PHYSICAL REVIEW VOLUME 133, NUMBER 4B 24 FEBRUARY 1964

Modified Khuri Series and its Convergence for a Single Yukawa Potential*

AKBAR AHMADZADEH
Lawrence Radiation Laboratory, University of California, Berkeley, California
(Received 23 September 1963)

The Khuri series for the partial-wave amplitude has been modified in such a way as to explicitly single out the Born term. In deriving this modified series it is shown that one needs weaker asymptotic conditions on the partial-wave amplitude than those used by Khuri. The convergence of these series has been investigated for the case of a single Yukawa potential. It is found that the modified series converges considerably faster than the Khuri series.

I. INTRODUCTION

It has been shown by Regge[1] that for a superposition of Yukawa potentials of the form

$$V(r)=\int_{m_1}^{\infty}\sigma(\mu)(e^{-\mu r}/r)d\mu \tag{1}$$

the partial-wave amplitude is meromorphic in the right-half λ plane and has the asymptotic form

$$A(\lambda,s)\sim C(s)e^{-\lambda\xi_1}/\sqrt{\lambda},\quad \text{Re}\lambda>0,\quad |\lambda|\to\infty, \tag{2}$$

where $\lambda=l+\frac{1}{2}$, s is the energy, and

$$\xi_1=\cosh^{-1}(1+m_1^2/2s),$$

m_1 being the lower limit of the integral in Eq. (1). From this, using the Sommerfeld-Watson transformation, Regge obtained, for the scattering amplitude, the representation

$$f(s,z)=-i\int_{-i\infty}^{i\infty}\lambda d\lambda P_{\lambda-\frac{1}{2}}(-z)\frac{A(\lambda,s)}{\cos\pi\lambda}+\pi\sum_{n=1}^{N}\frac{2\beta_n(s)\lambda_n P_{\lambda_n-\frac{1}{2}}(-z)}{\cos\pi\lambda_n}, \tag{3}$$

where β_n are the residues of the poles $A(\lambda,s)$ at $\lambda=\lambda_n=\alpha_n+\frac{1}{2}$. Using the above results and considering the subclass of potentials (1) for which the amplitude is also meromorphic in the left-half λ plane, with the additional assumption

$$A(\lambda,s)\sim C'(s)e^{-\lambda\xi_1}/\sqrt{\lambda},\quad \text{Re}\lambda<0,\quad |\lambda|\to\infty, \tag{4}$$

Khuri[2] has found the following expansion for the partial-wave amplitude:

$$A(l,s)=\sum_{\text{all poles}}\beta_n(s)\frac{e^{-(l-\alpha_n)\xi_1}}{l-\alpha_n},\quad \text{with } l \text{ an integer.} \tag{5}$$

In the remainder of this section we examine certain aspects of Eq. (5). In Sec. II we modify this formula, starting from weaker asymptotic conditions on $A(\lambda,s)$. Finally, in Sec. III, we examine the rate of convergence of Eq. (5) as well as the modified expansion of Sec. II for the case of a single Yukawa potential.

Now for the sake of simplicity let us consider Eq. (5) for a single Yukawa potential,

$$V(r)=-g(e^{-m_1 r}/r). \tag{6}$$

The ideas can be easily generalized if the potential is of the form of Eq. (1) and behaves as $1/r$ near the origin. It is well known[3] that as $s\to\infty$, $\alpha_n\to -n$, n being a positive integer, which is the asymptotic solution of the Coulomb potential case. Furthermore, as $s\to\infty$ one needs to consider only the behavior of the potential near the origin, so that the values of the residues as well as the poles for potential (6) approach that of the Coulomb case for sufficiently large s. The residues of the partial-wave amplitude in the case of the Coulomb potential $V=-g/r$ are[4]

$$\beta_n(s)=\frac{1}{2i\sqrt{s}}\frac{(-1)^n/(n-1)!}{\Gamma(-n+1+ig/\sqrt{s})}\underset{s\to\infty}{\longrightarrow}\frac{g}{2s}, \tag{7}$$

so that asymptotically the residues for potential (6) are

$$\beta_n(s)\underset{s\to\infty}{\longrightarrow} g/2s. \tag{8}$$

By assuming that for $s\to\infty$ the series (5) reduces to the Born term

$$A(l,s)=\frac{g}{2s}Q_l\left(1+\frac{m_1^2}{2s}\right), \tag{9}$$

Khuri[2] was able to find the correct asymptotic behavior of the residues [Eq. (8)]. Thus the series (5) does indeed converge to the Born term at high energies. For practical purposes, however, the series (5) is not suitable at high energies because in that case it reduces to

$$A(l,s)\underset{s\to\infty}{\longrightarrow}\frac{g}{2s}\sum_{n=1}^{\infty}\frac{\exp[-(l+n)m_1/\sqrt{s}]}{l+n}, \tag{10}$$

and the convergence is very slow. One might argue that, in contrast to the high-energy behavior (8), at intermediate energies the residues $\beta_n(s)$ may decrease for

