Neutrino Absorption in Dense Matter*

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Neutrino absorption in matter up to densities of 10^{15} g/cm^3 is discussed. The number density of neutrons, protons, and electrons in the degenerate matter is found. The effect of the Pauli exclusion principle on the absorption is then analyzed under the idealization of zero-temperature material composed of noninteracting nucleons. For example, the threshold neutrino energy, below which the degenerate matter is completely transparent to neutrinos, is found. The various results are presented in terms of tables, graphs, and simple equations.

INTRODUCTION

NEUTRINOS interact so weakly with other matter that it is customarily assumed that no absorption of neutrinos need ever be considered in physical problems. Once they are created near the center of a star or in a nuclear pile by one of the nuclear reactions $n\rightarrow p+e+\bar{\nu}$, $p\rightarrow n+e^{+}+\nu$, or $p+e\rightarrow n+\nu$, they travel almost unimpeded through any amount of matter.

This lack of interaction of neutrinos with matter has very important consequences for astrophysics. For example, neutrinos efficiently transmit energy away from the center of stars. This function of neutrinos was first utilized by Gamow in his famous Urca process.[1] Gamow explains the very rapid collapse of stars, leading perhaps to novae or supernovae, by means of a "neutrino pump." The reactions $n_{\text{hot}}\rightarrow p_{\text{cool}}+e_{\text{cool}}+\bar{\nu}$ and $p_{\text{hot}}+e_{\text{hot}}\rightarrow n_{\text{cool}}+\nu$ form a cycle whereby hot neutrons decay into cool protons and electrons, with the neutrino carrying off the energy difference. The hot protons also combine with hot electrons to decay into cool neutrons, with the neutrino again carrying off the excess energy. Gamow estimates that whole stars can collapse within half an hour by means of this cycle.

The general subject of neutrino equilibria in degenerate matter thus has some interest for astrophysics.[2] Obviously, for efficient transfer of energy, a relatively long mean free path for neutrinos is required. Therefore, a study of the absorption of neutrinos in matter of varying density will constitute the remainder of this paper.

For usual densities, neutrino absorption is very small. For example, it will be found that the mean free path of 1-MeV neutrinos in the sun (density about 1.4 g/cm^3) is of the order of 10^{19} cm, as compared to 7×10^{10} cm, the radius of the sun. What about neutrino absorption at higher densities? Observed densities of white dwarf stars go up to 10^8 g/cm^3, with a corresponding radius of 10^8 cm. Densities up to nuclear densities, around 10^{14} g/cm^3, have been studied theoretically by several independent groups.[3–5]

The primary mechanisms for neutrino absorption are the reactions

$$\bar{\nu}+p\rightarrow n+e^{+},$$

$$\nu+n\rightarrow p+e,$$

and at very high densities (greater than 10^{13} g/cm^3), the reaction

$$\bar{\nu}+e\rightarrow \mu+\bar{\nu}.$$

(Other reactions such as $\nu+p\rightarrow\nu+p$ may also be important in absorbing neutrinos. This paper treats only the old, well-established reactions and should be considered to give the contribution to the absorption coefficient of these reactions alone.) The simplifying assumption will be made that the neutrino interacts with free nucleons. Actually, of course, the nucleons are either in more complex nuclei at low densities, or form degenerate nuclear matter at high densities. This independent particle approximation neglects the mutual interaction between nucleons. For example, in beta decay, the energy of the emitted neutrino is not the same as in free-nucleon decay, but is that energy characteristic of the nucleus as a whole. The first-order correction to the independent particle model would be the consideration of correlations between two nucleons. This correction will be very briefly discussed later.

The low-energy absorption coefficient is easy to compute. It is altered at high densities due to the degenerate nature of the matter. The exclusion principle prevents final-state momenta below that of the Fermi sea for neutrons, protons, and electrons. This effect tends to make the matter more transparent than would at first be expected. The primary contribution of this paper will be the detailed calculation of the exclusion principle inhibiting factor from the kinematics of the reactions. The resulting mean free paths for neutrinos can then be compared to the equilibrium radius of a corresponding star to obtain an estimate of the relative importance of neutrino absorption.

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[2] J. A. Wheeler, *Geometrodynamics* (Academic Press Inc., New York, 1962), pp. 1–6.
[3] S. Chandrasekhar, *An Introduction to the Study of Stellar Structure* (Dover Publications, Inc., New York, 1939), Chap. 11.
[4] J. Oppenheimer and G. Volkoff, Phys. Rev. **55**, 347 (1939).
[5] B. Harrison, M. Wakano, and J. Wheeler, Onzième Conseil de l'Institute International de Physique Solvay, *La Structure et l'evolution de l'universe* (Edition Stoops, Brussels, 1958).

NEUTRINO ABSORPTION AT LOW DENSITIES

There will be two absorption coefficients, one for neutrinos (which can only interact with neutrons), and one for antineutrinos (which can only interact with protons and electrons).

