Perhaps the most interesting observation which can be made is that the structure persists to the more probable energies where the final excitation energy is larger. This can partly be attributed to the fact that at scission, most of what appears as final excitation energy is tied up in deformation energy. The existence of structure at the higher excitation energies implies that the primary fission fragments may be predominantly even-even nuclei. This enhancement of even-even primary fragment yields will be modified by neutron emission. Calculations based on Terrell's parameters[18] for neutron emission probability distributions indeed indicate that there are approximately equal probabilities for a primary fragment to emit an even or an odd number of neutrons. This means that the yield of even-even secondary fragments will not show this enhancement and that the structure will be obscured in the radiochemical yield measurements.

[18] James Terrell, Phys. Rev. **108**, 783 (1957).

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Comparison to Experiment of the Amatic-Leader-Vitale Multipion-Exchange Contributions*

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The phase shifts obtained by Amati, Leader, and Vitale (ALV), using the Cini-Fubini approach to the Mandelstam representation, are compared to proton-proton scattering data at 51.8, 96.5, 142, 210, and 310 MeV. The ALV phases yield a considerably better fit to the data than do pure one-pion exchange, but considerably poorer than modified phase analyses. The unusual behavior of the ALV $^3F_2$ and ϵ_4 is not confirmed. The lower angular momentum phases from ALV+data are generally not greatly different from those of modified phase analyses.

I. INTRODUCTION

IN a recent series of papers, Amati, Leader, and Vitale[1] (hereafter referred to as ALV) have applied the Cini-Fubini approach (to the Mandelstam representation) to the problem of nucleon-nucleon scattering. Within this framework, ALV were able to obtain predictions for the *total* (correlated and uncorrelated) two-pion exchange contribution for nucleon-nucleon phase shifts with orbital angular momentum $L\geq 2$ and incident laboratory energy $E_L\leq 300$ MeV. ALV added in one-pion exchange (OPE) and three-pion exchange (ω) as pole terms, and computed the resulting phase shifts. Those phases were then compared to YLAM and SMMN curves from energy-dependent phase-shift analyses of experimental data. The predicted "theory" curves in the most recent ALV reference show that the ALV multipion contributions generally correct OPE toward the "experimental" YLAM and SMMN curves.

The present work was motivated by two observations: (1) There are now more experimental data available than when the YLAM fit was made. This may also be true of the SMMN fit. (2) The SMMN curves have no errors shown, so there is no way of estimating how closely they should be matched by theoretical predictions. This is remedied here by computing the standard deviations for the phase shifts deduced from the experimental data, and by fitting the ALV phases directly to the data.

Section II specifies the data used, and Sec. III the method of analysis. IV defines the modified phase-shift analyses used for comparison. In Sec. VA the ALV fit to the data is compared to that of the pure one-pion exchange, and to the fits of the modified phase analyses. The extent to which the result depends on the $J=2$ phases is examined in Sec. VB. Finally, in Sec. VC, other current two-nucleon models are examined for the strange energy dependence of the ALV $^3F_2$ and ϵ_4. Also, an attempt is made there to confirm or refute the behavior by data analysis.

II. DATA USED

A compilation of the data which were used is shown in Table I. There were 222 pieces of proton-proton scattering data near 51.8, 96.5, 142, 210, and 310 MeV. All of the data were treated as though measured at the nearest energy in the above list. This probably did not introduce a significant amount of error, since the (absolute) cross section and polarization normalizations were treated as data separate from the relative angular distributions, and the shapes of angular distributions do not change rapidly with energy. Small-angle cross-section shapes probably are more rapidly varying with energy, but the only forward-angle cross sections included in this work were used at the measured energies.

* Supported in part by the U. S. Atomic Energy Commission.

[1] D. Amati, E. Leader, and B. Vitale, Phys. Rev. **130**, 750 (1963), and previous publications cited therein.

