Summation Over Feynman Histories: Charged Particle in a Uniform Magnetic Field*

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(Received 26 August, 1963; revised manuscript received 27 September 1963)

Using a particular parametrization of paths, the nonrelativistic propagator for a charged particle in a uniform magnetic field is derived by the Feynman method of summation over histories. It is shown that this sum is independent of the parameterization as long as the classical path is included. The result is used to obtain the density matrix for the system.

I. INTRODUCTION

IN Feynman's Lagrangian formulation of quantum mechanics,[1] the Schrödinger equation is replaced by an integral equaion which describes how the wave function "propagates" in space and time,

$$\psi(\mathbf{r}'',t)=\int K(\mathbf{r}'',\mathbf{r}',t)\psi(\mathbf{r}',0)d^3r'. \quad (1)$$

It is postulated that the kernel or propagator K is given by

$$K(\mathbf{r}'',\mathbf{r},T)=A\sum_H \exp(iS_H/\hbar), \quad (2)$$

where

$$S_H=\int_0^T L[\mathbf{r}(t')]dt' \quad (3)$$

is the classical action function, L being the Lagrangian, evaluated with respect to a possible trajectory $\mathbf{r}(t)$ satisfying $\mathbf{r}(t)=\mathbf{r}''$, $\mathbf{r}(0)=\mathbf{r}'$. The sum is to be taken over all "physical" trajectories or histories connecting the initial and final points. The normalizing factor A is determined by the requirement that the transformation (1) be unitary. Beginning with the Schrödinger equation it is straightforward to prove that

$$K(\mathbf{r}'',\mathbf{r}',t)=\sum_n \Phi_n(\mathbf{r}'')\exp(-i\mathcal{H}t/\hbar)\Phi_n(\mathbf{r}')^*, \quad (3a)$$

when $\mathcal{H}$ is the Hamiltonian and the sum is over the eigenstates of $\mathcal{H}$.

The problem of properly defining and carrying out the summation in (2) has been discussed by several authors and is reviewed by Brush.[2] In particular, Davies[3] has given a convenient prescription for parameterizing the trajectories and has applied his method to the cases of a free particle and a one dimensional oscillator. The purpose of this article is to apply his prescription to the case of a charged particle in a uniform magnetic field.

In Sec. III we show that the evaluation of the propagator requires only knowledge of the classical path. [*Note added in proof.* This approach has been used by F. Erdogan, Parke Mathematical Laboratories Report AFCRL-TN-60-1109 (unpublished).] However, in view of the possibility of applying Davies' procedure to classically intractable problems, it appears instructive to carry out the integrations. We conclude by using our result to rederive Sondheimer and Wilson's expression for the density matrix.

* This study was supported in part by U. S. Air Force Office of Scientific Research Grant No. AF-AFOSR-262-63.

[1] R. P. Feynman, Rev. Mod. Phys. **20**, 367 (1948).
[2] S. G. Brush, Rev. Mod. Phys. **33**, 79 (1961).
[3] H. Davies, Proc. Cambridge Phil. Soc. **53**, 199 (1957).

II. THE CALCULATION

The Langrangian for a particle of charge $-e$ in an electromagnetic field given by the vector potential $\mathbf{A}$ is

$$L=\tfrac{1}{2}m\dot{\mathbf{r}}^2-(e/c)\mathbf{A}\cdot\dot{\mathbf{r}}.$$

To represent a constant uniform magnetic field $\mathbf{H}_0$ in the Z direction we choose $\mathbf{A}=(-H_0y,0,0)$. The propagator is not gauge invariant and may be obtained for any other choice of gauge by a suitable unitary transformation. Thus

$$L=\tfrac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)+(eH_0/c)y\dot{x}.$$

Motion in the z direction represents free particle propagation; we shall neglect it here but include it at the end of the calculation by properly modifying K. We therefore consider the two-dimensional problem of motion from the point $(x',y';t=0)$ to $(x'',y'';t=T)$. Following Davies'[3] prescription we represent the path in terms of a cosine series

$$x(t)=\sum_{n=0}^{\infty}a_n\cos\frac{n\pi t}{T},\quad y(t)=\sum_{n=0}^{\infty}b_n\cos\frac{n\pi t}{T}.$$

The summation over paths is to be effected by integrating over the coefficients a_n, b_n. It is possible that paths having nonphysical characteristics such as discontinuities may be introduced in this way; this problem will be dealt with in Sec. III.

