Gravitational Lenses*

SIDNEY LIEBES, JR.
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey
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A stellar gravitational lens has the capacity to intensify by a factor in excess of 1000 the image of another star suitably aligned behind it. This capability, reported in 1936 by A. Einstein in response to a suggestion by R. W. Mandl, appears today not to be generally well known. In the present paper, the imaging properties of the gravitational lens are discussed, and mechanisms whereby lens effects might manifest themselves are evaluated. The relative proper motions of the stars in our galaxy will occasionally give rise to transitory alignments producing time symmetric pulses of light intensity. Typically, for two stars aligned to better than $\sim 10^{-8}$ rad, the image configuration of the more distant, object star will be characterized by an intensification varying inversely with the angular separation of the stars. This inverse angular dependence persists until the disk of the object star, subtending perhaps $\sim 10^{-11}$ rad, passes behind the center of the deflector star. It is estimated, for events involving the $\sim 10^9$ observable stars of the Milky Way, that $\tau_I/A_M \approx 3\times 10^6$ sec, and $\tau_W A_M \approx 5\times 10^7$ sec, where τ_I represents the mean interval between all events which will exhibit at closest alignment an intensity gain for the object star in excess of A_M, and τ_W denotes the temporal width of that pulse characterized by A_M. For the case of a globular cluster of $\sim 3\times 10^5$ deflector stars moving between the earth and the relatively dense object stellar population in the direction of the galactic center, we find $\tau_I/A_M \approx 3\times 10^8$ sec, and $\tau_W A_M \approx 7\times 10^6$ sec. For events involving stars entirely within a single globular cluster of $\sim 10^6$ stars, we find $\tau_I/A_M \approx 8\times 10^7$ sec, and $\tau_W A_M \approx 3\times 10^6$ sec, with A_M restricted, in this case, to be less than ~ 60. Unaccounted for white dwarf and neutron deflector stars are not likely to contribute more than a factor of ~ 2 increase in the frequency of events. No planets of the sun are useful for producing image intensifications. As a result of their gravitational lens action upon the light from other stars, all stars are surrounded by radial spikes of concentrated light intensity. Such spikes might occasionally manifest themselves as luminous beams in cosmic dust or gas clouds. A statistical analysis of the images of distant galaxies might reveal spectral anomalies explicable in terms of red shifted object galaxies intensified by the lens action of nearer deflector galaxies.

I. INTRODUCTION

A. Preliminary Remarks

It follows from the general theory of relativity that a ray of light, passing at distance of closest approach r to the center of a spherically symmetric body of mass M, should be deflected toward the body through an angle which, for small deflections, is of magnitude

$$\epsilon = 4GM/rc^2\,, \tag{1}$$

where G is the gravitational constant and c is the speed of light.[1] The most familiar application of this expression is to the deflection of starlight passing near the sun. In this case the maximum deflection is 1.75″. The eclipse observations of the past half-century have indicated the existence of deflections having approximately the predicted value. The significance of the tendency[2,3] for the observed deflections to exceed the theoretical ones remains to be established.

It is curious that the properties of the gravitational lens, which are a direct consequence of the gravitational deflection of light, and which were first recognized many years ago, are today no longer generally appreciated. For, as a consequence of the gravitational lens-like action of one star, it is possible for the image of another more distant star within our galaxy to be intensified by a factor in excess of 1000. The lack of a general awareness of this capability is perhaps a consequence of a past judgment that there is little likelihood that any given generation of scientists will witness a verifiable display of gravitational lens effects. The writer has, however, been unable to find in the literature a record of any attempt either to consider the possible modes by which lens effects could manifest themselves or to assign order of magnitude estimates to the frequencies with which conceivably detectable events might occur. Regardless of the expected ease or frequency with which lens events might be observed, it would seem that the image intensification capability is remarkable enough to be of general interest.

The principal aims of the present paper will be twofold: first, to discuss the properties of the gravitational lens, in slightly more general form and further detail than has appeared in the past; and second, to examine possible modes of display and to obtain rough estimates for the frequencies with which events of various types might occur.

B. Historical Remarks

We review, briefly, the references which have been found in the literature to gravitational lens phenomena, apologizing for those oversights which undoubtedly have been made.

In a discussion of the radial dependence of the refractive index that would be required to provide the relativistic deflection of light in its passage by the sun,

* This work supported by the National Science Foundation.
[1] A. Einstein, Sitzber. Kgl. Preuss. Akad. Wiss. 831 (1915).
[2] E. Finlay-Freundlich, *Vistas in Astronomy* (Pergamon Press, London, 1955), Vol. 1, p. 239.
[3] A. A. Mikhailov, Monthly Notices Roy. Astron. Soc. **119**, 593 (1959).

Lodge[4] remarked that it would be "impermissible to say that the solar gravitational field acts as a lens, for it has no focal length." Indeed, the effect of the gravitational field is to condense the light, from any sufficiently distant point source, onto a semi-infinite line. The "gravitational lens" terminology has, however, subsequently been independently introduced by different writers.[5]

Eddington[6] noted that if two stars were at sufficiently different distances from an observer, but were seen in the same line of vision to within about 1″, there ought to appear, in addition to the primary image of the more distant star, a weaker secondary image on the opposite side of the nearer star. He observed that as the alignment improved, the divergence of the rays from the primary image should increase in the direction parallel to the plane containing the observer and the two stars. As he overlooked an associated convergent tendency in the orthogonal direction, he mistakenly concluded that the primary image should fade.

In 1935, R. W. Mandl, a Czech electrical engineer, observed in a letter to Albert Einstein that a star should be expected to act as a "gravitational lens" for the light passing by it from another star.[7] Apparently, in the course of a subsequent visit, Mandl requested of Einstein that he publish the results of the analysis he had made in response to Mandl's suggestion. Einstein compiled by summarizing his findings in a note[8] in which he reported that the lens-like action of one star could lead to an intensification of the image of another star suitably aligned behind it. He obtained an expression, valid in the limit that a point star is located at relatively great distance beyond a deviating star, for the dependence of the image intensity upon the displacement of the observer from the extended line of centers of the two stars.

Russell[9] considered a hypothetical situation in which the lens-like action of a body would be observable to the naked eye. Zwicky[10–12] has considered the concept of the gravitational lens as it applies to galaxies and has stated,[11,12] on the basis of galactic mass and diameter estimates, that the observability of gravitational lens effects among galaxies would seem assured. The existence of such effects has, however, not yet been established. Zwicky has also noted[11] that, among others, E. B. Frost as early as 1923 outlined a program for the search for lens effects among the stars. At the end of his George Darwin Lecture, A. A. Mikhailov[3] remarked that the Soviet astronomer Tikhov had theoretically investigated lens effects. Darwin[13] has considered the action of the gravitational field of a star in producing multiple images of another star located behind it. Idlis and Gridneva[14] have discussed the derivation of galactic masses from observations of the gravitational lens-type distortions that these galaxies impose upon the images of more distant galaxies located behind them.