TABLE I. Data used. σ indicates relative cross-section angular distribution data; P, relative polarization; N_σ, (absolute) cross-section normalization for the σ's which follow; N_p, polarization normalization; σ_{abs}, absolute cross section; σ_{int}, integrated cross section. The rest are in standard notation.

Lab energy of analysis (MeV)	Experimental lab energy (MeV)	Type	No. data	Reference	Remarks
51.8	50	D	1	a	
	51.5	N_σ	1	b	
		σ	9	b	12.2° not used
	51.8	N_σ	1	b	
		σ	9	b	
		σ_{abs}	1	c	Interpolated
		P_{abs}	1	d	Interpolated
	52	C_{KP}	1	e	
		C_{NN}	1	f	
96.5	95	σ	13	g	
		N_p	1	g	
		P	14	g	
	96.5	σ_{int}	1	h	Interpolated
	98	D	5	i	
142	137.5	R	5	j	
	139	A	6	k	
	140.5	R	6	l	
	142	σ_{int}	1	h	Interpolated
	142	D	8	m	
	147	σ	27	g	$H1$ and $H8$ combined
		N_p	1	g	
		P	28	g	$H1$, $H8$, $H14$ combined
210	210	N_p	1	n	
		P	6	n	
	213	σ_{abs}	1	o	Interpolated value in Ref. o
		N_σ	1	o	
		σ	7	o	
		D	7	p	
		R	7	q	
		E	7	q	
		R'	4	r	
310	310	σ_{int}	1	s	
		σ_{abs}	1	s	Interpolated value in Ref. s
		N_p	1	s	
		P	7	s	
		R	6	s	
		D	6	s	
310	315	σ	7	s	Table II of Ref. s
		N_p	1	s	
		P	6	s	
	316	A	3	t	
	320	C_{NN}	1	u	

[a] T. C. Griffith, D. C. Imrie, G. J. Lush, and A. J. Metheringham, Phys. Rev. Letters 10, 444 (1963).
[b] K. Nisimura *et al.*, Institute for Nuclear Studies, University of Tokyo, Tokyo, Japan, Report No. INSJ-45, 1961 (unpublished); and K. Nisimura (private communication).
[c] L. H. Johnston and Y. S. Tsai, Phys. Rev. 115, 1293 (1959).
[d] C. J. Batty, G. H. Stafford, and R. Gilmore, Phys. Letters 2, 109 (1962).
[e] K. Nisimura *et al.*, Prog. Theoret. Phys. (Kyoto) 29, 616 (1963); and K. Nisimura (private communication).
[f] K. Nisimura (private communication), Prog. Theoret. Phys. (Kyoto) (to be published).
[g] J. N. Palmieri, A. M. Cormack, N. F. Ramsey, and R. Wilson, Ann. Phys. (N.Y.) 5, 299 (1958). All N_σ's in this reference have been withdrawn (private communication from R. Wilson). Small angle points were omitted because of possible multiple-scattering corrections (private communication from A. M. Cormack). P(4.13°) has been withdrawn (private communication from R. Wilson).
[h] J. N. Palmieri and R. Golaskie (private communication) (to be published).
[i] E. H. Thorndike and T. R. Ophel, Phys. Rev. 119, 362 (1960).
[j] S. Hee, R. Wilson, Harvard Cyclotron Report, 31 May 1963 (to be published).
[k] S. Hee, R. Wilson, Harvard Cyclotron Report, 1 June 1962 (to be published).
[l] E. Thorndike, J. LeFrancois, and R. Wilson, Phys. Rev. 120, 1819 (1960).
[m] C. F. Hwang, T. R. Ophel, E. H. Thorndike, and R. Wilson, Phys. Rev. 119, 352 (1960).
[n] J. H. Tinlot and R. E. Warner, Phys. Rev. 124, 890 (1961).
[o] A. Konradi, thesis, University of Rochester, 1961 (unpublished).
[p] K. Gotow, F. Lobkowicz, and E. Heer, Phys. Rev. 127, 2206 (1962).
[q] A. C. England, W. A. Gibson, K. Gotow, E. Heer, and J. Tinlot, Phys. Rev. 124, 561 (1961).
[r] F. Lobkowicz and K. Gotow (private communication).
[s] O. Chamberlain, E. Segre, R. D. Tripp, C. Wiegand, and T. Ypsilantis, Phys. Rev. 105, 288 (1957).
[t] J. Simmons, Phys. Rev. 104, 416 (1956).
[u] J. V. Allaby, A. Ashmore, A. N. Diddens, and J. Eades, Proc. Phys. Soc. (London) 74, 482 (1959).