* Work done under the auspices of the U. S. Atomic Energy Commission.

[1] A. Bottino, A. M. Longoni, and T. Regge, Nuovo Cimento **23**, 954 (1962).

[2] N. N. Khuri, Phys. Rev. **130**, 429 (1963).

[3] See, for example, S. Mandelstam, Ann. Phys. (N. Y.) **19**, 254 (1962).

[4] V. Singh, Phys. Rev. **127**, 632 (1962).

poles further to the left in the λ plane, improving the rate of convergence. Our numerical solution of the residues, utilized in testing the convergence of the series, shows that this is not the case, and at any given energy the different residues are generally of the same order of magnitude. Aside from this difficulty, it seems plausible that, for large $|\lambda|$, $A(\lambda,s)$ should approach the Born approximation, which for negative λ will be dominated by the largest masses in the exponential. So, for a superposition of Yukawa potentials the asymptotic condition (4), which emphasizes the longest rather than the shortest range component, seems to be too strong an assumption. These arguments suggest the need for a modification of the Khuri series in such a way as to single out the Born term and also de-emphasize the contribution of the pole terms further out in the left-half λ plane. In the next section we give a derivation of such a modified formula.

II. MODIFICATION OF THE KHURI SERIES

In this section it is assumed that the reader is familiar with Khuri's paper. For the sake of simplicity we consider a potential of the form

$$V(r)=-\sum_{i=1}^{k}\frac{g_i e^{-m_i r}}{r},\quad \text{for}\quad m_{i+1}>m_i. \tag{11}$$

Instead of assumptions (2) and (4), we make the weaker assumption that

$$A(\lambda,s)-\sum_{i=1}^{k}\frac{g_i}{2s}Q_{\lambda-\frac{1}{2}}\left(1+\frac{m_i^2}{2s}\right)\sim C(s)e^{-\lambda\xi}/\sqrt{\lambda},\quad |\lambda|\to\infty, \tag{12}$$

where $\xi=\cosh^{-1}(1+m^2/2s)$, m being arbitrary for the moment. Starting with Eq. (3) and adding and subtracting the Born term, we obtain

$$f(s,z)=\frac{-i}{2s}\int_{-i\infty}^{i\infty}\lambda d\lambda\frac{P_{\lambda-\frac{1}{2}}(-z)}{\cos\pi\lambda}\sum_{i=1}^{k}g_iQ_{\lambda-\frac{1}{2}}\left(1+\frac{m_i^2}{2s}\right)$$
$$-i\int_{-i\infty}^{i\infty}\lambda d\lambda\frac{P_{\lambda-\frac{1}{2}}(-z)}{\cos\pi\lambda}\left[A(\lambda,s)-\sum_{i=1}^{k}\frac{g_i}{2s}Q_{\lambda-\frac{1}{2}}\left(1+\frac{m_i^2}{2s}\right)\right]$$
$$+\pi\sum_{n=1}^{N}\frac{2\beta_n(s)P_{\lambda_n-\frac{1}{2}}(-z)\lambda_n}{\cos\pi\lambda_n}. \tag{13}$$

For the first integral in Eq. (13) we close the contour to the right and immediately obtain

$$f_1(s,z)=\sum_{i=1}^{k}\sum_{l=0}^{\infty}(2l+1)P_l(z)\frac{g_i}{2s}Q_l\left(1+\frac{m_i^2}{2s}\right)=\sum_{i=1}^{k}\frac{g_i}{m_i^2-t}, \tag{14}$$