The action (3) is now given by

$$S=\frac{m\pi^2}{4T}\sum_{n=0}^{\infty}n^2(a_n^2+b_n^2)-\frac{2eH_0}{c}\sum_{j+n\ \text{odd}}\frac{j^2a_jb_n}{j^2-n^2}.$$

Representing the sum over all paths by integrating over the a's and b's we see that the propagator (2) is proportional to the integral:

$$H=\lim_{N\to\infty}\int_{-\infty}^{\infty}da_0da_1\cdots da_N\int_{-\infty}^{\infty}db_0db_1\cdots db_N$$

$$\times\exp\Big[i\sigma\sum_{n=0}^{\infty}n^2(a_n^2+b_n^2)\Big]\exp\Big[is\sum_{\substack{n+j\\ \text{odd}}}\frac{j^2a_jb_n}{n^2-j^2}\Big]$$

$$\times\delta\Big[\sum{}''a_j-(x'+x'')/2\Big]\delta\Big[\sum{}'a_j-\Delta x\Big]$$

$$\times\delta\Big[\sum{}''b_j-\frac{(y'+y'')}{2}\Big]\delta\Big[\sum{}'b_j-\Delta y\Big],$$

where $\sigma=m\pi^2/4\hbar T$, $s=2eH_0/\hbar c$, $\Delta x=(x'-x'')/2$, $\Delta y=(y'-y'')/2$, and $\sum'$ denotes summation over odd j, $\sum''$ over even j.

We shall not attempt to make the calculation mathematically rigorous. However, most of the integrations and summations can be formally justified by suitable limiting processes such as giving the mass an infinitesimal imaginary part.

Since a_0 does not appear in the exponent, the a_0 integration is trivial. In carrying out the integrations we shall neglect any factor which does not depend on the end coordinates. These factors contribute only to the normalization which will be determined at the end of the calculation by the requirement of unitarity. The remaining even a integrals give simply

$$\exp\Big\{-\frac{is^2}{\sigma}\sum_{k=1}^{\infty}k^2B_k^2\Big\}=\exp\Big\{-i\frac{s^2\pi^2}{64\sigma}\sum_{j=1}^{\infty}b_j^2\Big\},$$

where

$$B_k=\sum_j{}'\frac{k_j}{(j^2-4k^2)}.$$

This formula is proven in Appendix A. The b_0 integration is now easily performed and gives

$$H\sim\exp\Big\{-is\Delta x\Big(\frac{y'+y''}{2}\Big)\Big\}H',$$

where

$$H'=\int_{-\infty}^{\infty}da_1da_3\cdots\int_{-\infty}^{\infty}db_1db_3\cdots\exp\Big\{i\sigma\sum{}'n^2a_n^2\Big\}$$

$$\times\exp\Big\{i\sigma\sum_{n=1}^{\infty}n^2b_n^2\Big\}\exp\Big\{is\sum_{\substack{n\text{ even}\\ j\text{ odd}}}\frac{j^2a_jb_n}{n^2-j^2}\Big\}$$

$$\times\exp\Big\{-\frac{is^2\pi^2}{64\sigma}\sum_n{}'b_n^2\Big\}\exp\Big\{is\Delta x\sum_j{}''b_j\Big\}$$

$$\times\delta\Big[\sum_j{}'a_j-\Delta x\Big]\delta\Big[\sum_j{}'b_j-\Delta y\Big].$$

At this point, the b_{2k} integrals are easily evaluated, and their contribution is simply

$$\exp\{-i(s^2/16\sigma)(\Lambda_1+2\Delta x\Lambda_2)\}\exp\{-(is^2\pi^2/96\sigma)\Delta x^2\},$$

where

$$\Lambda_1=\sum_{k=1}^{\infty}\frac{A_k^2}{k^2},\quad\Lambda_2=\sum_{k=1}^{\infty}\frac{A_k}{k^2},\quad\text{and}\quad A_k=\sum_j{}'\frac{j^2a_j}{4k^2-j^2}.$$

These sums are evaluated in Appendix B. We now observe that

$$H\sim\exp\{-\tfrac{1}{2}is\Delta x(y'+y'')\}K_aK_b,$$

where

$$K_\alpha=\int_{-\infty}^{\infty}d\alpha_1d\alpha_3\cdots\exp\Big\{i\sigma\sum_n{}'n^2\alpha_n^2-i\frac{s^2\pi^2}{64\sigma}\sum_n{}'\alpha_n^2\Big\}$$

$$\times\delta\Big[\sum{}'\alpha_n-\Delta x_\alpha\Big](x_a=x,\ x_b=y).$$

These last integrals are done by introducing the integral representation for the δ function:

$$K_\alpha=\int_{-\infty}^{\infty}e^{-ip\Delta x}dp\prod_n{}'\int_{-\infty}^{\infty}d\alpha_n\exp\{\lambda_n\alpha_n^2+ip\alpha_n\}$$

$$\sim\exp\{\Delta x_\alpha^2/\xi\},$$

where

$$\lambda_n=i\sigma\Big[n^2-\Big(\frac{s\pi}{8\sigma}\Big)^2\Big],\quad\xi=\sum_{k=1}^{\infty}(\lambda_{2k-1})^{-1}.$$