Thus the number of data which demanded accuracy in energy was rather small; in addition, it was possible to accurately interpolate values for most of these. Details are given in Table I.

The Harwell data sets near 95 and 140 MeV were not included in the analyses reported here, since, when included, they yielded exceedingly high contributions to the least-squares sum (χ^2) from isolated data points. Such data are incompatible with either the method of analysis or with the other data which have reasonable contributions to χ^2. This will be examined in detail in separate comprehensive reports on phase-shift analyses at those energies. Also, the cross-section measurement on the forward Coulomb rise at 51.5 MeV was not used. It gave a very high contribution to χ^2, and the experimental report contained no mention of how the reported center-of-mass angle was computed.

III. METHOD OF ANALYSIS

The method of analysis used was that usually referred to as the "modified phase analysis,"[2] where by the higher angular momentum phases are taken from theory and the lower ones are adjusted so as to yield a least-squares fit to the data. The least-squares fitting was here accomplished by the nonlinear method used by Lietzke.[3] This method accelerates in convergence toward a minimum in the least-squares sum χ^2 and provides a test of whether such a minimum was actually reached. In almost all cases to be reported here, a minimum *was* reached. The sole exception was the case designated ALV(6) at 96.5 MeV. There the method failed, so other search techniques were used to reach a minimum.

The standard deviations were obtained in the usual fashion[2] from the diagonal elements of the error matrix. The statistical theory upon which this relation is based, however, demands that the χ^2 surface be quadratic in the space of the searched-upon parameters. Each analysis was checked to make sure that the quadratic approximation was indeed sufficiently accurate in the volume bounded by the computed standard deviations to determine the latter to within 5–10%. The only exception was again the ALV(6) run. There the χ^2 surface was quadratic only over a volume about $\frac{1}{3}$ the needed radii. Thus the standard deviations shown for that case are only rough estimates.

[2] M. H. MacGregor, M. J. Moravcsik, and H. P. Stapp, Phys. Rev. 116, 1248 (1959), and previous publications cited therein.
[3] M. H. Lietzke, Oak Ridge National Laboratory Report ORNL-3259, April 1962 (unpublished).

TABLE II. A sample run. Least-squares fit to the 41 pieces of 210 MeV data, with 7 free and 7 ALV phases. The phases are nuclear bar, in degrees. All phases of higher angular momentum than those shown were represented by their contributions to the OPE amplitude, with $g^2=14.4$, $\mu=135.1$ MeV. Minimized $\chi^2=68$. Expected $\chi^2=34$.

Free (searched-on) phases	Fixed (at ALV values) phases
${}^1S_0=+6.12\pm0.49$	${}^3F_3=-2.25$
${}^3P_0=+0.04\pm0.50$	${}^3F_4=+2.98$
${}^3P_1=-22.31\pm0.54$	${}^1G_4=+1.27$
${}^3P_2=+15.74\pm0.29$	$\epsilon_4=-0.38$
$\epsilon_2=-2.72\pm0.16$	${}^3H_4=+0.45$
${}^1D_2=+6.94\pm0.30$	${}^3H_5=-0.80$
${}^3F_2=+1.74\pm0.24$	${}^3H_6=+0.34$

