}\frac{1}{2\pi^2}\left[\int_0^{k_M}\frac{dkk^2\hbar\omega_k}{1+(\hbar\omega_k/k_BT)-1}+\int_{k_M}^{\infty}\frac{dkk^2Dk^2}{\exp(Dk^2/k_BT)-1}\right],$$

where we define

$$Dk_M^2=\hbar\omega_M.$$

By writing the second integral as $\int_0^\infty-\int_0^{k_M}$, we find

$$C=\frac{d}{dT}\frac{1}{2\pi^2}\left[k_BT\frac{k_M^3}{3}+\frac{D}{2}\left(\frac{k_BT}{D}\right)^{5/2}\Gamma(\tfrac{5}{2})\zeta(\tfrac{5}{2})-\frac{D}{2}\left(\frac{k_BT}{D}\right)^{5/2}\int_0^{Dk_m2/k_BT}dx\frac{x^{3/2}}{e^x-1}\right].$$

Approximating the last integral by expanding e^x yields

$$C\cong\frac{d}{dT}\frac{1}{2\pi^2}\left[\frac{D}{2}\left(\frac{k_BT}{D}\right)^{5/2}\Gamma(\tfrac{5}{2})\zeta(\tfrac{5}{2})+\frac{D}{2}\left(\frac{k_BT}{D}\right)^{5/2}\frac{1}{5}\left(\frac{Dk_M^2}{k_BT}\right)^{5/2}+\frac{D}{2}\left(\frac{k_BT}{D}\right)^{5/2}\frac{5}{42}\left(\frac{Dk_M^2}{k_BT}\right)^{7/2}\right].$$

Then with $Dk_M^2=\hbar\omega_M$

$$C\cong C_0\left[1-0.027\left(\frac{\hbar\omega_M}{k_BT}\right)^{7/2}\right]. \quad (6)$$

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In case (B), (2) and (5b) give

$$C=\frac{d}{dT}\frac{1}{2\pi^2}\left[\int_0^{k_M} dk k^2 k_B T + \int_{k_M}^{\infty} \frac{dk k^2 D(k^2-k_M^2)}{\exp[D(k^2-k_M^2)/k_B T]-1}\right].$$

Let

$$x=D(k^2-k_M^2)/k_B T$$

and

$$\epsilon=Dk_M^2/k_B T.$$

Then

$$C=\frac{d}{dT}\frac{1}{2\pi^2}\left[k_B T\frac{k_M^3}{3}+\frac{D}{2}\left(\frac{k_B T}{D}\right)^{5/2}\int_0^{\infty}\frac{dx x(x+\epsilon)^{1/2}}{e^x-1}\right].$$

This integral is approximated by breaking it into two parts. For $x<\epsilon$ and for $x>\epsilon$:

$$x(x+\epsilon)^{1/2}\approx x\sqrt{\epsilon}\left(1+\frac{1}{2}\frac{x}{\epsilon}\right),$$

$$x(x+\epsilon)^{1/2}\approx x^{3/2}\left(1+\frac{1}{2}\frac{\epsilon}{x}\right).$$

Then

$$\int_0^{\infty} dx\frac{x(x+\epsilon)^{1/2}}{e^x-1}\approx\int_0^{\epsilon} dx\frac{x\sqrt{\epsilon}[1+\frac{1}{2}(\epsilon/x)]}{x[1+(2/x)]} + \int_0^{\infty} dx\frac{x^{3/2}[1+\frac{1}{2}(\epsilon/x)]}{e^x-1}-\int_0^{\epsilon}\frac{x^{3/2}[1+\frac{1}{2}(\epsilon/x)]}{x[1+(2/x)]}.$$

The evaluation of the integrals is not difficult:

$$\int_0^{\infty} dx\frac{x(x+\epsilon)^{1/2}}{e^x-1}\approx-\tfrac{1}{4}\epsilon^{3/2}+\Gamma(\tfrac{5}{2})\zeta(\tfrac{5}{2})+\tfrac{1}{2}\epsilon\Gamma(\tfrac{3}{2})\zeta(\tfrac{3}{2}).$$

With (2), this gives

$$C\cong\frac{d}{dT}\frac{1}{2\pi^2}\left\{k_B T\frac{k_M^3}{3}+\frac{D}{2}\left(\frac{k_B T}{D}\right)^{5/2}\left[-\frac{1}{4}\left(\frac{Dk_M^2}{k_B T}\right)^{3/2}\times\Gamma(\tfrac{5}{2})\zeta(\tfrac{5}{2})+\tfrac{1}{2}\left(\frac{Dk_M^2}{k_B T}\right)\Gamma(\tfrac{3}{2})\zeta(\tfrac{3}{2})\right]\right\}.$$