Let ρ be the density of the material which is composed of a fraction f_n of neutrons, and a fraction f_p of protons ($f_p+f_n=1$). The probability that a neutrino is absorbed in one centimeter of material is

$$\kappa_\nu = \text{effective area for absorption per scatterer} \times \frac{\text{number of scatterers}}{\text{cm}^3}$$

$$= \sigma(\nu,n:p,e) f_n \rho / m_n$$

$$\kappa_{\bar{\nu}} = \sigma(\bar{\nu},p:n,e^+) f_p \rho / m_p .$$

The cross section for the reaction $\bar{\nu}+n \rightarrow p+e$ in lowest order perturbation theory for the center-of-mass system is

$$\sigma = 4\pi^{-1}(G/\hbar c)^2 p^0 e e^0/(n^0+\nu^0),$$

where

$$(G/\hbar c) = 4.5\times 10^{-33}\ \text{cm}^2 .$$

Similarly, the center-of-mass cross section for the reaction $\nu+p \rightarrow n+e^+$ is

$$\sigma = 4\pi^{-1}(G/\hbar c)^2 e(n^0\nu^0 p^0 e^0 + \tfrac{1}{3} n\nu p e)(\nu^0+p^0)^{-1} \times (p^0\nu^0 - \mathbf{p}\cdot\boldsymbol{\nu})^{-1}.$$

The result for low energies is then

$$(\kappa_\nu,\kappa_{\bar{\nu}}) = 3.9\times 10^{-20} e e^0 (f_n,f_p)\rho\ \text{cm}^{-1},$$

where ρ is in g/cm^3 and e and e^0 are in MeV/c and MeV, respectively. For the free neutron-neutrino reaction, $e^0=\nu^0+1.3$ MeV, whereas for the free proton-antineutrino reaction, $e^0=\nu^0-1.3$ MeV.

Here, and throughout the paper, e, p, and n stand for the magnitude of the appropriate momentum, while e^0, p^0, and n^0 stand for the corresponding total relativistic energy. The velocity of light is taken equal to one throughout.

The mean free path for neutrino absorption is thus

$$\lambda = \kappa^{-1} = 2.56\times 10^{19}(e e^0 f \rho)^{-1}\ \text{cm}.$$

For the sun, the density is approximately 1.4 g/cm^3. Hence for 10-MeV neutrinos, the mean free path is about 10^{17} cm. No appreciable absorption takes place in such stars.

CONSTITUTION OF DEGENERATE MATTER AS A FUNCTION OF DENSITY

While densities in visible white dwarf stars go up to 10^8 g/cm^3, even higher densities are possible. The radius-density relationship for stellar equilibrium has been worked out through densities of the order of 10^{16} g/cm^3.[3–5] What are the relative ratios of protons, neutrons, and electrons in such degenerate matter?

In the following, n_f, p_f, and e_f will stand for the Fermi momentum of the neutron, proton, and electron. The Fermi momentum at absolute zero is that momentum above which no particle states are filled, and below which all particle states are filled. The Fermi momentum can be computed from the number density of particles.

$$N = \int_0^{p_f} 2\times 4\pi p^2 dp/h^3 = (8/3)\pi p_f^3/h^3 ,$$

$$p_f(\text{MeV}/c) = 5.15\times 10^{-3}(\rho \text{ in g/cm}^3)^{1/3}.$$

Even at densities of 10^{14} g/cm^3, the Fermi momentum is only 200 MeV/c for protons and neutrons. Consequently, nonrelativistic mechanics will be used for the nucleons, while relativistic mechanics must be used for the electrons.

Two conditions uniquely determine the equilibrium constitution of a degenerate star. First, the star must be electrically neutral. Consequently $N_e=N_p$. Second, in order for the two reactions $n\rightarrow p+e+\bar{\nu}$ and $p+e\rightarrow n+\nu$ to be in equilibrium, the total neutron Fermi energy must equal the sum of the Fermi energies of the proton and electron. In symbols, $n_f^0=p_f^0+e_f^0$, or neglecting the difference between the rest mass of the neutron and proton, $n_f^2/2m=p_f^2/2m+e_f^0$. Now let f represent the ratio of the proton density ρ_p to the total density ρ: $\rho_p=f\rho$, $\rho_n=(1-f)\rho$. The two requirements lead to the following expression for the density as a function of f for $f\ll 1$:

$$\rho^{1/3} = 3.62\times 10^5 f^{1/3}/[(1-f)^{2/3}-f^{2/3}].$$

Figure 1, which shows the density as a function of the fractional number of protons, was obtained from this equation.

ABSORPTION DUE TO THE REACTION $\nu+n\rightarrow p+e$

The reaction $\nu+n\rightarrow p+e$ will take place as long as both the resulting proton and electron have a momentum greater than that of their surrounding Fermi sea. Because the neutron Fermi momentum is so much greater than the proton Fermi momentum, almost all created protons will be permitted by the exclusion principle. However, low-energy neutrinos will create low-energy electrons which might very well be below the level of the electron Fermi momentum. Consequently, in the following analysis, the inhibition of the reaction due to the protons will be neglected, while the inhibition due to the electrons will be taken into account. The momenta and energies in the following discussion will be normalized to unit neutron and proton mass, while the electron will be assumed to have zero rest mass (i.e., to be in the highly relativistic region) unless the reverse is stated.