ξ is evaluated in Appendix C. So including a factor

$$\exp\{(im/2\hbar T)(z'-z'')^2\}$$

for free z propagation, we find finally, that

$$K(\mathbf{r}'',\mathbf{r}',T)$$

$$=B(T)\exp\Big\{i\Big(\frac{2m}{\hbar^2}\Big)\Big[\frac{\beta H_0}{4}\cot\Big(\frac{\beta H_0T}{\hbar}\Big)$$

$$\times[(x''-x')^2+(y''-y')^2]$$

$$+\tfrac{1}{2}\beta H_0(x''-x')(y''+y')+\frac{\hbar}{4T}(z''-z')^2\Big]\Big\},\quad(4)$$

where β is the Bohr magneton $e\hbar/2mc$.

We shall now determine the normalization $B(t)$ (up to a phase factor). Unitarity requires:

$$U=\int d^3\mathbf{r}'K(\mathbf{r}'',\mathbf{r}',t)K^*(\mathbf{r}',\mathbf{r},t)=\delta(\mathbf{r}''-\mathbf{r}).$$

Let ($a=2m/\hbar^2$), $b=\frac{1}{4}\beta H_0\cot(\beta H_0t/\hbar)$, $c=\frac{1}{2}\beta H_0$, $d=\hbar/4t$. Then from (4) we easily find

$$U=(8\pi^3/2a^2d)|B(t)|^2\exp\{ia[b\Delta x(x''+x)$$

$$+b\Delta y(y''+y)+c(x''y''-xy)+d\Delta z(z''+z)]\}$$

$$\times\delta(\Delta z)\delta(2b\Delta x+c\Delta y)\delta(2b\Delta y+c\Delta x),$$

where $\Delta x=x''-x$, etc. By the theorem in Appendix D

$$\delta(2b\Delta x+c\Delta y)\delta(c\Delta x+2b\Delta y)=(4b^2-c^2)^{-1}\delta(\Delta x)\delta(\Delta y).$$

Hence, since $x''y''-xy=\frac{1}{2}[\Delta x(y''+y)+(x''+x)\Delta y]$,

$$U=\frac{4\pi^3}{a^3d(4b^2-c^2)}|B(t)|^2\delta(\Delta z)\delta(\Delta x)\delta(\Delta y).$$

Therefore, unitarity requires that

$$\begin{aligned}|B(t)|&=[a^3d(4b^2-c^2)/4\pi^3]^{1/2}\\&=(m/2\pi\hbar t)^{3/2}(\beta H_0t/\hbar)\csc(\beta H_0t/\hbar).\end{aligned}$$

To obtain the propagator in the symmetric gauge $\mathbf{A}=\frac{1}{2}H_0(-y,x,0)=\frac{1}{2}\mathbf{H}_0\times\mathbf{r}$, we need only multiply by the phase factor $\exp\{i(eH_0/2\hbar c)(x'y'-x''y'')\}$. Therefore, for this gauge

$$\begin{aligned}K(\mathbf{r}'',\mathbf{r}',t)=B(t)\exp\{&i(m/2\hbar^2)\\&\times[\beta H_0\cot(\beta H_0t/\hbar)[(x''-x')^2+(y''-y')^2]\\&+(\hbar/t)(z''-z')^2+2\beta H_0(x''y'-x'y'')]\}.\quad(5)\end{aligned}$$

III. DISCUSSION

In the previous section we pointed out that Davies' prescription for evaluating the sum over histories introduced a number of unphysical paths, such as those with discontinuities. We avoid this difficulty by showing that only the classical path contributes to the spatial part of the propagator, all other paths contributing to the normalization only.