With $Dk_M^2=\hbar\omega_M$ this can be rewritten as

$$C\cong C_0[1+0.39(\hbar\omega_M/k_B T)+0.094(\hbar\omega_M/k_B T)^{3/2}]. \quad (7)$$

In case (C), (2) and (5c) give

$$C=\frac{d}{dT}\frac{1}{2\pi^2}\int_0^{\infty} dk k^2\frac{(Dk^2+\hbar\omega_M)}{\exp[(Dk^2+\hbar\omega_M)/k_B T]-1}.$$

With $x=Dk^2$ this becomes

$$C=\frac{d}{dT}\frac{1}{4\pi^2 D^{3/2}}\int_0^{\infty} dx\frac{x^{3/2}+\hbar\omega_M x^{1/2}}{\exp(\hbar\omega_M/k_B T)\exp(x/k_B T)-1}.$$

These integrals are of the form

$$\int_0^{\infty} dx\frac{x^s}{(1/\xi)\exp(x/\tau)-1}=\Gamma(s+1)\tau^{s+1}\sum_{m=1}^{\infty}\xi^m\frac{1}{m^{(s+1)}}$$

for $\xi\leq 1$; hence

$$C=\frac{d}{dT}\frac{1}{4\pi^2 D^{3/2}}\left\{\Gamma(\tfrac{5}{2})(k_B T)^{5/2}\sum_{m=1}^{\infty}\exp\left[-\frac{m\hbar\omega_M}{k_B T}\right]\frac{1}{m^{5/2}} + \hbar\omega_M\Gamma(\tfrac{3}{2})(k_B T)^{3/2}\sum_{m=1}^{\infty}\exp\left[-\frac{m\hbar\omega_M}{k_B T}\right]\frac{1}{m^{3/2}}\right\}. \quad (8)$$

With

$$\zeta(\tfrac{5}{2})=\sum_{m=1}^{\infty}\frac{1}{m^{5/2}}=15\pi^{1/2}/8,$$

this reduces properly to (3) when $\hbar\omega_M=0$. For $\hbar\omega_M\ll k_B T$ the first and second summations on the right-hand side of this equation have approximately the values $\zeta(\frac{5}{2})$ and $\zeta(\frac{3}{2})$, respectively; thus with $\zeta(\frac{5}{2})=15\pi^{1/2}/8$ and $\zeta(\frac{3}{2})=\pi^{1/2}/2$, we have

$$C=C_0\left[1+\frac{4}{25}\frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{5}{2})}\left(\frac{\hbar\omega_M}{k_B T}\right)\right]. \quad (9)$$

We should mention in passing that the (C) dispersion relation (5c) is a good first approximation to dispersion relations in the presence of an applied magnetic field H_0 if $\hbar\omega_M$ is replaced by $\hbar\gamma H_0$. Thus (8) or the approximation (9) gives the heat capacity in the presence of an applied field. Kunzler, Walker, and Galt[4] have also obtained this result (8) for the effect of a magnetic field on the heat capacity.

III. CONCLUSIONS

All three models investigated above lead to the same general result that for $\hbar\omega_M/k_B T\ll 1$ the leading term in the heat capacity is C_0, the heat capacity for the approximation $\hbar\omega_k=Dk^2$. The first correction term is smaller than this leading term by a factor of the order of $(\hbar\omega_M/k_B T)^n$, where $n=\frac{7}{2}$, 1, 1 for cases (A), (B), and (C), respectively. Hence the leading term C_0 is a good approximation in the high-temperature limit $\hbar\omega_M/k_B T\ll 1$. Since these results are the same for all three approximations, it is expected that the results are also valid for the true dispersion relation (4).

It is concluded that the approximate dispersion relation $\hbar\omega_k=Dk^2$ leads to a good approximation to the specific heat when $\hbar\omega_M/k_B T\ll 1$. This is physically reasonable, for when $k_B T\gg\hbar\omega_M$, many high-energy magnons having $\hbar\omega_k\gg\hbar\omega_M$ are excited, and for these magnons $\hbar\omega_k=Dk^2$ is a good approximation to (4). For YIG at room temperature $4\pi M=1750$ G and $\hbar\omega_M/k_B=0.2$°K. Shinozaki's lowest temperature was 1.5°K, Kunzler *et al.* was 1.45°K, Meyer and Harris's was 1.4°K, and Edmond's and Peterson's was 1°K. Thus the high-temperature approximation $\hbar\omega_M/k_B T\ll 1$ is well satisfied in all the experiments, and the approximation $\hbar\omega_k=Dk^2$ is good for all four experiments.