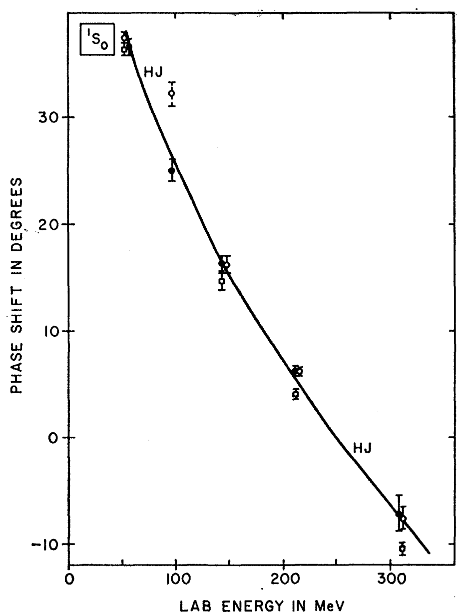

FIG. 1. The ${}^1S_0$ phase shifts from the various models. ALV(6,7) (see text) predictions are denoted by open circles; OPE(6,7) by boxes; and the modified phase analyses OPE(5–12) by filled circles. "HJ" denotes the Hamada-Johnston potential prediction (see text).

Two models for the fixed (higher L) phases were used. In the first (OPE model) they were taken to be exactly one-pion exchange, represented by the OPEC amplitude of Ref. 2, from which the lower L OPE contributions had been subtracted. In the second, ALV model, the ALV phases as given in their latest paper were used. For phases of higher L than those computed by ALV, a subtracted OPEC amplitude was again used. Throughout, the OPE constants were taken to be $g^2=14.4$, $\mu=135.1$ MeV, where g^2 is the pion-nucleon coupling constant and μ is the neutral pion mass. The detailed results of a typical analysis are shown in Table II.

IV. COMPARISON PHASES

In order to provide comparisons, modified phase-shift analyses of the usual type[2] were made. Such analyses are mainly as in the model already labeled OPE(N), where

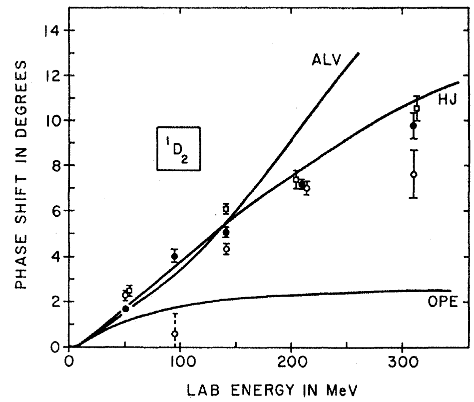

FIG. 2. The ${}^1D_2$ phases from the various models. (○)=ALV(6,7), (□)=OPE(6,7), (●)=modified phase analysis.

N is the number of free (searched-upon) phases, and the higher L phases are taken to be exactly OPE. Now, however, N is taken to be an energy-dependent number to be determined from the analysis itself and/or from plausibility arguments.

At 51.8 MeV, N was taken to be 5, since recent potential models agree that all phases with $L\geq 2$ (except ${}^1D_2$) can be fairly accurately represented by their OPE values at that energy. The value $N=6$ at 96.5 MeV was chosen for the same reason. In addition, the ratio of χ^2 to its expected value was plotted versus N. As N is increased from a small value, the ratio first drops precipitously, then reaches a plateau. For both of the above values of N, the ratio had reached the plateau. At the higher energies, the criterion was that N should be large enough that the aforementioned χ^2 ratio was in the plateau region, but not so high that the highest L phases searched upon would have drifted to values far from those given by potentials, OPE or ALV. No effort was made to have the ratio of χ^2 to its expected value come out to any particular number or in a particular range. The higher energy values of N finally used were: 142 MeV, 11; 213 MeV, 11; 310 MeV, 12.