where $t=-2s(1-z)$. For the remaining part of Eq. (13),

$$f_2(s,z)=-i\int_{-i\infty}^{i\infty}\lambda d\lambda\frac{P_{\lambda-\frac{1}{2}}(-z)}{\cos\pi\lambda}\times\left[A(\lambda,s)-\sum_{i=1}^{k}\frac{g_i}{2s}Q_{\lambda-\frac{1}{2}}\left(1+\frac{m_i^2}{2s}\right)\right]+\pi\sum_{n=1}^{N}\frac{2\beta_n\lambda_nP_{\lambda_n-\frac{1}{2}}(-z)}{\cos\pi\lambda_n}, \tag{15}$$

we follow the same procedure as Khuri. Equation (15) can be written as

$$f_2(s,z)=\frac{-i}{\pi(2)^{3/2}}\times\int_{-i\infty}^{i\infty}d\lambda\left[A(\lambda,s)-\sum_{i=1}^{k}\frac{g_i}{2s}Q_{\lambda-\frac{1}{2}}\left(1+\frac{m_i^2}{2s}\right)\right]\times\int_{\xi}^{\infty}\frac{e^{\lambda x}\sinh x dx}{(\cosh x-z)^{3/2}}-\frac{i}{\pi(2)^{3/2}}$$
$$\times\int_{-i\infty}^{i\infty}d\lambda\left[A(\lambda,s)-\sum_{i=1}^{k}\frac{g_i}{2s}Q_{\lambda-\frac{1}{2}}\left(1+\frac{m_i^2}{2s}\right)\right]\times\int_{-\infty}^{\xi}\frac{e^{\lambda x}\sinh x dx}{(\cosh x-z)^{3/2}}+\pi\sum_{n=1}^{N}\frac{2\beta_n\lambda_nP_{\lambda_n-\frac{1}{2}}(-z)}{\cos\pi\lambda_n}. \tag{16}$$

For the first term on the right-hand side, we close the contour to the left in the λ plane, and in addition to the pole terms of $A(\lambda,s)$ we pick up the poles of the Q functions. For the second term we close the contour to the right in the λ plane, and we pick up only the poles of $A(\lambda,s)$. The result is

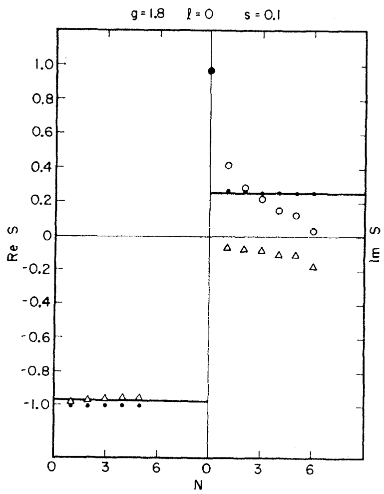
$$f_2(s,z)=\frac{1}{\sqrt{2}}\sum_{\text{left poles}}\beta_n\int_{\xi}^{\infty}\frac{e^{\lambda_n x}\sinh x dx}{(\cosh x-z)^{3/2}}-\frac{1}{\sqrt{2}}\sum_{i=1}^{k}\frac{g_i}{2s}\sum_{n=1}^{\infty}P_{n-1}\left(1+\frac{m_i^2}{2s}\right)\int_{\xi}^{\infty}\frac{e^{-(n-\frac{1}{2})x}\sinh x dx}{(\cosh x-z)^{3/2}}$$
$$-\frac{1}{\sqrt{2}}\sum_{\text{right poles}}\beta_n\int_{-\infty}^{\xi}\frac{e^{\lambda_n x}\sinh x dx}{(\cosh x-z)^{3/2}}+\pi\sum_{n=1}^{N}2\beta_n\lambda_n\frac{P_{\lambda_n-\frac{1}{2}}(-z)}{\cos\pi\lambda_n}. \tag{17}$$

The partial-wave amplitude is given by

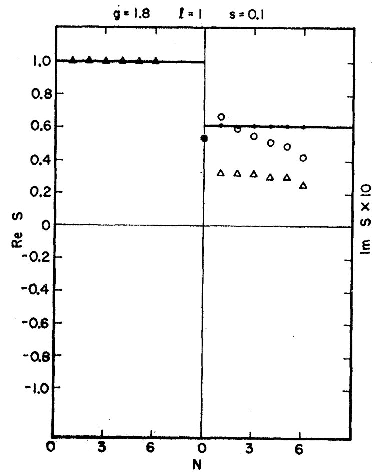
$$A(l,s)=\frac{1}{2}\int_{-1}^{1}f(s,z)P_l(z)dz,\quad \text{with } l \text{ an integer}, \tag{18}$$