The present writer recently reconsidered the properties of the gravitational lens and presented estimates for the frequencies of stellar-stellar events among the observable stars of our galaxy.[15,15a]

II. IMAGING PROPERTIES OF A GRAVITATIONAL LENS

A. Geometrical Considerations

Located at O in Fig. 1 is an object which is a source of electromagnetic radiation; centered at D is a localized spherically symmetric deflecting body of mass M; and stationed at R is a receiver which detects the radiation emitted by the object. It will be assumed throughout the present discussion that the wavelengths involved are small enough that the principles of geometrical optics may appropriately be applied in treating the action of the deflector. Under favorable conditions of alignment there will exist two distinct paths, shown by the curved solid lines in the figure, along which light can propagate from each point on the object to the receiver. We wish to find expressions for the angles θ_1 and θ_2, in Fig. 1, in terms of $(G, c, M, \alpha, l_D, l_{OD})$, valid for the small deflections generally to be expected in practice. It will be assumed that the distances l_D and l_{OD} are large compared to the distances of closest approach of the light beams to the deflecting body, and that the light paths remain external to the mass distribution of the deflector. Referring to Fig. 1, we observe that

$$\begin{aligned}\theta_1&=\alpha+\beta,\\ \epsilon_1&=\beta+\beta',\\ \beta'&=\beta(l_D/l_{OD}).\end{aligned}\tag{2}$$

Expressions (2) may be combined to yield

$$\theta_1=\alpha+(\epsilon_1/\mu),\tag{3}$$

[4] O. J. Lodge, Nature **104**, 354 (1919).
[5] See Refs. 8 and 15.
[6] A. S. Eddington, *Space, Time, and Gravitation* (Cambridge University Press, New York, 1920), pp. 134 and 308 (reprinted by Harper and Brothers, Inc., New York, 1959).
[7] See Refs. 8, 10, and 12.
[8] A. Einstein, Science **84**, 506 (1936).
[9] H. N. Russell, Sci. Am. **156**, 76 (1937).
[10] F. Zwicky, Phys. Rev. Letters **51**, 290 (1937).
[11] F. Zwicky, Phys. Rev. Letters **51**, 679 (1937).
[12] F. Zwicky, *Morphological Astronomy* (Springer-Verlag, Berlin, 1957), p. 215.

[13] C. Darwin, Proc. Roy. Soc. (London) **A249**, 180 (1959).
[14] G. M. Idlis and S. A. Gridneva, Izv. Astrofiz. Inst. Acad. Nauk. Kaz. SSR **9**, 78 (1960).
[15] The principal findings of the present paper were summarized in a postdeadline paper, "Observable Anomalous Electromagnetic Imaging Properties of Stellar Gravitational Lenses," delivered to the Washington Meeting of the American Physical Society, April 1963. As the writer's initial search of the literature had failed to reveal evidence of a previous consideration of gravitational lens effects, the postdeadline paper was derelict in acknowledgment.
[15a] *Note added in proof*. The following articles relevant to the subject matter of the present paper have recently appeared: Yu. G. Klimov, Dokl. Akad. Nauk. SSSR **148**, 789 (1963) [English transl.: Soviet Phys.—Doklady **8**, 119 (1963)]; A. W. K. Metzner, J. Math. Phys. **4**, 1194 (1963).

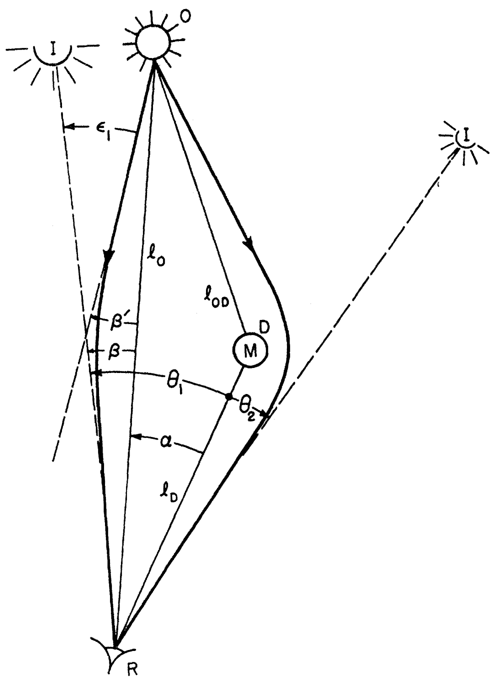

FIG. 1. The gravitational deflection of light by the deflector D of mass M causes the receiver R to detect two images I of the object O.

where

$$\mu \equiv 1+(l_D/l_{OD}). \tag{4}$$

Since, from (1),

$$\epsilon_1 = 4GM/\theta_1 l_D c^2, \tag{5}$$

it follows that

$$\theta_1^2 - \alpha\theta_1 - \theta_0^2 = 0, \tag{6}$$

where

$$\theta_0 \equiv (4GM/\mu l_D c^2)^{1/2}. \tag{7}$$

Thus,

$$\theta_1 = +\tfrac{1}{2}\alpha + [(\tfrac{1}{2}\alpha)^2 + \theta_0^2]^{1/2}, \tag{8}$$

and similarly,

$$\theta_2 = -\tfrac{1}{2}\alpha + [(\tfrac{1}{2}\alpha)^2 + \theta_0^2]^{1/2}. \tag{9}$$

Therefore,

$$\theta_1 - \theta_2 = \alpha, \tag{10}$$

and

$$\theta_1\theta_2 = \theta_0^2. \tag{11}$$

The lens-like action of the deflector is demonstrated by the sequence of Figs. 2(a), (b), (c), and (d), in which there is shown the transformation that occurs in the image configuration I of a spherical object O as the object, subtending angular radius φ_O with respect to the receiver, passes from left to right behind the gravitational lens associated with a spherically symmetric deflecting body D of mass M and angular radius φ_D. The dotted circle in each of the figures represents the cone of inversion, of angular radius θ_0, defined in (7). As the object approaches alignment behind the deflector, Fig. 2(a), the primary image I lags to the left behind the true position of the object O, while a developing secondary image I emerges from behind the right limb of the deflector D. By the time the object is well inside the cone of inversion, Fig. 2(b), the images have acquired considerable distortion. When the right limb of the object becomes aligned behind the center of the deflector, Fig. 2(c), the primary image establishes contact from the outside with the left half of the cone of inversion, and the secondary image establishes contact from the inside with the right half of the cone. The two images thereby coalesce. When the object becomes precisely aligned behind the deflector, Fig. 2(d), there appears a single annular image of the object concentric with the deflector and spanning the cone of inversion. It will subsequently be shown that this ring image generally has an angular thickness approximately equal to φ_O.