In a typical encounter, a neutrino of some fixed energy will encounter a neutron with a momentum n less than the Fermi neutron momentum n_f. Looking

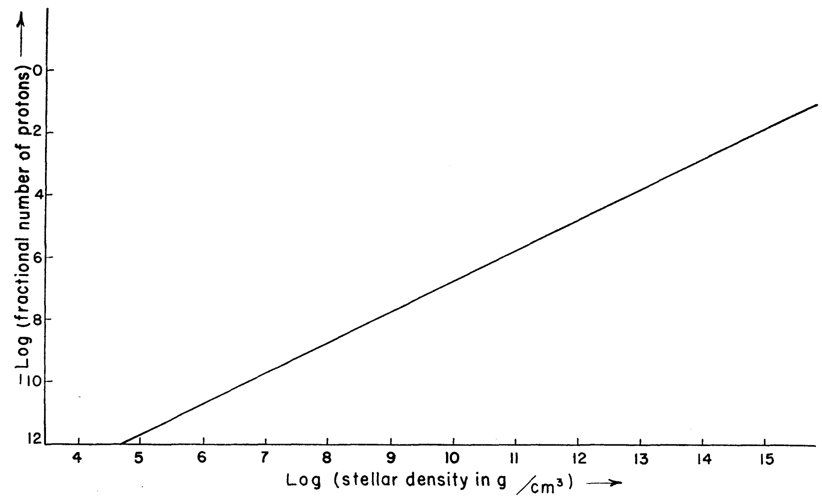

FIG. 1. The ratio of the equilibrium proton density to the total density of degenerate stellar matter is shown as a function of the total density. The demands for equilibrium are that the matter be electrically neutral (equal number densities of protons and electrons) and that the star be in equilibrium under the beta-decay reactions $n \rightarrow p+e+\bar{\nu}$, and $p+e \rightarrow n+\nu$. Consequently, the sum of the Fermi energies of the proton sea and the electron sea must equal the Fermi energy of the neutron sea.

at the encounter in the center-of-momentum system, we see a particle of zero rest mass and a particle with unit rest mass collide with equal and opposite momenta, and fly apart again with new names—now a proton and electron—but with the same energies. All directions are equally possible from dynamical considerations in the center-of-mass system. The simplifying assumptions are made that all directions are also equally possible in the laboratory system and that the cross section is essentially constant for all neutrons in the Fermi sea interacting with a given incident neutrino. The deviation resulting from either of these assumptions is proportional to v/c for the incident neutrons. However, the energy of the emerging electron now depends on the direction of the electron relative to that of the neutron and neutrino. This energy is highest in the direction of the total incident momentum and decreases as the angle with this direction increases. There are three cases. All emerging electrons have less energy than the Fermi energy, in which case the reaction cannot occur; all emerging electrons have a greater energy than the Fermi energy, in which case no correction has to be made; or there is some angle θ beyond which all electrons have less energy than that of the Fermi sea. In the last case, the reaction probability is reduced by the solid angle factor $\frac{1}{2}(1-\cos\theta)$. The probability that the neutrino will be absorbed by any neutron in the sea is then a simple average of the

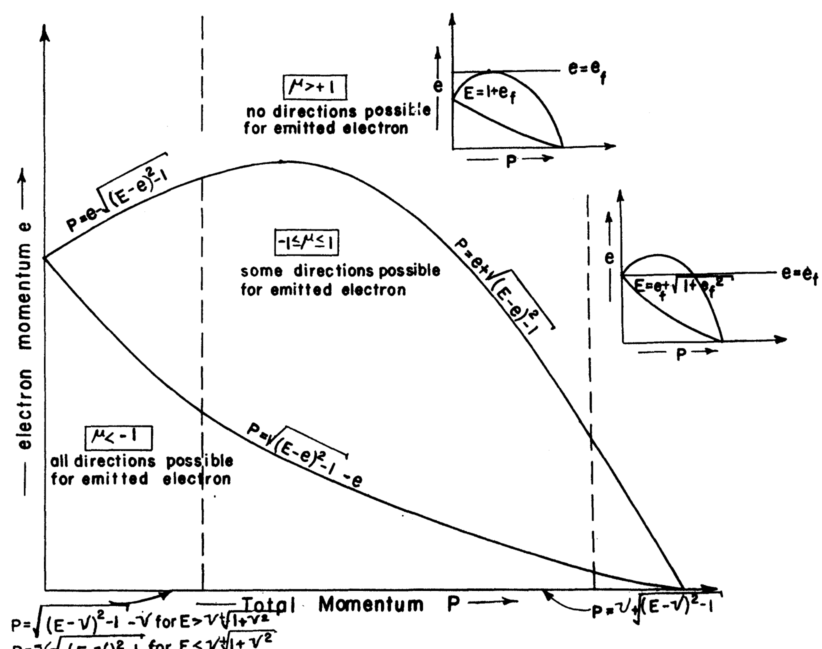

FIG. 2. The effects of the Pauli exclusion principle on the reaction $\nu+n \rightarrow p+e$ are indicated for fixed total energy $E(=n^0+\nu^0=p^0+e^0)$. For a specified total momentum $\mathbf{P}(=\mathbf{n}+\mathbf{v}=\mathbf{p}+\mathbf{e})$, the momentum of the electron in the direction of $\mathbf{P}$, $[\mu=\cos(\mathbf{P},\mathbf{e})=1]$ and in the opposite direction $(\mu=-1)$ are indicated. Drawing a horizontal line at the electron Fermi momentum then indicates the limits on P for fixed E for which all electrons are above the electron Fermi sea $(\mu<-1)$, electrons in some but not all directions are above the sea $(-1\leq\mu\leq1)$, and the region in which no electrons have sufficient energy $(\mu>1)$. The vertical dashed lines bracket the permissible total momentum. The two small inserts show the appearance of the large diagram for two different values of the total energy E.

probability for the individual neutrons. The cross section will then be reduced by this inhibiting factor which will be called j in the following discussion.