where $f(s,z)=f_1(s,z)+f_2(s,z)$. From (18), (17), and (14) we obtain

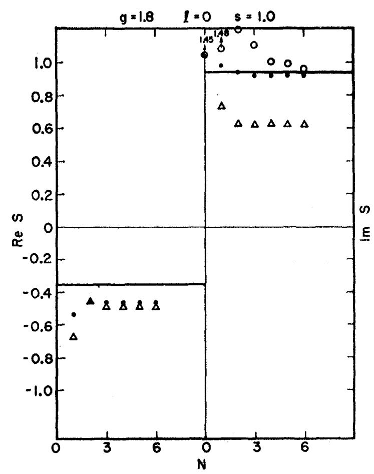
$$A(l,s)=\sum_{\text{all poles}}\frac{\beta_n e^{-(l-\alpha_n)\xi}}{l-\alpha_n}+\sum_{i=1}^{k}\frac{g_i}{2s}Q_l\left(1+\frac{m_i^2}{2s}\right)-\sum_{i=1}^{k}\frac{g_i}{2s}\sum_{n=1}^{\infty}P_{n-1}\left(1+\frac{m_i^2}{2s}\right)\frac{e^{-(l+n)\xi}}{l+n}. \tag{19}$$


g = 1.8 ℓ = 0 s = 0.1
Re S
Im S
N

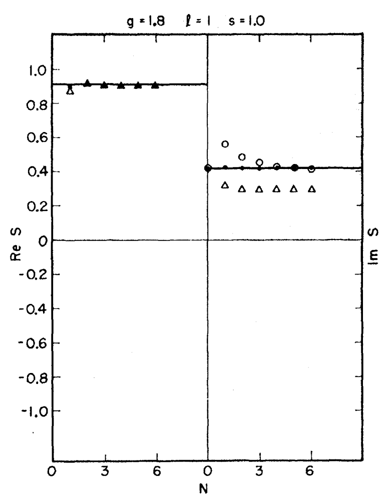
(a)


g = 1.8 ℓ = 1 s = 0.1
Re S
Im S × 10
N

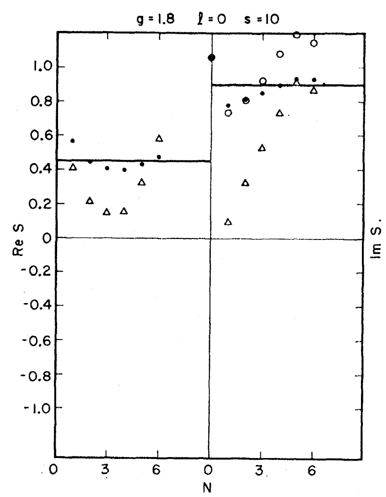
(b)


g = 1.8 ℓ = 0 s = 1.0
1.45
1.48
Re S
Im S
N

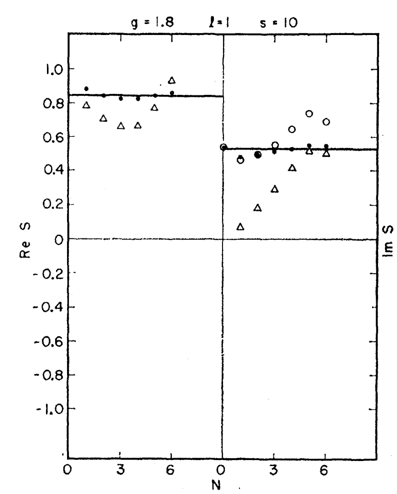
(c)


g = 1.8 ℓ = 1 s = 1.0
Re S
Im S
N

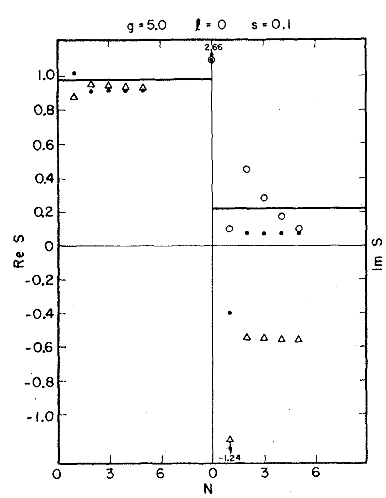
(d)