It is anticipated that diffraction effects related to the finite aperture of the receiving telescope and seeing conditions associated with fluctuations in the atmosphere will generally preclude the resolution of stellar image configurations such as those illustrated. However, evaluation of the image intensification will be facilitated by reference to the image configurations of Fig. 2.

If the entire surface of the object is characterized by a uniform brightness, then throughout the transformation illustrated in Fig. 2, the image configuration will exhibit a brightness identical to that of the object. That this is so may be demonstrated by a thermodynamic argument. Imagine the object to be a section of the surface of an isothermal enclosure. If the introduction of the present transparent lens were to result in an alteration of the brightness of any portion of the wall, then the resulting power-flux imbalance would enable the second law of thermodynamics to be violated. It follows, then, from the invariance of the image brightness that the intensity of the radiation from the object detected by the receiver, will be proportional to the solid angle subtended by the virtual image of the object.

B. Image Intensification

It will be convenient, for the remainder of the discussion, to have at hand the following:

$$\begin{aligned}
G &= 6.67\times10^{-8}\ \text{dyne cm}^2\ \text{g}^{-2} \quad \text{(gravitational constant)}\\
c &= 3.00\times10^{10}\ \text{cm sec}^{-1} \quad \text{(speed of light)}\\
1\ \text{l.y.} &= 9.46\times10^{17}\ \text{cm} \quad \text{(1 light year)}\\
1\ \text{sec} &= 4.85\times10^{-6}\ \text{rad (angular measure)}\\
1\ \text{day} &= 8.64\times10^{4}\ \text{sec}\\
1\ \text{yr} &= 3.15\times10^{7}\ \text{sec}\\
M_\odot &= 1.99\times10^{33}\ \text{g} \quad \text{(solar mass)}\\
R_\odot &= 6.96\times10^{10}\ \text{cm} \quad \text{(solar radius)}\\
\epsilon_\odot &= 4GM_\odot/R_\odot c^2 = 8.47\times10^{-6}\ \text{rad}\\
&= 1.75\ \text{sec} \quad \text{(glancing solar deflection)}.
\end{aligned} \tag{12}$$

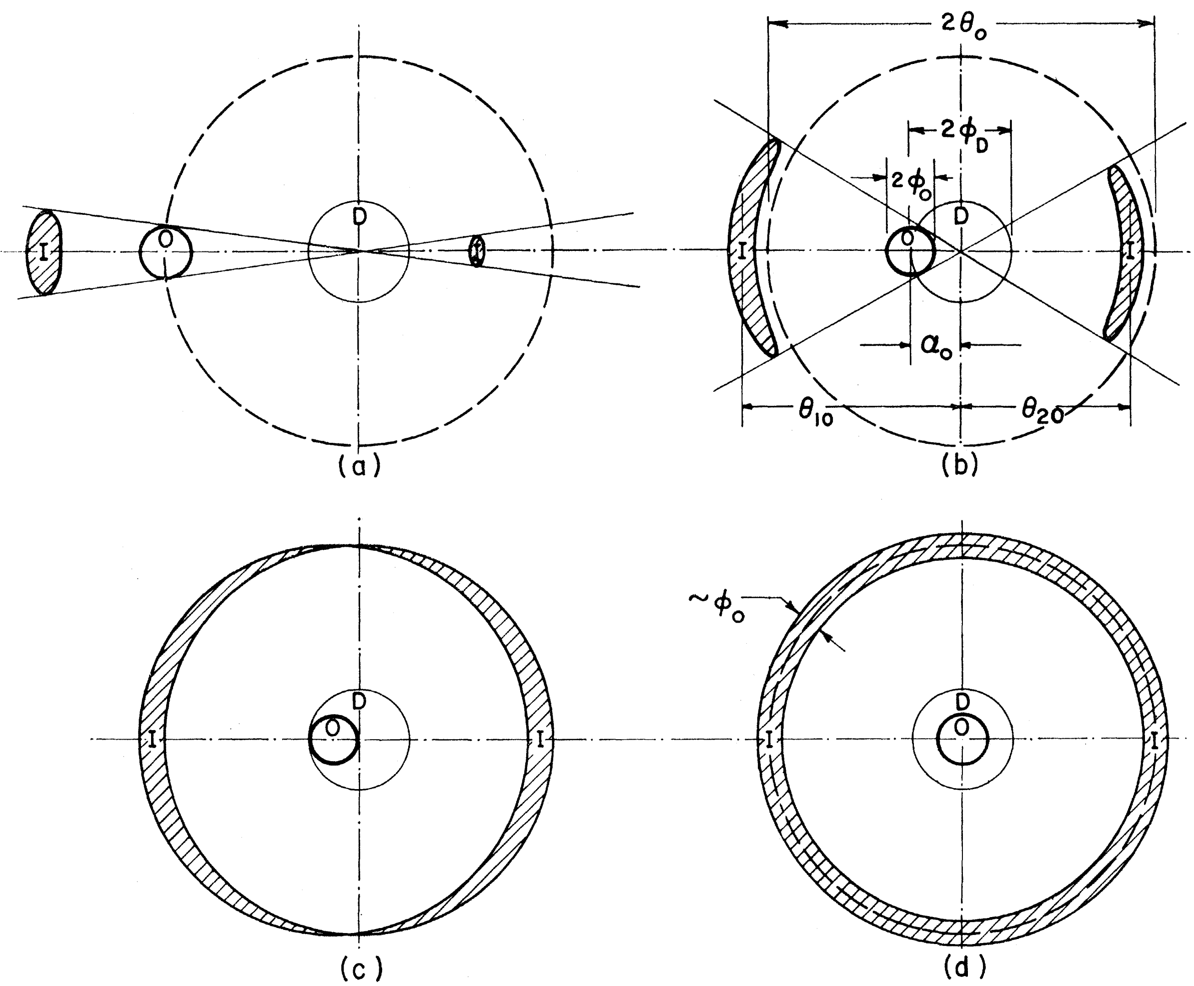

FIG. 2. Transformation of the image configuration I of the object O as the object moves from left to right behind the deflector D.

