High-Frequency Region of the Spectrum of Electron and Positron Bremsstrahlung. II*

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The high-frequency limit of the bremsstrahlung spectrum from very high energy electrons has been calculated exactly with numerical methods. It is found that, even in the heaviest elements, only the s- and p-state partial waves of the outgoing zero-energy electron contribute significantly to the cross section: For Th($Z=90$) the s state gives 56% of the total and the p state 42%, while the d-state contribution is 2% and the f-state contribution 0.1%. It is also found that the s- and p-state cross sections are well represented by analytic expressions which we have given previously.

I. INTRODUCTION

WE have previously reported[1] an analytical calculation of the high-frequency limit of the bremsstrahlung spectrum from very high-energy electrons. This "tip" cross section, corresponding to a complete transfer of energy from incoming high-energy electron to outgoing photon, can be expressed[2] as a sum of partial cross sections $\sigma_{\rm tip}=\sum\sigma_{jl}$ corresponding to the possible angular momentum states of the outgoing zero-energy electron. To lowest order in $a\equiv\alpha Z$, where Z is the atomic number of the scatterer,[3] each partial cross section contributes in relative order a^{2l}. In our previous paper we obtained an expression for the s-state cross section σ_s as a power series in a, neglecting only terms of relative order a^4. Expressions were also given for the p-state cross sections neglecting relative order a^2, and so $\sigma_{\rm tip}$ itself could be given neglecting $O(a^4)$ terms.

Two basic questions arise concerning these analytic expressions. First, how well do the power series in a represent the partial wave cross sections? Second, how many partial waves make a significant contribution to $\sigma_{\rm tip}$? The purpose of this paper is to answer these questions by calculating the σ_{jl} exactly with numerical methods.

The general formalism needed for this problem was developed in I, to which we refer. Each σ_{jl} can be expressed as a double integral, as in Eq. 8 in I, but with $\epsilon=1$ since we are now only discussing the tip limit. In Sec. II we summarize the analytical results which are obtained for s-, p-, and d-state cross sections by expanding the integrands of these double integrals as power series in a; d-state results are new. In Sec. III we present the results we have obtained in an exact numerical evaluation of the σ_{jl} on the IBM 7090 computer and then use them to answer our two basic questions.

II. ANALYTIC RESULTS

Here we give analytic expressions for the cross sections σ_{jl} for s, p, and d final angular momentum states as power series in $a\equiv\alpha Z$.

s States

From Eq. (29) in I we have

$$\sigma_s=\frac{8\pi e^2a^{2\gamma_1+1}}{k\Gamma(2\gamma_1+1)}e^{-\pi a}(1-0.837a+0.318a^2+0.564a^3)\,, \quad (1)$$

where $\gamma_1=(1-a^2)^{\frac{1}{2}}$ and k here is the photon energy. The factors not expanded were chosen in analogy with previous work on the photoeffect.[4]

p States

Neglecting terms of relative order a^2, the p-state cross sections may be obtained either from corresponding photoeffect calculations[2] or directly[1] from Eq. (8) in I; they may be written

$$\sigma_{p1/2}=\frac{64\pi e^2\delta_2{}^{2\gamma_1+3}}{k\Gamma(2\gamma_1+1)}e^{-\pi a}\left(1+\frac{4\pi}{9}a\right),$$
$$\sigma_{p3/2}=\frac{8192\pi e^2\delta_3{}^{2\gamma_2+1}}{3k\Gamma(2\gamma_2+1)}e^{\pi a}\left(1-\frac{33\pi}{140}a\right), \quad (2)$$

where

$$\delta_2=a(2+2(1-a^2)^{1/2})^{-1/2},\quad \delta_3=\tfrac{1}{2}a,$$
$$\gamma_1=(1-a^2)^{1/2},\quad \gamma_2=(4-a^2)^{1/2}. \quad (3)$$

The factors not expanded are a combination of those used previously,[1,2] and have been chosen to give best agreement with our numerical calculations.

d States

Starting from Eq. (8) in I, setting $l=2$, the tip cross section for both d states has been calculated to the lowest

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[1] R. J. Jabbur and R. H. Pratt, Phys. Rev. **129**, 184 (1963), hereafter referred to as I. See Paper I for notation and references to previous work and experimental results.