g = 1.8 ℓ = 0 s = 10
Re S
Im S.
N

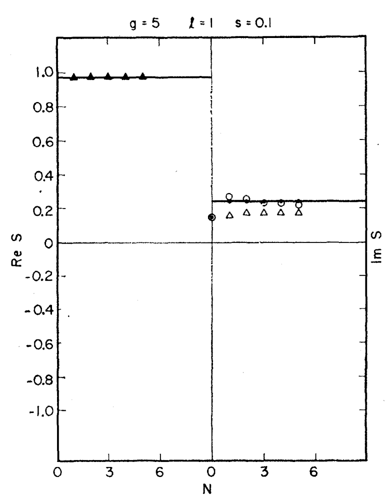
(e)


g = 1.8 ℓ = 1 s = 10
Re S
Im S
N

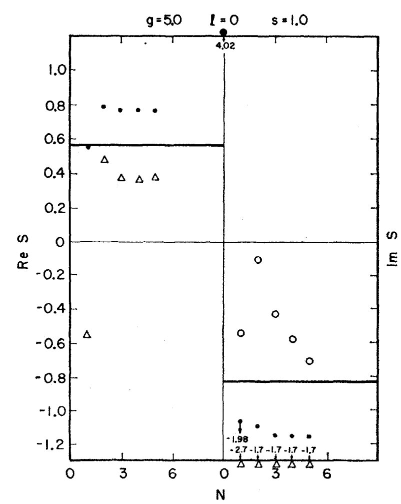
(f)


g = 5.0 ℓ = 0 s = 0.1
2.66
-1.24
Re S
Im S
N

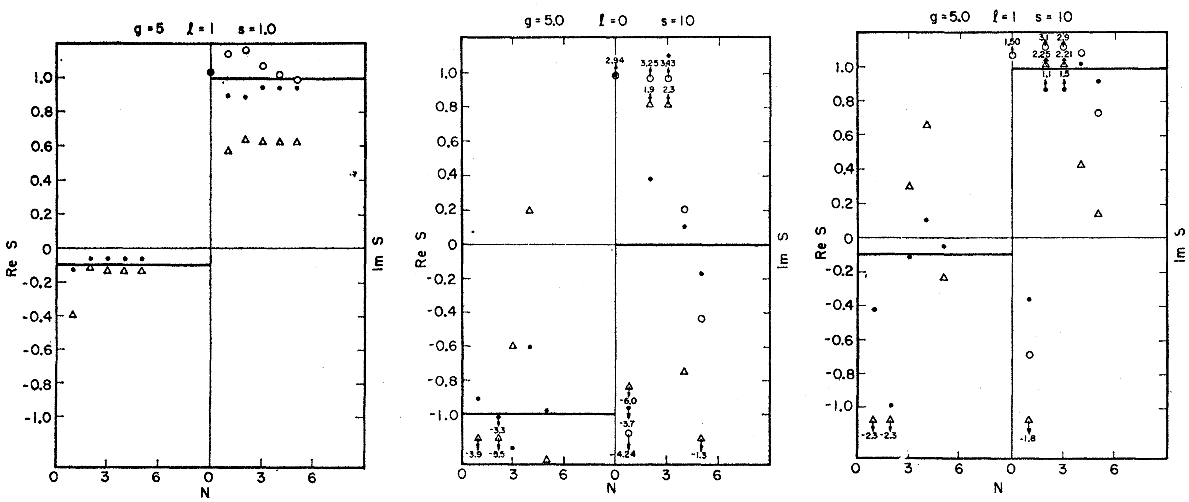
(g)


g = 5 ℓ = 1 s = 0.1
Re S
Im S
N

(h)


g = 5.0 ℓ = 0 s = 1.0
4.02
-1.98
-2.7 -1.7 -1.7 -1.7 -1.7 -1.7
Re S
Im S
N

(i)



(j) (k) (l)

FIG. 1. (a)–(l). Real and imaginary parts of S versus the number of terms in the expansion. △ Eq. (5′); ○ Eq. (21′) with $m=m_1$; ● Eq. (21′) with $m=2m_1$.