Our nearest stellar neighbor other than the sun is at a distance of 4.2 l.y. Let us consider, for the purpose of illustration, the relatively unfavorable situation in which a deflector star and an object star, each of which is similar to the sun, are located at distances $l_D=4$ l.y. and $l_O=8$ l.y., respectively, from us. For this circumstance we have $\varphi_D=R_\odot/l_D=1.84\times10^{-8}$ rad, $\varphi_O=R_\odot/l_O=0.92\times10^{-8}$ rad, and from (7), $\theta_0=2.79\times10^{-7}$ rad, so that $\theta_0/\varphi_D=15.2$ and $\theta_0/\varphi_O=30.4$. Thus, the cone of inversion is large enough to be unobscured by the deflector, and furthermore the angle subtended by the object is small compared to the half-apex angle θ_0 of the cone of inversion. From (7) we observe that, for fixed μ, $\theta_0/\varphi_D\propto l_D{}^{1/2}$. It should also be noted that the mean distance to the observable stars in our galaxy is approximately three orders of magnitude greater than the values employed in the present example. Furthermore, typical stellar masses and radii are comparable with the values for the sun. We therefore conclude that, for randomly paired stars within the galaxy, it will generally be the case that $\theta_0/\varphi_D\gg1$ and $\theta_0/\varphi_0\gg1$. We shall assume, for the purpose of the following discussion, that these conditions are satisfied.

We will henceforth denote by α_0 the angle between the line of sight to the center of the deflector and that to the center of the object. For $\alpha_0\gtrsim\varphi_O$, both the primary and secondary images shown in Fig. 2 may be treated as bent ellipses. It will be seen, from a consideration of the geometrical relationships in Fig. 2, that the primary image subtends a solid angle

$$\Omega_1\approx\pi\varphi_1\varphi_O\theta_{10}/\alpha_0\,, \tag{13a}$$

wherein the quantity

$$\varphi_1\approx\varphi_O\frac{d\theta_1}{d\alpha}\bigg|_{\alpha=\alpha_0}, \tag{13b}$$

which may be evaluated from (8), represents the semiminor axis of the ellipse; and θ_{10} denotes the angular displacement from the center of the deflector of that point in the primary image which corresponds to the central point, at α_0, of the object. We obtain from (13a, b)

$$\Omega_1 \approx \tfrac{1}{4}\pi\varphi_O^2(2+\xi+\xi^{-1}), \tag{14a}$$

where

$$\xi \equiv [1+(2\theta_0/\alpha_0)^2]^{1/2}. \tag{14b}$$

Therefore, the ratio of the primary image intensity to the intensity which would be observed in the absence of the deflector is

$$A_1 \approx \tfrac{1}{4}(2+\xi+\xi^{-1}). \tag{15}$$

It follows in similar fashion that the equivalent ratio for the secondary image is

$$A_2 \approx \tfrac{1}{4}(-2+\xi+\xi^{-1}). \tag{16}$$

We thus find that when $\alpha_0 \gtrsim \varphi_O$, and when neither of the images is obscured by the limb of the deflector, the intensity gain $A \equiv A_1+A_2$ characterizing the total image configuration of the object is

$$A \approx \tfrac{1}{2}(\xi+\xi^{-1}). \tag{17}$$

This gain may be expressed more conveniently for a broad range of values of α_0. When $\alpha_0 \gtrsim 4\theta_0$, that is, when the object is well outside the cone of inversion, (17) reduces to

$$A \approx 1+2(\theta_0/\alpha_0)^4. \tag{18}$$

When $\theta_0 \gtrsim \alpha_0 \gtrsim \varphi_O$, that is, in the important case when the object is inside the cone of inversion but with its limb not yet aligned behind the center of the deflector, (17) may be approximated by

$$A \approx \theta_0/\alpha_0. \tag{19}$$

It will be noted from (8) that

$$\lim_{\alpha/\theta_0 \to 0} \frac{d\theta_1}{d\alpha} = \tfrac{1}{2}. \tag{20}$$

Thus the ring image, that prevails when the object is perfectly aligned behind the deflector, as illustrated in Fig. 2(d), generally has an angular thickness very nearly equal to the angular radius φ_O of the object. Since the mean angular radius of the ring is equal to θ_0, the intensity gain at perfect alignment is approximately

$$A \approx 2\theta_0/\varphi_O. \tag{21}$$

Figure 3 shows, for various angular impact parameters Ψ_0, the gain A as a function of the angle χ_0 by which the object is displaced along its trajectory from the point of closest approach to the deflector. These curves pertain to the case where $\Psi_0 > \varphi_O$.

III. APPLICATIONS

A. Stars of the Milky Way

A typical randomly selected pair of observable deflector and object stars within our galaxy might be located at distances $l_D \approx 0.5\times10^4$ l.y. and $l_O \approx 1.0\times10^4$ l.y., respectively, from us. In order to illustrate the orders of magnitude of the various parameters involved, let us assume that such a pair is constituted of stars similar to the sun. Thus, the angular radius of the object star will be $\varphi_O \approx 7.36\times10^{-12}$ rad, and the half-apex angle of the cone of inversion for the system will be $\theta_0 \approx 7.90\times10^{-9}$ rad. If the relative proper, or transverse, motion of these stars were to cause them to appear to pass by one another with an angular impact parameter less than θ_0, then a relatively significant time symmetric pulse of light intensity should ensue. If the stars were to move on a course carrying them through perfect alignment, then the intensity of the image of the object star would, in accordance with (19), vary inversely as the angular separation α_0, for $\varphi_O \lesssim \alpha_0 \lesssim \theta_0$. And as the disk of the object star passed behind the center of the deflector, the amplification of the image intensity would round off, passing through a peak value of $\sim$2150 for $\alpha_0=0$. There thus exists the potentiality for a vivid demonstration of the gravitational deflection of light.

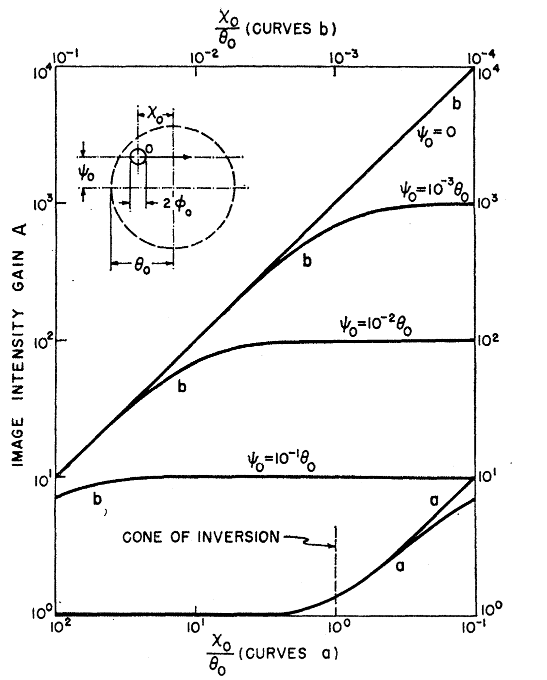

FIG. 3. The intensity gain A as a function of the angle χ_0 by which the object is displaced along its trajectory from the point of closest approach to the deflector. The various curves are associated with different angular impact parameters $\Psi_0 \theta_0$ represents the half-apex angle of the cone of inversion.