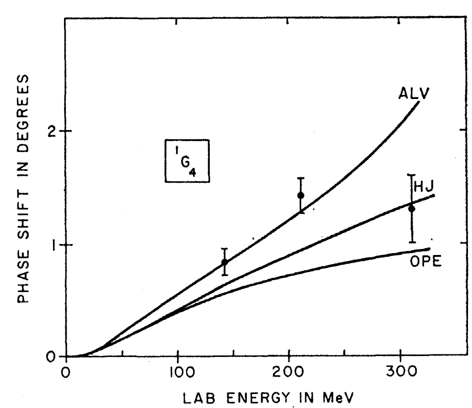

FIG. 3. The ${}^1G_4$ phases from the various models. (●)=modified phase analysis.

The above criteria are somewhat subjective, but it is hoped that they have produced phase shifts which are interesting for comparisons.

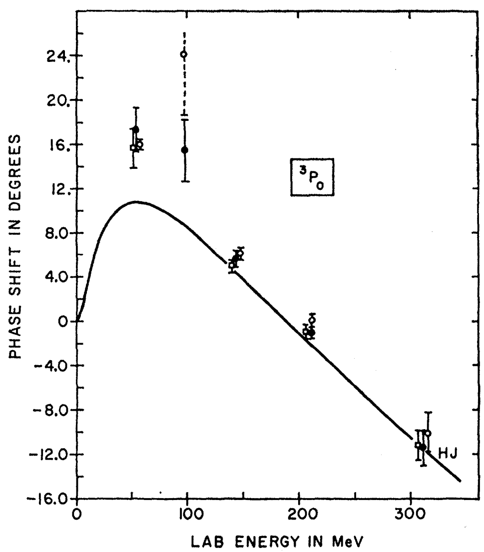

FIG. 4. The $^3P_0$ phases from the various models. (○) = ALV(6,7), (□) = OPE(6,7), (●) = modified phase analysis.

V. COMPARISON TO EXPERIMENT OF THE ALV PHASES

A. Phases for $L \geq 2$

As mentioned in the Introduction, ALV computed all phases with $L \geq 2$. This left unspecified five PP phases; $^1S_0$, $^3P_0$, $^3P_1$, $^3P_2$, and ϵ_2. Using the data of Sec. II and the method of Sec. III, these phases and their corresponding least-squares sums χ^2 were evaluated at the various energies. The resulting χ^2 ratios are shown in Table III for the models: (1) higher L phases from ALV, (2) higher L phases from OPE, and (3) the modified phase-shift analysis [denoted by OPE(●) here] described in Sec. IV. Values were not computed at 310 MeV because ALV did not compute $^1D_2$ there.

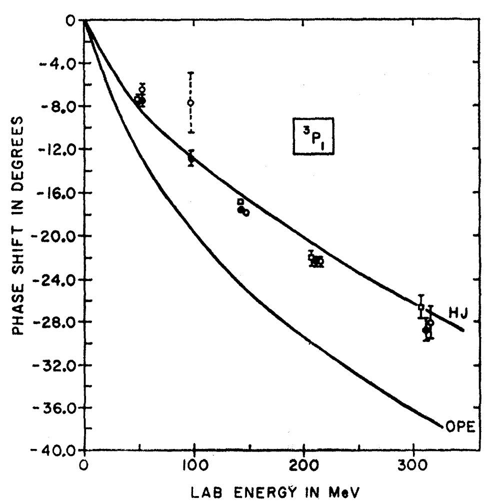

FIG. 5. The $^3P_1$ phases from the various models. (○) = ALV(6,7), (□) = OPE(6,7), (●) = modified phase analysis.

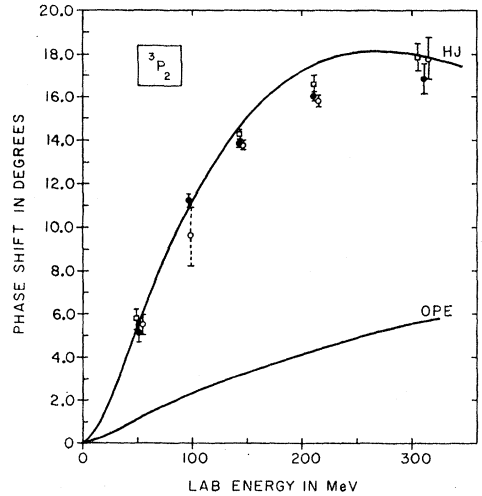

FIG. 6. The $^3P_2$ phases from the various models. (○) = ALV(6,7), (□) = OPE(6,7), (●) = modified phase analysis.