Now let us consider Eq. (19) for the case of a single Yukawa potential $V(r)=-ge^{-m_1r}/r$:

$$A(l,s)=\sum_{\text{all poles}}\frac{\beta_n e^{-(l-\alpha_n)\xi}}{l-\alpha_n}-\frac{g}{2s}\sum_{n=1}^{\infty}P_{n-1}\left(1+\frac{m_1^2}{2s}\right)\times\frac{e^{-(l+n)\xi}}{l+n}+\frac{g}{2s}Q_l\left(1+\frac{m_1^2}{2s}\right). \quad (20)$$

For $\xi=\xi_1$, the last two terms exactly cancel, so that, mathematically, series (20) is identical with Khuri's series if $\xi=\xi_1=\cosh^{-1}(1+m_1^2/2s)$. In deriving Eq. (20), however, we have used a weaker assumption than the one used in Khuri's paper. Two immediate advantages of Eq. (20) over the Khuri series are immediately apparent. At high energies the first two terms on the right-hand side cancel, and we simply obtain the Born term. Also at large values of l the two summations are small, and the Born term stands out as it should.

There is a one-to-one correspondence between the terms in the two summations, and in practice the first N terms of each summation are used. By considering the first N terms we obtain the approximate expression

$$A(l,s)\simeq\sum_{n=1}^{N}\beta_n\frac{e^{-(l-\alpha_n)\xi}}{l-\alpha_n}-\frac{g}{2s}\sum_{n=1}^{N}P_{n-1}\left(1+\frac{m_1^2}{2s}\right)\times\frac{e^{-(l+n)\xi}}{l+n}+\frac{g}{2s}Q_l\left(1+\frac{m_1^2}{2s}\right). \quad (21)$$

Here we make the conjecture that[5] in Eq. (12), $m^2=4m_1^2$. For $\text{Re}\lambda>0$, this seems to be correct, because once the Born term is taken out of $f(s,t)$, the dispersion integral[6] in t starts at $t=4m_1^2$. The asymptotic behavior of $A(\lambda,s)$ for $\text{Re}\lambda<0$, $|\lambda|\to\infty$ is not known, and Eq. (12) with $m^2=4m_1^2$ is the weakest asymptotic behavior that we can afford and still be correct in the right-half λ plane. It is seen from the numerical calculations given at the end of this paper that for a single Yukawa potential we obtain a rapidly convergent series, which supports the above conjecture. Here we would also like to remark that $m^2=4m_1^2$ correctly implies that the left-hand cut in s for $A(l,s)$ in the region $-m_1^2<s<-\frac{1}{4}m_1^2$ is entirely due to the Born term.

Finally, in the case of potential (1), Eq. (19) should be generalized to

$$A(l,s)=\sum_{\text{all poles}}\frac{\beta_n e^{-(l-\alpha_n)\xi}}{l-\alpha_n}-\frac{1}{2s}\int_{m_1}^{\infty}\sigma(\mu)d\mu Q_l\left(1+\frac{\mu^2}{2s}\right)+\frac{1}{2s}\int_{m_1}^{\infty}\sigma(\mu)d\mu\sum_{n=1}^{\infty}P_{n-1}\left(1+\frac{\mu^2}{2s}\right)\frac{e^{-(l+n)\xi}}{l+n},$$

where

$$\xi=\cosh^{-1}(1+(4m_1^2/2s)). \quad (22)$$

III. NUMERICAL CALCULATIONS

In this section we shall present the results of our numerical calculations applied to series (5) as well as to series (21). For our purposes it is more convenient to work with the S matrix rather than with the amplitude. Series (5) for the S matrix, taking the first N terms, is

$$S(l,s)=1+\sum_{n=1}^{N}\beta_n\frac{e^{-(l-\alpha_n)\xi_1}}{l-\alpha_n}, \quad (5')$$

[5] Geoffrey F. Chew (private communication).

[6] R. Blankenbecler, M. L. Goldberger, N. N. Khuri, and S. B. Treiman, Ann. Phys. (N. Y.) 10, 62 (1960).

and instead of (21) we have

$$S(l,s)=1+\sum_{n=1}^{N}\frac{\beta_n e^{-(l-\alpha_n)\xi}}{l-\alpha_n}-\frac{ig}{\sqrt{s}}\sum_{n=1}^{N}P_{n-1}\left(1+\frac{m_1^2}{2s}\right)\times\frac{e^{-(l+n)\xi}}{l+n}+\frac{ig}{\sqrt{s}}Q_l\left(1+\frac{m_1^2}{2s}\right),$$

since

$$S(l,s)=1+2i\sqrt{s}A(l,s). \qquad (21')$$

In Eq. (5′) and (21′) $\beta_n(s)$ are now the residues of the partial-wave S matrix rather than the partial-wave amplitude.