We wish now to obtain an order of magnitude estimate of the frequency of occurrence of conceivably detectable pulses of light intensity arising from chance transitory alignments among the observable stars of our galaxy. For this purpose we shall utilize a simplified galactic model which will be developed with reference to Figs. 4(a) and (b).

Obscuration by dust clouds renders optically inaccessible to us the vast majority of the estimated $\sim10^{11}$ stars in our galaxy, most of which reside in the nucleus.

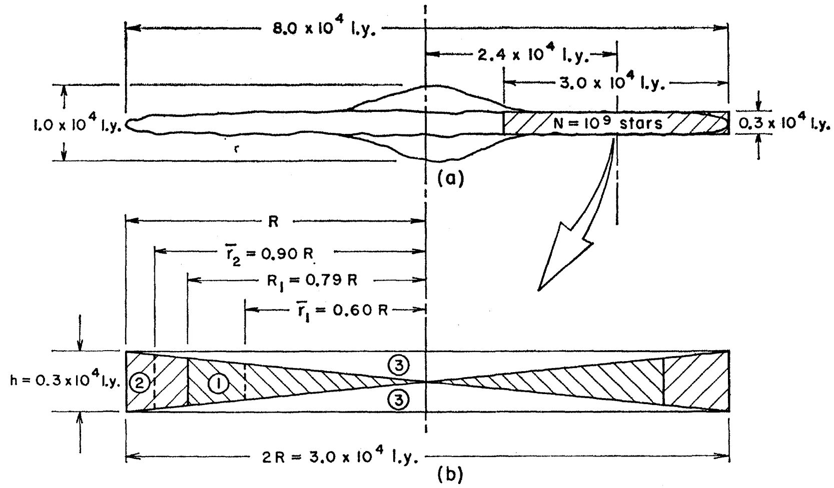

FIG. 4. (a) Model of our galaxy. (b) Model of the visible portion of our galaxy.

We shall initially assume that the $N\approx 10^9$ observable stars within the galaxy are distributed approximately uniformly throughout the volume of a pancake, having thickness $h\approx 0.3\times 10^4$ l.y. and radius $R\approx 1.5\times 10^4$ l.y., within which we are centrally located. Region 3, in Fig. 4(b), consists of the pair of cones having bases coincident with the faces of the pancake and apexes at the center of the pancake. This region contains one-third of the volume V of the entire pancake and therefore, also, one-third of the stars. The relatively large solid angle over which the stars of region 3 are distributed renders alignments among these stars relatively unlikely; we shall, therefore, neglect events of this type.

In a more precise treatment than the present one an attempt would be made to account for the observed nonuniformities in the visible stellar distribution, such as result from the spiral structure, obscuration, attenuation, and the unobservability of the more distant of the fainter stars. We shall undertake several simplifying operations which are considered consistent with the level of the approximations of the present treatment. First, having excluded region 3, we divide the remaining volume, by the insertion of a cylindrical surface of radius $R_1\approx 0.79R\approx 1.19\times 10^4$ l.y., into two regions, 1 and 2, each of equal volume $\frac{1}{3}V$. We now limit our attention to the class of interactions involving alignments among deflector stars in region 1 and object stars in region 2. In view of the fact that the actual distribution of observed stars is characterized by a nonuniformity not here accounted for, we shall take the liberty of redistributing, for convenience, the stars of our model. We shall assume that all of the stars within region 1 are reassembled at the mean radius $\bar{r}_1\approx 0.60R\approx 0.90\times 10^4$ l.y. of this volume, and that all of the stars within region 2 are reassembled at the mean radius $\bar{r}_2\approx 0.90R\approx 1.35\times 10^4$ l.y. of this volume. We now inquire into the frequency of coincidences to be expected among these redistributed stars as a consequence of their relative proper motions. Contributions to the observed relative motions of the stars come from the random stellar velocities $\sim$30 km/sec, the orbital velocity of the earth about the sun $\sim$30 km/sec, the random component of the sun's velocity, and shear associated with the $1/r$-dependent bulk velocity distribution in the galactic plane. The rotational velocity of the galaxy at the sun's distance from the center is $\sim$300 km/sec. Our simplified method of accounting for the relative motions of the stars will be to assume that the stars of region 1 all rotate about the axis of the pancake with a speed $v\approx 30$ km/sec relative to the field of stars in region 2. The mean free angle through which a star in region 1 must move across the field of stars in region 2, between alignments characterized by angular impact parameters less than Ψ_0, is $\Theta\approx\Omega/2N_2\Psi_0$, where $\Omega\approx 2\pi h/R$ is the solid angle subtended by the "Milky Way" regions 1 and 2 as viewed from the earth, and $N_2=\frac{1}{3}N$ is the number of stars in region 2. The mean time between such coincidences among all stars in regions 1 and 2 is $\tau_I\approx\Theta\bar{r}_1/N_1v$, where $N_1=\frac{1}{3}N$ is the number of stars in region 1. We shall now make the assumption that each of the stars in our model has the same mass as the sun; in point of fact, stars are rarely observed to have masses more than a factor of 10 larger or smaller than that of the sun. The half-apex angle of the cone of inversion associated with each deflector in region 1 paired with an object in region 2 is, then, from (7), $\theta_0\approx 4.8\times 10^{-9}$ rad. If we now confine our attention to that class of coincidences which have angular impact parameters Ψ_0 in the range $\varphi_O\lesssim\Psi_0\lesssim\theta_0$, we can obtain a simple relationship between τ_I and a quantity A_M that denotes the maximum gain, that is, the gain at closest approach, exhibited by an object star passing with apparent angular impact parameter Ψ_0 behind a deflector star. For, from (19) we have $A_M\approx\theta_0/\Psi_0$. We thus obtain the result

$$\frac{\tau_I}{A_M}\approx\frac{\pi h\bar{r}_1}{N_1N_2R\theta_0v},\tag{22a}$$

wherein τ_I represents the mean interval between pulses exhibiting intensity gains in excess of A_M. The temporal width, measured between points at half height, of the pulse characterized by A_M is $\tau_W \approx 2\sqrt{3}\Psi_0\bar{r}_1/v$; consequently, we have