It is convenient to use as variables of integration over the neutron sea, not n and μ [$=\cos(\mathbf{n},\boldsymbol{\nu})$], but the total energy $E=n^0+\nu^0=p^0+e^0$ and the magnitude of the total momentum $\mathbf{P}=\mathbf{n}+\boldsymbol{\nu}=\mathbf{p}+\mathbf{e}$. Computing the Jacobian of the transformation shows that the neutron density, which, for a Fermi distribution, is proportional to $n^2dnd\mu$ for neutron energies less than the Fermi energy, is also proportional to $(E-\nu)dEdp^2$ for $E<n_f^0+\nu^0$. Eliminating the proton variables from the energy momentum equations gives the relationship

$$(p^0)^2=(E-e^0)^2=1+(P-e)^2=1+P^2-2Pe\mu_{eP}+e^2.$$

The graph of this relationship for fixed E is shown in Fig. 2. The various limits on E are also shown in Fig. 2. For $E<1+e_f$, no electrons have enough energy, while for $E=e_f+(1+e_f^2)^{1/2}$, for the first time it is possible that all electrons have enough energy near $P=0$. However, these diagrams only deal with the final state of the reaction. The initial state also puts restrictions on E and P. The total energy E must range between $1+\nu$ and $n^0+\nu$, while for E fixed, P has the limits $|\nu-\{(E-\nu)^2-1\}^{1/2}|$ and $\nu+\{(E-\nu)^2-1\}^{1/2}$. The appropriate limits on the averaging integrals thus depend upon the relative positions of the various limits at the $e=e_f$ line. The resulting integrations are straightforward, but complicated to set up. The inhibiting factor j for a density of 10^{15} g/cm^3 is plotted in Fig. 3. The inhibiting factor at any density can be roughly deduced from Fig. 4. The limits beyond which all emerging electrons are above the Fermi sea ($j=1$) is given by the formula

$$\nu=e_f\frac{2(n_f^0-e_f^0)(n_f^0+n_f)-1}{1-4e_f(n_f^0-e_f)}.$$

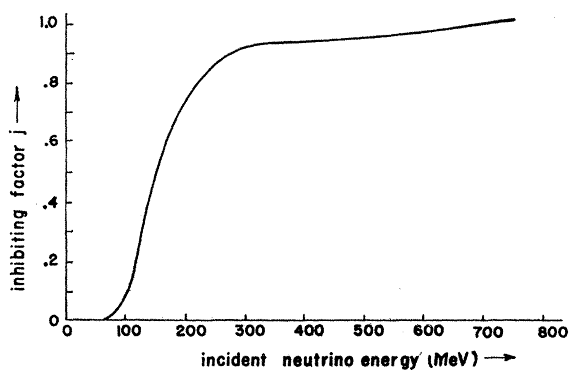

FIG. 3. The effect of the Pauli exclusion principle on the reaction $\nu+n\rightarrow p+e$ is indicated for a stellar density of 10^{15} g/cm^3. The mean free path for neutrinos is obtained by dividing the uninhibited mean free path obtained from Table I by the value of j corresponding to that neutrino energy. For example, for 100-MeV neutrinos, Table I indicates an uninhibited mean free path of $2.6\times10^4/10^4=2.6$ cm. The actual mean free path is then roughly $2.6/0.09=30$ cm.

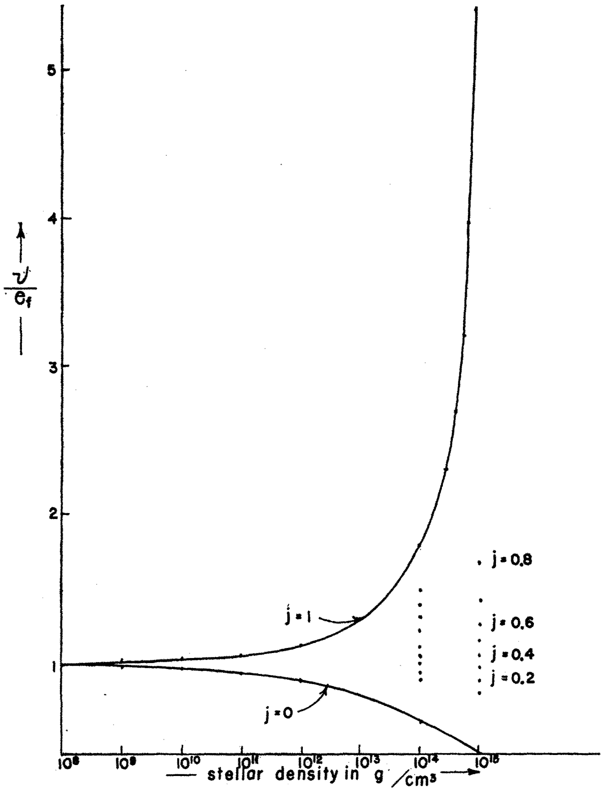

FIG. 4. The value of the inhibiting factor j for various stellar densities is shown as a function of the neutrino momentum normalized by the electron Fermi momentum corresponding to that density for the reaction $\nu+n\rightarrow p+e$. The lines $j=1$ and $j=0$ are drawn in, while the points $j=0.1$, 0.2, etc. have been computed for stellar densities of 10^{14} and 10^{15} g/cm^3.