We have

$$K\sim\sum_H\exp(iS_H/\hbar),\quad S_H=\int Ldt,$$

where $L=\frac{1}{2}m(\dot{x}^2+\dot{y}^2)+(H_0/c)y\dot{x}$. We write

$$(x,y)=(x_c(t),y_c(t))+(\xi(t),\eta(t)),$$

where (x_c,y_c) denotes the classical path and $(\xi,\eta)=(0,0)$ at $t=0$, T. Then the Lagrangian takes the form:

$$\begin{aligned}L=L_c+\tfrac{1}{2}m(\dot{\xi}^2+\dot{\eta}^2)&+m(\dot{x}_c\dot{\xi}+\dot{y}_c\dot{\eta})\\&+(eH_0/c)(y_c\dot{\xi}+\eta\dot{x}_c+\eta\dot{\xi}),\end{aligned}$$

where L_c is the Lagrangian computed for the classical path. Hence

$$\begin{aligned}S=S_c&+m\int_0^T(\dot{x}_c\dot{\xi}+\dot{y}_c\dot{\eta})dt\\&+\frac{eH_0}{c}\int_0^T(y_c\dot{\xi}+\eta\dot{x}_c+\eta\dot{\xi})dt+\tfrac{1}{2}m\int_0^T(\dot{\xi}^2+\dot{\eta}^2)dt.\end{aligned}$$

Now, integrating by parts, we find

$$m\int_0^T(\dot{x}_c\dot{\xi}+\dot{y}_c\dot{\eta})dt=-m\int_0^T(\ddot{x}_c\xi+\ddot{y}_c\eta)dt.$$

However, the classical equations of motion are

$$\begin{aligned}m\ddot{x}_c+(eH_0/c)\dot{y}_c&=0,\\m\ddot{y}_c-(eH_0/c)\dot{x}_c&=0,\end{aligned}$$

so the action takes the form

$$S=S_c+\int_0^T\{\tfrac{1}{2}m(\dot{\xi}^2+\dot{\eta}^2)+(eH_0/c)\eta\dot{\xi}\}dt=S_c+S'$$

and S' is independent of the endpoints. Therefore,

$$K\sim e^{iS_c/\hbar}\sum_H e^{iS'/\hbar},$$

and since S' is independent of the endpoints the sum contributes only to the normalization. This means that if we choose the proper normalization the sum may be taken over any set of paths which includes the classical trajectory.

As an application of (5) we shall obtain the density matrix for an electron in a uniform magnetic field. For nonrelativistic quantum mechanics $K(\mathbf{r}'',\mathbf{r}',t)$ is given by (3a). Sondheimer and Wilson[4] have evaluated the quantity

$$\psi(\mathbf{r}'',\mathbf{r}',\gamma)=\sum_n\Phi_n^*(\mathbf{r}'')\exp(-\gamma\mathcal{H})\Phi_n(\mathbf{r}')$$

for the case we consider here. To obtain their result we need only make the replacement $t=-i\hbar\gamma$ and interchange the initial and final points of the paths. When these replacements are made, (5) becomes

$$\begin{aligned}\psi(\mathbf{r}'',\mathbf{r}',\gamma)&\\=\left(\frac{m}{2\pi\hbar^2\gamma}\right)^{3/2}&\frac{(\beta H_0\gamma)}{\sinh(\beta H_0\gamma)}\exp\left\{-\left(\frac{m}{2\pi\hbar^2\gamma}\right)\right.\\&\times[\beta H_0\gamma\coth(\beta H_0\gamma)[(x''-x')^2+(y''-y')^2]\\&\left.+(z''-z')^2+2i\beta H_0\gamma(x''y'-x'y'')]\right\}.\end{aligned}$$

This is precisely the result obtained in Ref. 4.

ACKNOWLEDGMENTS

The author wishes to thank Dr. R. P. Kenan for several stimulating conversations and Battelle Memorial Institute for its supportof this work.

APPENDIX

$$\begin{aligned}\text{A:}\quad S&=\sum_{k=1}^{\infty}k^2B_k^2\\&=\tfrac{1}{16}\sum_{n\neq j}{}'\,b_nb_jS(n/2,j/2)+\tfrac{1}{16}\sum_n{}'\,n_n^2S(n/2),\end{aligned}$$

where

$$S(a,b)=\sum_{k=1}^{\infty}\frac{k^2}{(k^2-a^2)(k^2-b^2)},\quad S(a)=\sum_{k=1}^{\infty}\frac{k^2}{(k^2-a^2)^2}.$$

[4] E. H. Sondheimer and A. H. Wilson, Proc. Roy. Soc. (London) **A210**, 173 (1951).