$$\tau_W A_M \approx 2\sqrt{3}\bar{r}_1\theta_0/v. \tag{22b}$$

Substituting the appropriate numbers into the above expressions, we obtain the following order of magnitude results:

$$\tau_I/A_M \approx 3\times10^6 \text{ sec}, \tag{23a}$$

$$\tau_W A_M \approx 5\times10^7 \text{ sec}, \tag{23b}$$

$$\tau_I\tau_W \approx 2\times10^{14} \text{ sec}^2, \tag{23c}$$

$$n_1 \approx \tau_W/\tau_I \approx 10/A_M^2, \tag{23d}$$

$$n_2 \approx n_1 A_M \approx 10/A_M, \tag{23e}$$

where n_1 represents the mean number of events for which the intensity gain instantaneously exceeds $A_M/2$, and n_2 denotes the mean number of events for which the intensity gain instantaneously exceeds ~ 2. Specific values for the various quantities appearing in (23) are presented in Table I. There has been included in this table the intensity gain A_{CM} which represents the ratio of the combined object and deflector intensities to the combined intensities that would exist in the absence of lens action; we have assumed that the two stars have equal luminosities.

As a rough check on the results of the present section, let us note that with each of the $N\approx10^9$ observable stars there are associated roughly $N\approx10^9$ radial spikes of concentrated light intensity which result from the gravitational condensing action of each star upon the light radiated by other stars. If a plane were passed through the earth in a direction orthogonal to the galactic disk, we might expect $n\approx N^2/10$ spikes to penetrate the central area $a\approx10^8$ l.y.^{2} of this plane. Thus, the mean distance between points where spikes intersect the plane would be $d\approx(a/n)^{1/2}\approx10^{13}$ cm, which is of the order of the diameter of the earth's orbit. These points of intersection might be expected to move with speeds typically of the order of $v\approx30$ km/sec, about the same as the speed of the earth in its orbit about the sun. The core of maximum intensity in a typical spike will have a radius comparable to that of the sun. Thus, the mean distance that the earth must move between successive cores is very roughly $\lambda\approx a/2R_\odot n\approx10^{16}$ cm. The mean time between such events, corresponding to the observation of a pulse of maximum intensity, would thus be $\tau\approx\lambda/v\approx100$ years. This figure is consistent with the result that may be obtained from (23a) for the case of a pulse of peak intensification.

It should be noted that the $N\approx10^9$ stars employed in the discussion of this section are those which are currently photographically accessible. These stars vastly exceed in number those among which the search for novae is presently conducted. It should furthermore be realized that the identification of a gravitational lens event must be made from a background consisting, in part, of a great many variable stras and eclipsing binaries. We must therefore conclude, on the basis of the model employed, that there is little likelihood of identifing any particular one of the typical lens events involving interactions among the random stars of the Milky Way.

TABLE I. Image intensification events among the random stars of the Milky Way. Derived from expressions (23a, b).

A_M	A_{CM}	τ_I	τ_W
1000	300	100 yr	14 h
100	30	10 yr	6 days
10	4	1 yr	2 months
2	1.3	3 months	1 yr

The frequency of occurrence of detectable events within a given element of solid angle is proportional to the number of object stars contained, to the number of deflector stars contained, and to the mean transverse speeds of the stars involved. There will, therefore, be regions of the Milky Way which lend themselves very much more favorably than others to the detection of image intensification events. It is also of interest to inquire into the existence of particularly useful subgroupings of stars. We are led, thereby, to a consideration of the utility of globular clusters.

B. Globular Clusters

Somewhat more than 100 globular clusters, believed to consist of from $\sim10^4$ to $\sim10^7$ stars, have been detected within our galaxy. Among the characteristics of a typical globular cluster are its approximation to spherical symmetry and the high number density of stars contained near its core; roughly half of the stars in a cluster may be within ~10 l.y. of its center. Typical internal stellar velocities are ~30 km/sec. Indications are that the globular clusters are approximately spherically distributed about the nucleus of the galaxy, and have average velocities of ~300 km/sec. Three possibilities present themselves as regards the observation of gravitational lens events involving globular clusters: (1) a field of nearby deflector stars residing before a globular cluster of object stars, (2) a globular cluster of deflector stars moving before the relatively dense distribution of object stars in the vicinity of the galactic center, and (3) events involving only stars within a single globular cluster.

For case (1), let us consider a globular cluster that contains $N_C\approx3\times10^5$ object stars, and is located at a distance $l_O\approx2.0\times10^4$ l.y. Let there be $dN_D/d\Omega\approx1\times10^8$ deflector stars per steradian located at a characteristic distance of $l_D\approx0.7\times10^4$ l.y. If we assign to each deflector the mass of the sun, then the associated cone of inversion will have a half-apex angle $\theta_0\approx7.6\times10^{-9}$ rad. Let us assume that an appropriate mean relative transverse velocity is $v\approx200$ km/sec. Rough considerations, analo-

TABLE II. Case (1): Image intensification events involving a globular cluster of object stars behind nearer deflection stars. Derived from expressions (24a, b).

A_M	A_{CM}	τ_I	τ_W
1000	110	10^5 yr	8 h
100	12	10^4 yr	3 days
10	2	10^3 yr	1 month
2	1.1	10^2 yr	6 months

gous to those of the preceding section, then yield

$$\frac{\tau_I}{A_M} \approx \left(\frac{dN_D}{d\Omega}\right)^{-1} \frac{l_O}{2N_C\theta_0 v} \approx 2\times10^9 \text{ sec}, \tag{24a}$$

and

$$\tau_W A_M \approx 2\sqrt{3}\, l_O\theta_0/v \approx 3\times10^7 \text{ sec}. \tag{24b}$$

Values for the quantities A_M, τ_I, and τ_W are presented in Table II; the combined object-deflector gain A_{CM}, which is also tabulated, is evaluated under the assumption that the object and deflector stars exhibit identical luminosities.