Below the energy

$$\nu=e_f\frac{2(n_f^0-e_f)(n_f^0-n_f)-1}{1-4e_f(n_f^0-e_f)}$$

no electrons have enough energy to rise above the Fermi sea ($j=0$). These results are all shown in Fig. 4. Intermediate values of the inhibiting factor j for any density can be obtained from this figure in combination with Table I. The mean free path can then be obtained by dividing the uninhibited mean free path by the corresponding value of j. Results for two densities are shown in Fig. 5.

ABSORPTION DUE TO THE REACTION $\nu+p\rightarrow n+e^+$

Because the proton Fermi momentum is far below the neutron Fermi momentum, the exclusion principle plays an important part for the reaction $\bar{\nu}+p\rightarrow n+e^+$. Concentrating on the final state, introducing the total energy E and the total momentum $\mathbf{P}$ and eliminating the positron variables gives the equation

$$e^2=(E-n^0)^2=(\mathbf{P}-\mathbf{n})^2=P^2-2Pn\mu_{Pn}+n^2.$$

TABLE I. Relevant information for various degenerate stars is tabulated. The source of these numbers is discussed in the text. The mean free path of a neutrino can be found by dividing the mean free path obtained by neglecting the effects of the exclusion principle $\lambda_{\text{uninhibited}}$ by the value of the inhibiting factor j. The uninhibited mean free path is thus a lower limit on the mean free path of a neutrino.

Density (g/cm^3)	Stellar radius (cm)	Proton density / Total density	Neutron Fermi momentum (MeV/c)	Proton (and electron) Fermi momentum (MeV/c)	$ee^0\,\lambda_{\text{uninhibited}}(\nu,n:p,e)$ (cm MeV^2/c)	$ee^0\,\lambda_{\text{uninhibited}}(\bar{\nu},p:n,e^+)$ (cm MeV^2/c)
10^5	3×10^9	2.1×10^{-12}	0.24	3.1×10^{-5}	2.6×10^{14}	1.2×10^{26}
10^6	2×10^9	2.1×10^{-11}	0.52	1.4×10^{-5}	2.6×10^{13}	1.2×10^{24}
10^7	1×10^9	2.1×10^{-10}	1.1	6.6×10^{-4}	2.6×10^{12}	1.2×10^{22}
10^8	7×10^8	2.1×10^{-9}	2.4	3.1×10^{-3}	2.6×10^{11}	1.2×10^{20}
10^9	4×10^8	2.1×10^{-8}	5.2	1.4×10^{-2}	2.6×10^{10}	1.2×10^{18}
10^{10}	1×10^8	2.1×10^{-7}	11	6.6×10^{-2}	2.6×10^9	1.2×10^{16}
10^{11}	4×10^7	2.1×10^{-6}	24	3.1×10^{-1}	2.6×10^8	1.2×10^{14}
10^{12}	1×10^7	2.1×10^{-5}	52	1.4	2.6×10^7	1.2×10^{12}
10^{13}	4×10^6	2.2×10^{-4}	110	6.7	2.6×10^6	1.2×10^{10}
10^{14}	2×10^6	2.0×10^{-3}	240	30	2.6×10^5	1.3×10^8
10^{15}	1×10^6	1.7×10^{-2}	520	130	2.6×10^4	1.5×10^6

Fixing the total energy E gives the characteristic diagram of Fig. 6. The initial state then adds limits of its own. The total energy E is restricted to be between $1+\nu$ and $p^0+\nu$, while the total momentum P must be between $\nu-\{(E-\nu)^2-1\}^{1/2}$ and $\nu+\{(E-\nu)^2-1\}^{1/2}$. (See Fig. 6.) The same approximations that were used in the last section can be made, and the inhibiting factor j can be calculated in the same way.

The threshold neutrino energy (the highest energy for which $j=0$) corresponds to the situation where the neutrino interacts with a proton moving in the same direction as the neutrino at the Fermi surface. The resulting neutron escapes in the same direction with the positron going in the opposite direction. The threshold energy is found to be

$$\nu_{\text{threshold}}=(n_f^0+n_f-p_f^0-p_f+e^0-e)/2\,,$$

which simplifies to

$$\nu_{\text{threshold}}=(n_f^0+n_f-p_f^0-p_f)/2$$

for highly relativistic electrons. For high densities (greater than 10^8 g/cm^3), the expression simplifies to $\nu_{\text{threshold}}=n_f/2$. The inhibiting factor j is zero by definition at the threshold neutrino energy and approaches a value less than 1 as ν^0 approaches infinite