TABLE III. Ratios of χ^2 to the expected values of χ^2. The latter are the number of data minus the number of free (searched-on) parameters.

Model	Lab energies in MeV 51.8	96.5	142	210	Combined data
OPE(5)	1.03	1.80	7.0	29.2	8.3
ALV(5)	0.76	1.81	2.14	7.3	2.5
OPE(●)	0.84	0.82	0.82	0.96	0.86

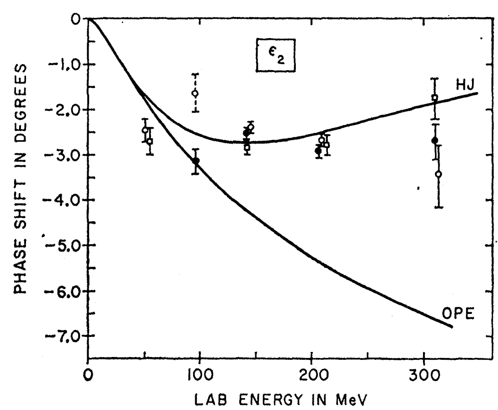

FIG. 7. The ϵ_2 phases from the various models. (○) = ALV(6,7), (□) = OPE(6,7), (●) = modified phase analysis.

It is obvious from Table III that ALV is a large improvement over OPE, the bare one-pion exchange phases, except at 96.5 MeV. It is also obvious that the modified phase analyses, OPE(●), are in turn a large improvement over ALV except at the lowest energy.

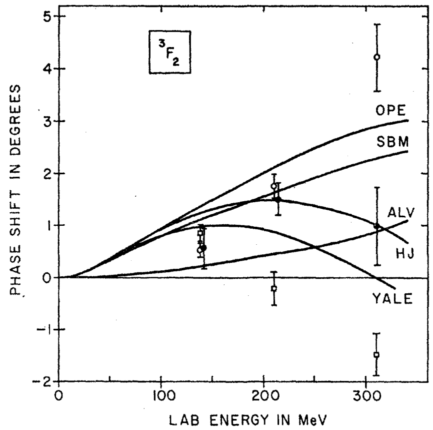

FIG. 8. The $^3F_2$ phases from the various models. (○)=ALV(6,7), (□)=OPE(6,7), (●)=modified phase analysis.

This indicates that the ALV phases differ from OPE in the right direction, but further refinement seems necessary, even at as low an energy as 96.5 MeV. Only at 51.8 MeV are the ALV phases statistically superior to the modified phase analyses as one would desire for a good theory.

TABLE IV. χ^2 ratios, as in Table III. N indicates the number of searched-upon phases. OPE(●) designates the modified phase analysis results.

Model	Lab energies in MeV (N)					Combined data
	51.8(6)	96.5(6)	142(7)	210(7)	310(7)	
OPE(N)(○)	0.71	0.82	1.83	3.31	1.20	1.72
ALV(N)(□)	0.50	1.21	1.25	2.00	3.51	1.70
OPE(●)	0.84	0.82	0.82	0.96	0.86	0.86

B. Phases for $J \geq 3$

To determine how much of the improvement of ALV over OPE depends on its lowest L phases, the calculations cited above were repeated, but with $^1D_2$ and $^3F_2$ now released from their model values. However, the data at the lowest two energies were insufficient to determine $^3F_2$ for the ALV model, so only $^1D_2$ was released there. The resulting χ^2 ratios are shown in Table IV. Examination of the Table shows that the ALV phases for $J \geq 3$ are not a decided improvement over the corresponding pure one-pion exchange values. One might wish for clearer trends versus energy in comparing the first two lines in Table IV.