For case (2), we consider a globular cluster of $N_C \approx 3\times10^5$ deflector stars located at a distance $l_D \approx 1.8\times10^4$ l.y., and a background field of $dN_O/d\Omega \approx 2\times10^9$ object stars per steradian located at a characteristic distance $l_O \approx 2.2\times10^4$ l.y. We assume that $v \approx 200$ km/sec represents a reasonable relative transverse stellar velocity. It follows that $\theta_0 \approx 2.6\times10^{-9}$ rad. For this case, we find that

$$\frac{\tau_I}{A_M} \approx \left(\frac{dN_O}{d\Omega}\right)^{-1} \frac{l_D}{2N_C\theta_0 v} \approx 3\times10^8 \text{ sec}, \tag{25a}$$

and

$$\tau_W A_M \approx 2\sqrt{3}\, l_D\theta_0/v \approx 7\times10^6 \text{ sec}. \tag{25b}$$

There is considerable flexibility in the choice of the parameters employed in both cases (1) and (2). It is primarily a result of the choices made for the stellar number densities outside the globular cluster that the frequency of events in case (2) exceeds by an order of magnitude that found in case (1). Selected values for A_M, A_{CM}, τ_I, and τ_W for case (2), as summarized in (25a, b), are presented in Table III; A_{CM} is again computed under the assumption of equal luminosities.

For the intraglobular case (3), we assume that there are $N_C \approx 1\times10^6$ stars within a globular cluster that has characteristic radius $R_C \approx 10$ l.y. and is located at distance $l_C \approx 2.0\times10^4$ l.y. We take $v \approx 30$ km/sec for the mean relative transverse stellar velocity. Because of the small relative separation between the object and deflector stars, the half-apex angle of the cone of inversion is only $\theta_0 \approx 1.2\times10^{-10}$ rad. We find, in this case, that

$$\frac{\tau_I}{A_M} \approx \frac{2\pi R_C^2}{N_C^2\theta_0 l_C v} \approx 8\times10^7 \text{ sec}, \tag{26a}$$

and

$$\tau_W A_M \approx 2\sqrt{3}\, l_C\theta_0/v \approx 3\times10^6 \text{ sec}, \tag{26b}$$

where (26a) is particularly sensitive to both the size and the population of the cluster. Selected values of A_M, A_{CM}, τ_I, and τ_W for case (3) are presented in Table IV. In the present model A_M is limited to be less than ~ 60. The combined gain A_{CM} has again been evaluated under the assumption that both the object and deflector stars exhibit equal luminosities.

The results of this section indicate that events involving selected single globular clusters can occur as frequently as several times per decade. The fact that the nuclei of the most populous clusters have not yet been completely resolved constitutes a serious practical problem. It seems likely that adequate resolution might only be attained by observations made from above the earth's atmosphere.

TABLE III. Case (2): Image intensification events involving a globular cluster of deflector stars residing before a background field of object stars. Derived from expressions (25a, b).

A_M	A_{CM}	τ_I	τ_W
1000	400	10^4 yr	2 h
100	40	10^3 yr	1 day
10	5	10^2 yr	1 week
2	1.4	20 yr	1 month

TABLE IV. Case (3): Image intensification events involving stars entirely within a single globular cluster. Derived from expressions (26a, b).

A_M	A_{CM}	τ_I	τ_W
50	25	130 yr	1 day
15	8	40 yr	2 days
5	3	13 yr	7 days
2	1.5	5 yr	17 days

C. Unobservable Stars as Deflectors

In arriving at the preceding estimates for the frequencies of gravitational lens events, we have assumed that the visible stars were the only bodies acting as deflectors. If a significant fraction of the mass of the galaxy were, in fact, in the form of unobservable bodies capable of functioning as deflectors, then our previous frequency estimates would be too low. An estimate of the mass density within the galactic plane in the vicinity of the sun has been obtained by examining the distribution of stellar velocity components in the direction normal to the plane.[16] Indications are that the mass contributed by neutron and faint dwarf stars cannot exceed by more than a factor of ~ 2 that due to visible stars and known interstellar matter. The factor of ~ 2 would also seem to be a reasonable upper limit for the number of unobserved deflectors within the globular clusters.

[16] M. Schmidt, Bull. Astron. Inst. Neth. **13**, 15 (1956).

D. Nonstellar Deflectors Within the Galaxy

Let us next examine the lens action of the asteroids, planets, and other similar aggregations of matter that are floating about the galaxy. In order that an opaque deflector have the capacity to produce stellar image intensification, it is necessary that the deflector not obscure the cone of inversion for the system; this condition may be expressed in the form

$$\theta_0/\varphi_D=(\epsilon_D/\mu\varphi_D)^{1/2}>1\,, \tag{27}$$

where ϵ_D denotes the angle through which a light ray is deviated upon its glancing passage by the limb of the deflector.

The large relative transverse velocities of the asteroids and planets of the solar system would appear to qualify them as particularly attractive candidates for deflectors. However, of these bodies, Pluto comes closest to satisfying condition (27); and for this planet we find that, under even the most favorable orbital conditions, $\theta_0/\varphi_D=1/9$.

We next consider the lens action of the planets that orbit stars other than the sun. These planets might be expected to be an order of magnitude more numerous than the stars. If significant stellar image intensification is to occur, the condition

$$\theta_0/\varphi_O\gg 1 \tag{28}$$

must necessarily be satisfied, in addition to (27). For the case of a deflector planet similar to the earth and an object star similar to the sun, located, respectively, at the distances $\bar{r}_1\approx 0.90\times 10^4$ l.y. and $\bar{r}_2\approx 1.35\times 10^4$ l.y. employed in Sec. III A, we find that $\theta_0\approx 0.8\times 10^{-11}$ rad and $\varphi_O\approx 0.5\times 10^{-11}$ rad. Thus, the interaction angular cross section is small, as is also the maximum available intensification. It seems, therefore, that the primary effect of planetary deflectors bound to stars other than the sun will be to slightly perturb the lens action of these stars.

There appears little likelihood that unbound planet-sized bodies floating about the galaxy would contribute significantly to the frequency of detectable events. For, the associated pulses would be so weak and infrequent and of such fleeting duration—perhaps a few hours—as to defy detection.

E. Object Stars in the Andromeda Galaxy

At a distance of $\sim 1.5\times 10^6$ l.y., the stars of the great spiral galaxy M 31 (NGC 224) in Andromeda constitute a dense and populous, if distant, collection of objects upon which the stellar gravitational lenses of our galaxy can operate. Typical minimal intensifications would certainly go unnoticed. Even maximal events corresponding to intensity gains of $\sim 10^5$ for the image of the object are not likely to be recorded. For, not only would such events occur centuries apart, but also the combined object-deflector image would exhibit an intensity gain of only ~ 10 for the case that both stars have equal luminosity. Even if the deflector consisted of an unobservably faint dwarf or neutron star, some random object star in M 31 flaring up from perhaps 30th to 18th apparent magnitude for a few minutes, and being photographically accessible for but a few hours once every few centuries, is hardly likely to be noticed.