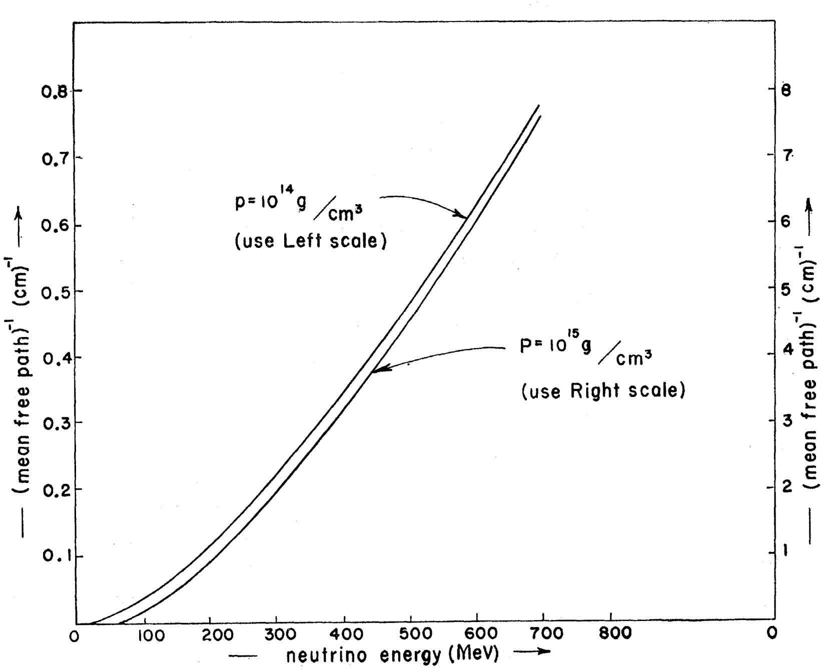

FIG. 5. Absorption coefficient of neutrinos (cm^{-1}) due to the reaction $\nu+n\rightarrow p+e$ as a function of neutrino energy for two values of stellar density. These results are based upon the independent particle picture and ignore nuclear correlations.

energy. Namely, for $\nu > (n_f{}^0 + n_f - p_f{}^0 + p_f)/2$,

$$\kappa = a - b/\nu ,$$

$$a = 3(16n_f)^{-1}\{p_f{}^0(2+p_f{}^{-2}) - p_f{}^{-3}\ln(p_f{}^0+p_f)\} - \tfrac{1}{2}(n_f{}^0 n_f{}^{-1} - 1) ,$$

$$b = n_f{}^{-1}\{\tfrac{1}{2} + p_f{}^2/10 - (3n_f{}^0/16)[p_f{}^0(2+p_f{}^{-2}) - p_f{}^{-3}\ln(p_f{}^0+p_f)]\} .$$

The inhibiting factor j for a density of 10^{15} g/cm^3 is shown in Fig. 7.

ABSORPTION DUE TO THE REACTION $\bar{\nu}+e \rightarrow \mu+\bar{\nu}$

The reaction whereby an electron from the Fermi sea and an antineutrino interact to give a μ meson and another antineutrino proves to be the dominant mechanism for antineutrino absorption at very high stellar densities (around 10^{15} g/cm^3). The reaction $\nu+e \rightarrow \mu+\nu$ seems to be ruled out by the recent discovery of the two independent types of neutrinos.[6]

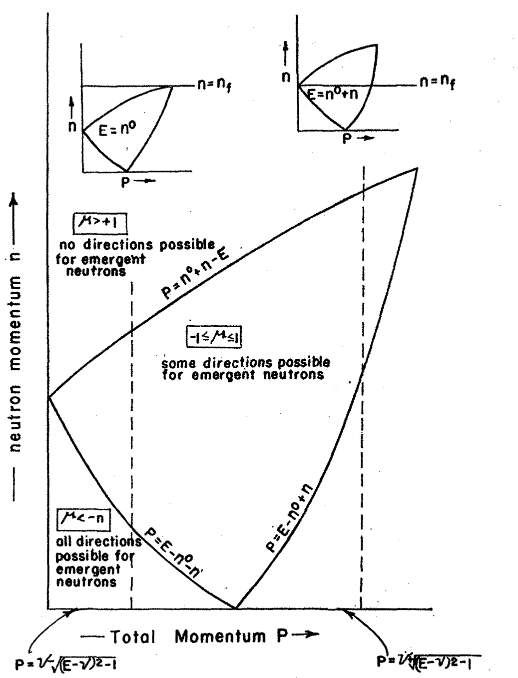

FIG. 6. The kinematics of the reaction $\bar{\nu}+p \rightarrow n+e$ are displayed for fixed total energy $E(=\nu^0+p^0=n^0+e^0)$. The emerging neutron momentum n in the direction of the total momentum $\mathbf{P}(=\bar{\nu}+\mathbf{p}+\mathbf{n}+\mathbf{e})$ is shown as a function of $\mathbf{P}$ [here $\mu=\cos(\mathbf{P},\mathbf{n})=1$] as is the neutron momentum in the direction opposite to $\mathbf{P}$ (here $\mu=-1$). By drawing a horizontal line at the neutron Fermi momentum, various limits on P can be seen for fixed E. For the region where $\mu<-1$, all emitted neutrons are above the level of the Fermi sea; for $-1 \leq \mu \leq 1$, only some directions for the emitted neutrons are possible; while for $\mu>1$, no neutrons have sufficient energy to rise above the sea. The vertical dashed lines show the limits on the possible total momentum values due to the initial momentum of the antineutrino and proton.

[6] G. Danby, J.-M. Gaillard, K. Goulianos, L. M. Lederman, N. Mistry, M. Schwartz, and J. Steinberger, Phys. Rev. Letters **9**, 36 (1962).

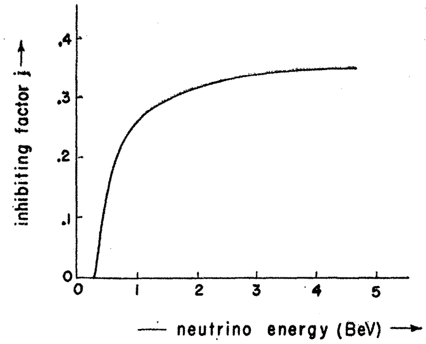

FIG. 7. The inhibiting factor j indicates the effects of the exclusion principle on the reaction $\bar{\nu}+p \rightarrow n+e^+$ at a stellar density of 10^{15} g/cm^3. The mean free path for an antineutrino in a star of this density is obtained by dividing the uninhibited mean free path obtained from Table I by j. For example, for 500-MeV neutrinos, $\lambda_{\text{uninhibited}} = 1.5\times10^6/2.5\times10^5 = 6$ cm. Dividing by $j=0.125$ gives a mean free path of 48 cm.