In Fig. 1 (a)–(l), a plot of the real and imaginary parts of the S matrix versus the number of terms in the expansion for the Khuri series as well as for our series (21) for both $m=m_1$ and $m^2=4m_1^2$ is given. The horizontal lines correspond to the actual values of the S matrix. The Regge parameters used in the series as well as the actual S-matrix values have been calculated by numerical integration of the Schrödinger equation.

The fact that for $g=5$ the agreement is not quite as good as for $g=1.8$ may be due to small errors in the residues. For stronger potentials our numerical calculation of the residues is less accurate. And for $g=5$ it turns out that, in some cases, only a few percent error in the residues introduces a considerable error in the values of the real or imaginary parts of the S-matrix calculated from the series.

ACKNOWLEDGMENTS

The author is indebted to Professor Geoffrey F. Chew for suggesting this problem and for his guidance throughout the course of this work. Thanks are also due Dr. E. Leader for a very interesting discussion.

PHYSICAL REVIEW VOLUME 133, NUMBER 4B 24 FEBRUARY 1964

Pseudoscalar and Vector Exchanges in the Production of Vector Mesons*

Robert W. Huff
Lawrence Radiation Laboratory, University of California, Berkeley, California

(Received 4 October 1963)

The cross section for the general reaction (pseudoscalar meson +nucleon → spin-$\frac{1}{2}$ baryon+vector meson) and the decay angular distributions for the final baryon and vector meson are calculated under the assumption that the reaction is dominated by the exchange of pseudoscalar and vector mesons. The results are applied to an analysis of the reaction $\pi^- p\to\Sigma^0 K^{*0}$.

I. INTRODUCTION

In this paper the cross section for the general reaction (pseudoscalar meson+nucleon → spin-$\frac{1}{2}$ baryon +vector meson) is calculated by assuming that the reaction is dominated by the exchange of pseudoscalar and vector mesons. In Sec. II, we derive expressions for this cross section, and for the decay angular distributions for the final baryon and vector meson. Section III contains a discussion of the structure of the form factors that appear in these expressions. In Sec. IV we use the results of the preceding sections in an analysis of the reaction $\pi^- p\to\Sigma^0 K^{*0}$, which analysis is an extension of one reported earlier.[1]

II. CALCULATION OF CROSS SECTIONS

We will use the conventions $\hbar=c=1$, $g^{\mu\nu}=(1,-1,-1,-1)$, $A_\mu B^\mu=A^0B^0-\mathbf{A}\cdot\mathbf{B}$, $\{\gamma^\mu,\gamma^\nu\}=2g^{\mu\nu}$, $\sigma^{\mu\nu}=\frac{1}{2}i\times[\gamma^\mu,\gamma^\nu]$, and $\gamma_5=\gamma^0\gamma^1\gamma^2\gamma^3$. Also, $\epsilon_{\mu\nu\lambda\sigma}$ is a completely antisymmetric tensor, which is $+1$ when $(\mu\nu\lambda\sigma)$ is an even permutation of (0123), -1 when it is odd, and zero otherwise. All spinors will be normalized so that $\sum_r u_r(p)\bar{u}_r(p)=p_\mu\gamma^\mu+m$.

Let us begin by considering the reaction $K^-p\to\Lambda\omega\to(\pi^-p)(\pi^+\pi^-\pi^0)$. Other reactions of the general form (pseudoscalar meson+nucleon → spin-$\frac{1}{2}$ baryon +vector meson) will have the same results, except for a possible over-all numerical factor for isotopic spin and a possible modification for different decay interactions for the final particles. Let p, r, H, Q be the momenta of the target nucleon, incident pseudoscalar meson, final baryon, and vector meson, respectively. Define two additional momenta, $k=H-p=r-Q$ and $s=p+r=H+Q$, so that k^2 and s^2 are the squares of the invariant momentum transfer and of the total center-of-mass energy, respectively. Let m, $\bar{m}$ be the masses of the target nucleon and incident pseudoscalar, M, $\bar{M}$ the masses of the final baryon and vector particle, and ν_p and ν_v the masses of the exchanged pseudoscalar (K) and vector (K^*) mesons, respectively. Then the most general Feynman amplitudes that can be written

* Work done under the auspices of the U. S. Atomic Energy Commission.

[1] Gerald A. Smith, Joseph Schwartz, Donald H. Miller, George R. Kalbfleisch, Robert W. Huff, Orin I. Dahl, and Gideon Alexander, Phys. Rev. Letters **10**, 138 (1963).