F. Gravitational Waves

The rare event characterized by a pair of stars briefly aligned well enough to produce an optimal intensification has been shown to have associated with it an extraordinarily high degree of angular resolution, of the order $\sim 10^{-11}$ rad.

Gross and Wheeler have investigated the effect which gravitational waves might have upon galactic and stellar images and have concluded that such effects would be unobservable.[17] One might raise the question again in the present context. For waves of sufficiently high density and short enough wavelength would lead to a fluctuation of the amplified stellar image intensity. It is found, however, that the frequencies and energy densities required to produce 10^{-11} rad deflection over a path length of 10^4 l.y., in the short time available, exceed by many orders of magnitude upper limits which can be independently established.[18]

G. Detection of Luminous Spikes

We have noted that every star has associated with it radial spikes of concentrated light intensity, each spike representing the gravitational lens action of the star upon the light emanating from some other star. It is conceivable that under suitable circumstances such spikes might manifest themselves by scattering off cosmic dust or gas clouds, in a manner analogous to that whereby a search light beam becomes visible in the atmosphere. Radial comet-shaped luminosities have been photographed[19] in the planetary nebula NGC 7293. Conditions in such a nebula might possibly be suitable for the exhibition of gravitational lens spikes. The small streaks of light which Gaposhkin was so startled to observe[20] in the globular cluster ω Centauri might also indicate the existence of suitable conditions. It is anticipated that lens-induced luminosities will be very faint, in which event they might most readily be detected when viewed in a direction nearly parallel to the axis of the luminous beam.

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[17] F. Gross and J. A. Wheeler (private communication).

[18] J. A. Wheeler, Onzième Conseil de l'Institut International de Physique Solvay, *La Structure et l'évolution de l'universe* (Editions Stoops, Brussels, 1958), p. 112.

[19] Mt. Wilson and Palomar Observatories.

[20] S. Gaposhkin, *Vistas in Astronomy* (Pergamon Press, New York, 1960), Vol. 3, p. 294.

H. Lens Effects Among Galaxies

As previously remarked, Zwicky has considered gravitational lens effects as they apply to galaxies.[10–12] We shall therefore comment only briefly on this topic.

In the case of galactic-galactic events, one would expect to find, as a consequence of the lens action of a deflector galaxy, either distorted or image intensified object galaxies. Let us consider the case that the deflector consists of the nucleus of a galaxy similar to our own. If the distortion of the object galaxy is to be readily observed, it is necessary that the cone of inversion for the system not be obscured by the deflector nucleus. If the mass of the deflector is $M_D=3\times10^{10}\ M_\odot$ and its radius is $R_D=0.5\times10^4$ l.y., it follows that for $\mu=2$, the deflector must be at a distance of at least $l_D\approx2.7\times10^9$ l.y. For a deflector located at this minimum acceptable distance, the half-apex angle θ_0 of the cone of inversion and the angle $\varphi_D=R_D/l_D$ subtended by the radius of the deflector are identically equal to 0.4″. As atmospheric fluctuations normally limit photographic resolution to $\sim$1″, image distortion would, in the present case, not be readily detected. At such great distances the red shift is large enough that one might hope to distinguish between the unresolved deflector and the intensified object images on the basis of the relative extent of their red shift. A statistical study of a large number of image intensification candidates might be expected to reveal anomalous spectral distributions for a significant number of the composite images.

In order that an optically opaque deflecting galactic nucleus be capable of producing significant image intensification, it is necessary that condition (27) be satisfied. For the critical case, then, that the cone of inversion just clears the limb of the deflector, we have

$$\theta_0=\varphi_D=\epsilon_D/\mu. \tag{29}$$

Zwicky has presented a table[21] of critical values of φ_D and $l_D=R_D/\varphi_D$ for various values of M_D and R_D, valid for the special case that μ, in (29), is equal to unity. Such a table is of particular interest in that no assumptions or approximations are involved in its preparation. Unfortunately, Zwicky's table appears to be slightly in error; the critical angles are too large and the critical distances too small, each by a factor of $\sim$3. A corrected version is presented here in Table V. For those cases where μ exceeds unity, the values of φ_D in Table V must be multiplied by a factor $1/\mu$ and the values of l_D increased in proportion to μ.

TABLE V. Galaxies as gravitational lenses. In order for significant lens action to occur, the deflector galaxy must be at a distance in excess of l_D, at which distance the half-angle subtended by the deflector is φ_D.

M_D \ R_D	10^3 l.y.	10^4 l.y.
$10^{10}\ M_\odot$	$\varphi_D=1.3''$. $l_D=1.6\times10^8$ l.y.	$\varphi_D=0.13''$. $l_D=1.6\times10^{10}$ l.y.
$10^{11}\ M_\odot$	$\varphi_D=13''$. $l_D=1.6\times10^7$ l.y.	$\varphi_D=1.3''$. $l_D=1.6\times10^9$ l.y.

[21] See Ref. 12, p. 216, Table XLVII.

IV. CONCLUDING REMARKS

We have seen that the gravitational lens action of a star condenses the light, passing by it from other stars, into semi-infinitely extensive spikes of concentrated light intensity. Thus, each of the $\sim10^9$ visible stars in our galaxy effectively emanates $\sim10^9$ such spikes. The intensity varies approximately inversely as the distance from the axis of a spike, over a large portion of any transverse cross section. As the earth passes through a spike, the intensity of the composite image of the pair of stars involved will vary in a characteristic time symmetric fashion.

The detection of intensifications associated with random pairings of stars in the Milky Way is rendered difficult by the faintness of the majority of images, and by the difficulty of determining toward which of the tremendous number of images our attention should be directed. The situation is further complicated by the great number of stellar images which exhibit intensity variations for unrelated reasons.

High-resolution examination of the stars within selected globular clusters may constitute a possible means of observing intensification events. Conceivably detectable events involving a single globular cluster may occur as frequently as several times a decade. These events will, however, generally be of rather short duration—usually less than one month.

There may exist galactic gas and dust concentrations of such a nature that a penetrating spike will manifest itself as a luminescence.

The detection of lens effects among the galaxies seems a possibility. But it is judged that such effects will only evidence themselves in response to a deliberate search of a kind which has not yet been performed.

The verification of a stellar gravitational lens image intensification event would, at the very least, constitute a dramatic qualitative confirmation of the deflection prediction of general relativity. A carefully monitored event would also enable a quantitative check of the form of the radial dependence of the deflection. Information relating to the radial dependence out to distances of tens of thousands of light years might result from the detection of events involving galactic deflectors. Events among galaxies would also provide mass information relating to the deflector, and enhanced spectral information relating to the intensified distant object.

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