The asymptote (the dashed line) being less than 1 reflects the fact that the proton Fermi energy is much less than the neutron Fermi energy. Even for very high incident neutrino energies, some emerging neutrons have energy less than that of the neutron Fermi sea. For example, a possible final scattering state has the emerging electron going in the same direction as the incident neutrino with the emerging neutron having the same direction as the incident proton and hence less energy than the exclusion principle requires for the reaction to take place.

Because the threshold neutrino energy for this reaction is inversely proportional to the electron Fermi momentum,

$$\nu_{\text{threshold}} = m_\mu{}^2/4e_f = 2800 \text{ MeV}/e_f \quad (e_f \text{ in MeV}/c) ,$$

this reaction is relatively unimportant at lower densities for reasonable neutrino energies. However, at a stellar density of 10^{15} g/cm^3, the neutrino threshold for this reaction is 21 MeV, while the antineutrino-proton threshold is at 250 MeV due to the exclusion principle. (These thresholds are obtained by neglecting the nucleon-nucleon interaction and the finite temperature, both of which tend to keep the mean free path finite, but large.) Hence, the neutrino-electron process is dominant here.

The analysis of the mean free path is rather simple for this reaction. Writing the conservation laws for energy and momentum in their four-dimensional form as $\nu_1+e=\mu+\nu_2$, and taking the invariant scalar product gives the relation

$$2\nu_1 e[1-\cos(\mathbf{v}_1,\mathbf{e})] = m_\mu{}^2 + 2m_\mu\nu_2 ,$$

where the electron mass is neglected when compared to the muon mass, and where the emerging neutrino energy is also considered small when compared to the muon mass. Putting the secondary neutrino energy equal to zero, and $\cos(\mathbf{v}_1,\mathbf{e})=1$ gives the above result for threshold.

For a fixed energy of the incoming neutrino, all electrons with energies from the threshold electron energy to the electron Fermi energy can interact with conservation of energy and momentum. For a fixed

TABLE II. The mean free path for the reaction $\bar{\nu}+e\rightarrow\mu+\bar{\nu}$ is shown as a function of antineutrino energy for stellar matter at about ten times nuclear density (10^{15} g/cm^3). The electrons form a degenerate Fermi sea with a Fermi momentum of 132 MeV/c. For neutrino energies below 21 MeV, no absorption can take place.

Neutrino energy (MeV)	Mean free path (cm)
21	∞
23	7.3×10^6
32	6.8×10^4
53	5.3×10^2
85	2.1×10
110	5.5
130	1.9
210	0.11

electron energy in this range, all electron angles (taken with respect to the direction of the incoming neutrino) between the limits $\mu=-1$ and $\mu=1-m_\mu^2/2\nu_1 e$ are allowed.

The cross section for the reaction near threshold is $\sigma=4\pi^{-1}(G/hc)^2\nu_2^2$. ν_2 is related to e and μ by

$$\nu_2=m_\mu(\tau/4)(1-2/\tau-\mu),$$

where τ is the ratio of the electron energy to the threshold electron energy for the incoming neutrino energy ν_1. The threshold electron energy is the lowest energy an electron can have and still react with the neutrino. $e_{\text{threshold}}^2=m_\mu^2/4\nu_1$. Averaging the cross section over all of the electrons in the Fermi sea then gives

$$\sigma_{\text{av}}=6.8\times10^{-41}\text{ cm}^2[\tfrac{1}{5}(\tau_t^5-1)-\tfrac{3}{4}(\tau_t^4-1)+(\tau_t^3-1)-\tfrac{1}{2}(\tau_t^2-1)],$$

$$\tau_t=e_f/e_{\text{threshold}}=\nu_1/\nu_{1,\text{ threshold}}.$$

The mean free path for antineutrinos is then found to be

$$\lambda^{-1}=\sigma_{\text{av}}f_p\rho/m_p,$$

ρ being the stellar density and f_p being the fractional number of protons. Some results are tabulated in Tables II, III, and IV.

ABSORPTION DUE TO THE REACTION $\bar{\nu}+p+e\rightarrow n$

It might be thought that the reaction whereby an antineutrino combines with a proton and electron in the Fermi sea to produce a neutron would also be possible. However, it is seen that this reaction is impossible for densities of interest here because of the low position of the Fermi level for protons and electrons with respect to the level of the neutron Fermi sea.

Conservation of energy and momentum results in the equations $n^0=p^0+e^0+\nu^0$, $\mathbf{n}=\mathbf{p}+\mathbf{e}+\boldsymbol{\nu}$. From the momentum equation, one obtains the inequality

$$n<p+e+\nu<\nu+2p_f,\quad\text{or}\quad \nu>n-2p_f.$$

From the energy equation, $\nu=n^0-p^0-e^0<n^0-m_n$. Combining these inequalities gives the inequality $n^0-m_n>n-2p_f$, which can be put in the form

$$n_f^0+n_f<n^0+n<m^2(m-2p_f)^{-1}\approx m+2p_f,$$

which never holds for densities of interest. Therefore, because of the relative sizes of the Fermi seas, the reaction cannot occur for any incoming neutrino energy.