Starting from the well-known formula

$$\sum_{k=1}^{\infty} \frac{1}{k^2-a^2}=\frac{1}{2a^2}-\frac{\pi}{2a}\cot\pi a, \tag{1A}$$

we easily find

$$S(a,b)=\frac{\pi a^2}{2(a^2-b^2)}\left[\frac{\cot\pi b}{b}-\frac{\cot\pi a}{a}\right]-\frac{\pi}{2b}\cot\pi b=0,$$

where the last equality is due to the fact that a and b are to be halves of odd integers; and

$$S(a)=\frac{\pi^2}{4}\csc^2\pi a-\frac{\pi}{4a}\cot\pi a=\frac{\pi^2}{4}.$$

Therefore,

$$S=\frac{\pi^2}{64}\sum_n{}' b_n{}^2.$$

B: $$\Lambda_1=\sum_{k=1}^{\infty}\frac{A_k{}^2}{k^2}=\tfrac{1}{16}\sum_{j\neq n}{}' j^2n^2a_ja_n\lambda(j/2,n/2)+\tfrac{1}{16}\sum_n{}' n^4a_n{}^2\lambda(n/2),$$

where

$$\lambda(a,b)=\sum_{k=1}^{\infty}\frac{1}{k^2(k^2-a^2)(k^2-b^2)},\quad \lambda(b)=\sum_{k=1}^{\infty}\frac{1}{k^2(k^2-b^2)}.$$

Again from (A.1) we find

$$\lambda(a,b)=\frac{\pi^2}{6a^2b^2}-\frac{a^2+b^2}{2a^4b^4}+\frac{\pi}{2(a^2-b^2)}\left[\frac{\cot\pi b}{b^3}-\frac{\cot\pi a}{a^3}\right]$$

so

$$\lambda(j/2,n/2)=\frac{8\pi^2}{3j^2n^2}-\frac{32(j^2+n^2)}{j^4n^4},$$

and

$$\lambda(b)=\frac{\pi^2}{6b^4}-\frac{1}{b^6}+\frac{3\pi}{4b^5}\cot\pi b+\frac{\pi^2}{4b^4}\csc^2\pi b$$

so

$$\lambda(n/2)=20\pi^2/3n^4-64/n^6.$$

Therefore,

$$\Lambda_1=\frac{\pi^2}{6}\Big(\sum_n{}' a_j\Big)^2+\frac{\pi^2}{4}\sum_j{}' a_j{}^2-4\left(\sum_j{}'\frac{a_j}{j^2}\right)\Big(\sum_j{}' a_j\Big)$$

$$=\frac{\pi^2}{6}\Delta x^2+\frac{\pi^2}{4}\sum_j{}' a_j{}^2-4\Delta x\left(\sum_i{}'\frac{a_j}{j^2}\right).$$

Again,

$$\Lambda_2=\sum_{k=1}^{\infty}\frac{A_k}{k^2}=\tfrac{1}{4}\sum_j{}' j^2a_j\mu(j/2)$$

where

$$\mu(a)=\sum_{k=1}^{\infty}\frac{1}{k^2(k^2-a^2)}$$

$$=\frac{1}{2a^4}-\frac{\pi^2}{2a^3}\cot a-\frac{\pi^2}{6a^2}.$$

Hence,

$$\mu(j/2)=8/j^4-2\pi^2/3j^2$$

and

$$\Lambda_2=2\sum_j{}'\frac{a_j}{j^2}-\frac{\pi^2}{6}\Delta x.$$

Combining, we find

$$\Lambda_1+2\Delta x\Lambda_2=\frac{\pi^2}{4}\sum_j{}' a_j{}^2-\frac{\pi^2}{6}\Delta x^2.$$

C: $$\xi=\sum_{k=1}^{\infty}\frac{1}{\lambda_{2k-1}}=\frac{1}{i\sigma}\sum_{k=1}^{\infty}\left[(2k-1)^2-\left(\frac{s\pi}{8\sigma}\right)^2\right]^{-1}.$$

From the well-known sum,

$$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2-x^2}=\frac{\pi}{4x}\tan\frac{\pi x}{2},$$

we find

$$\xi=2/is\tan(s\pi^2/16\sigma).$$

D: Theorem:

$$A=\prod_{j=1}^{n}\delta\Big[\sum_{i=1}^{n}a_i{}^{(j)}x^i\Big]=\frac{1}{\Delta}\prod_{i=1}^{n}\delta(x^i),$$

where $\Delta=\det|a_i{}^{(j)}|\neq 0$.

Proof:

$$A=(2\pi)^{-n}\int_{-\infty}^{\infty}dk_1\cdots dk_n e^{ik_ja_i{}^{(j)}x_i}$$

(we sum over all repeated indices). Let $K_i=k_ja_i{}^{(j)}$. The Jacobian of the inverse transformation is simply Δ. Hence,

$$A=(2\pi)^{-n}\Delta^{-1}\int_{-\infty}^{\infty}dK_1\cdots dK_n e^{iK_ix^i}=\Delta^{-1}\prod_{i=1}^{n}\delta(x^i).$$