[2] R. H. Pratt, Phys. Rev. **120**, 1717 (1960).

[3] Unrationalized charge units are used throughout the paper. Also $\hbar=c=m_e=1$, and $x=O(y)$ shall mean x is of order y.

[4] R. H. Pratt, Phys. Rev. **117**, 1017 (1960).

TABLE I. Tip electron bremsstrahlung partial wave cross sections Φ for various final angular momentum states as calculated by machine (upper line) and analytically (lower line) from Eqs. (5), (1), (2), and (4). Since Eq. (4) for d states is valid only to lowest order in a, we show the d-state analytic values only for $Z=1$.

Z	$s_{1/2}$	$p_{1/2}$	$p_{3/2}$	$d_{3/2}$	$d_{5/2}$	$f_{5/2}$	$f_{7/2}$
1	0.089103	1.2058×10^{-6}	4.2199×10^{-6}	9.6765×10^{-12}	4.3848×10^{-11}		
	0.089104	1.2057×10^{-6}	4.2200×10^{-6}	9.6007×10^{-12}	4.4405×10^{-11}		
13	0.84205	0.00236	0.00669				
	0.84204	0.00235	0.00672				
26	1.23569	0.01739	0.03817				
	1.23541	0.01723	0.03873				
47	1.4408	0.0965	0.1335	0.00133	0.00308		
	1.4400	0.0950	0.1389				
74	1.430	0.380	0.276	0.01050	0.01555		
	1.447	0.383	0.293				
82	1.408	0.524	0.313	0.01668	0.02155		
	1.442	0.541	0.333				
90	1.379	0.706	0.348	0.02532	0.02858	0.00069	0.00105
	1.441	0.754	0.367				

order in a with the methods of our previous paper. As discussed earlier, these cross sections are of relative order a^4 compared to s states. The results are

$$\sigma_{3/2}=\frac{4\pi e^2a^3}{k}e^{-\pi\alpha}\left(\frac{17}{450}a^4\right),$$
$$\sigma_{5/2}=\frac{4\pi e^2a^3}{k}e^{-\pi\alpha}\left(\frac{13}{75}a^4\right), \tag{4}$$

where the normalization has been expressed in its simplest form.[5]

III. NUMERICAL RESULTS

The numerical evaluation of the double integral Eq. (8) in I, performed with the aid of an IBM-7090 computer, is straightforward (although the complexity of the integrand makes programming somewhat tedious). One must integrate by parts to remove terms which are singular at the origin, and the resulting regular integral may be evaluated with a double application of Simpson's rule. We believe the numerical results which have been obtained in this way are accurate to at least 0.2%.

We may now compare these numerical results with the analytic predictions given in Sec. II. It is conventional to define the dimensionless parameter

$$\Phi=k(d\sigma/dk)/\alpha r_0^2Z^2, \tag{5}$$

where α is the fine structure constant and r_0 is the classical electron radius. (Note in our units $e^2a^2=\alpha r_0^2Z^2$.) Table I gives the results obtained for Φ from the analytic Eqs. (1), (2), (4), and from the numerical calculations.

From Table I we deduce the following results:

1. The analytic result Eq. (1) for s state cross sections is a good representation of the exact σ_s. The difference between the numerical and analytic results is only 5% for the high-Z element Th($Z=90$).

2. The p-state contribution to the total cross section is very significant for heavy elements. Indeed, for $Z=90$, the p states contribute about 42% of the total result. Since the p states are of relative $O(a^2)$, they contribute less significantly for lower values of Z. Again, the analytic expressions Eq. (2) for p-state cross sections seem to represent the actual values fairly well. The difference between the analytic and numerical calculations is within 1% for Ag($Z=47$) and is only 7% for a high-Z element such as Th($Z=90$).

3. Even for Th, for which the d states have their highest relative contribution, the d states contribute only 2% to the total cross section. The f-state contribution is entirely negligible, and we conclude that only s and p states contribute significantly to high-energy bremsstrahlung at the tip limit.

[5] In Ref. 1 it was stated that the relative magnitude of the d states was approximately $2a^4$. In fact these magnitudes, which are the terms in parentheses in Eq. (4), have appreciably smaller coefficients.