BRIEF DISCUSSION OF THE RESULTS

The results presented are based upon some very restrictive assumptions. Completely noninteracting particles were assumed to fill all states up to the Fermi level, with no states filled beyond the Fermi level. A finite temperature will modify the results by producing a Fermi tail of energetic particles and some holes below the Fermi level. Also, the interaction between nucleons will affect the results by changing the momentum distribution of the nucleons. The high-energy tail on the momentum distribution has been studied by examining such processes as deuteron pickup, meson capture, and the high-energy proton-carbon scattering. Brueckner *et al.*[7] discuss the available experimental evidence and compare the results with some theoretical studies. They come to the conclusion that the momentum distribution can be roughly represented by a Gaussian

$$|\varphi(p)|^2=\alpha^{-3}\pi^{-3/2}\exp(-p^2/\alpha^2),$$

where α corresponds to a mean kinetic energy of 19.3 MeV ($\alpha=160$ MeV/c) for the carbon nucleus. However, the variation of this momentum distribution for different densities of both protons and neutrons is not at

TABLE III. The mean free path due to the reaction $\bar{\nu}+e\rightarrow\mu+\bar{\nu}$ is shown for two neutrino energies, 1.1 times the threshold neutrino energy for the reaction, and 5 times the threshold for the reaction for various stellar densities.

Density (g/cm^3)	Electron Fermi momentum (MeV/c)	Threshold energy (MeV)	Mean free path (cm) for $\nu=1.1\ \nu_{\text{threshold}}$	Mean free path (cm) for $\nu=5\ \nu_{\text{threshold}}$
10^{11}	3.1×10^{-1}	0.91×10^4	5.8×10^{14}	4.3×10^8
10^{12}	1.4	2.0×10^3	5.8×10^{12}	4.3×10^6
10^{13}	6.7	420	5.5×10^{10}	4.1×10^4
10^{14}	30	93	6.1×10^8	4.5×10^2
10^{15}	130	21	7.3×10^6	5.4

[7] K. Brueckner, R. Eden, and N. Francis, Phys. Rev. 98, 1445 (1955).

TABLE IV. The effective cross section for the reaction $\bar{\nu}+e\rightarrow\mu+\bar{\nu}$, defined as the cross section averaged over all the electrons in the Fermi sea that are capable of reacting, is shown as a function of the ratio of the incoming neutrino energy to the threshold neutrino energy.

Neutrino energy / Threshold energy	Effective cross section in cm²
1	0
1.1	1.4×10^{-44}
1.5	1.5×10^{-42}
2	4.4×10^{-41}
3	7.8×10^{-40}
5	1.8×10^{-38}
7	1.3×10^{-37}
10	0.92×10^{-36}

all clear. It is enough to point out here that both of these effects will keep the mean free path finite even where the above simplified analysis implied that it was infinite. The analysis can, however, be considered to give a lower limit to the mean free path.

From these results, we see that neutrino absorption does become very important for high-energy neutrinos in extremely dense matter. Mean free paths of the order of 30 cm for 100-MeV neutrinos for stellar densities of 10^{15} g/cm³ are to be compared to the characteristic stellar diameter of 10^6 cm. For lower densities, however, or for lower energy neutrinos, it is seen that neutrinos have a remarkably great penetrating ability. Consequently, neutrinos carry energy away from the star with great efficiency.

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Semiphenomenological Solutions of Pion-Nucleon Partial-Wave Dispersion Relations*

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A variational method of solving π–N partial-wave dispersion relations is developed. Analytic trial functions are used, and their parameters are varied to obtain a unitary solution. A good solution for the $(\frac{3}{2},\frac{3}{2})$ resonance is obtained by using the Layson function. The shape of the resonance, as well as its position, is obtained. The only problem about the solution is the validity of the short-range interaction term which is used. Crossing of the real part of the amplitude verifies that it is accurate, but it is not obvious why the variation method should give such a good result. The explanation appears to be that in many cases the low-energy behavior of a partial wave is dominated by the long-range interactions, and a comparatively simple analytic function will give a good solution. An application of the variational method to confirm an earlier analysis of s-wave π-N scattering is also given.

I. INTRODUCTION

WE present a new method for solving partial-wave dispersion relations by using a variational technique. The N/D method[1] which is normally used suffers from certain disadvantages in practice. One disadvantage is that if we have a rough idea of the solution we cannot readily incorporate this information in the N/D method so as to make it easier to get an exact solution. Another disadvantage arises from the fact that the short-range part of the interaction between any pair of elementary particles is unknown (we refer to ranges less than 0.2 10^{-13} cm), and no obvious method exists by which it can be discovered. The ambiguity which this introduces in the N/D method cannot be conveniently handled in practice. Alternatively, we might try to make up for the ignorance about the short-range part of the interaction by imposing high-energy boundary conditions on the partial-wave solutions; for this purpose, some physically reasonable conjectures can be made. However, this approach is again not easy to incorporate in the N/D method. The variational method which is developed here can, to some extent, avoid these various difficulties.

In the variational method, we start with a unitary trial solution $F_p(s)$ which is in practice an algebraic function of s which contains some arbitrary parameters. This trial function is used to approximate the physical integral in the dispersion relation, and thereby give an approximation $\mathrm{Re}F'(s)$ to the real part of the amplitude for physical values of s. Next, the parameters are varied to make $F'(s)$ approach as close to unitarity as possible.

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[1] G. F. Chew and S. Mandelstam, Phys. Rev. 119, 467 (1960).