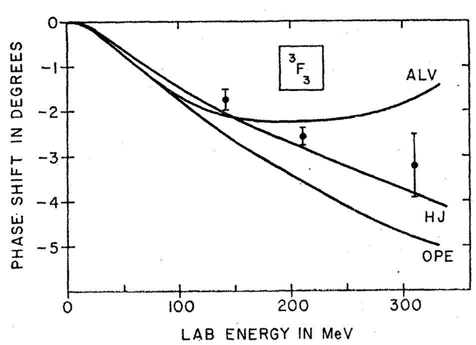

FIG. 9. The $^3F_3$ phases from the various models. (●)=modified phase analysis.

C. ALV+Data Predictions for $^1D_2$ and $^3F_2$

The phases from the models discussed in the previous section (VB) are plotted in Figs. 1–12. The predictions of the Hamada-Johnston[4] potential model are also shown, as typical of potential predictions; one could equally well have used the Yale[5] model. The ALV(6,7) are denoted by open circles, the OPE(6,7) by boxes, and the OPE modified phase analyses by solid circles. Note that the models are in fair agreement for the low angular momentum phases, except for the ALV(6) predictions at 96.5 MeV. If, instead, the ALV(5) predictions had been plotted (corresponding to a considerably higher χ^2) there would have been no such discrepancy.

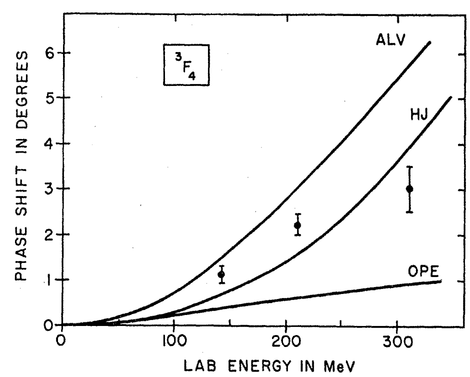

FIG. 10. The $^3F_4$ phases from the various models. (●)=modified phase analysis.

One of the purposes in plotting the ALV(6,7) points was to examine the $^1D_2$ and $^3F_2$ phases predicted by the higher L ALV phases combined with the experimental data. Those $J=2$ phases can then be compared to calculated ALV $J=2$ phases as a consistency check, and may also indicate above what energy the lowest L phases depart from their ALV values. Figure 2 shows that the ALV+ data predictions for $^1D_2$ are several standard deviations away from the calculated ALV curve at all energies. More important, perhaps, is the apparent pulling away from the points above about 150 MeV. The discrepancy between the ALV calculated and ALV+ data $^3F_2$ phases (Fig. 8) is very marked at the higher energies.

[4] T. Hamada and I. D. Johnston, Nucl. Phys. 34, 382 (1962).
[5] K. E. Lassila, M. H. Hull, H. M. Ruppel, F. A. McDonald, and G. Breit, Phys. Rev. 126, 881 (1962).

The $^3F_2$ phases corresponding to the model parameters of the Hamada-Johnston[4] potential, the Yale[5] potential-with-spin orbit-cutoff, and the Saylor-Bryan-Marshak[6] boundary-condition-plus-TMO potential, are plotted in Fig. 8. One observes that the centrifugal barrier forces all of the model phases to approach the OPE (potential tail) value at low energy. The calculated ALV phase-shift curve displays a quite different behavior. The latter was taken by ALV to be a major success, since it agreed with the result of a purely phenomenological energy-dependent phase-shift analysis. One must point out, however, that the present data are insufficient to distinguish between the ALV and OPE phases below 200 MeV.[7] Thus the shape of a purely phenomenological energy-dependent phase analysis curve for $^3F_2$ below 200 MeV is not very significant.

VI. CONCLUSIONS

The ALV phases have proved to be fully amenable to data analysis, except at 96.5 MeV. Lower angular momentum phases have been obtained for the ALV set by means of least-squares fitting to *PP* data, and standard deviations were obtained.

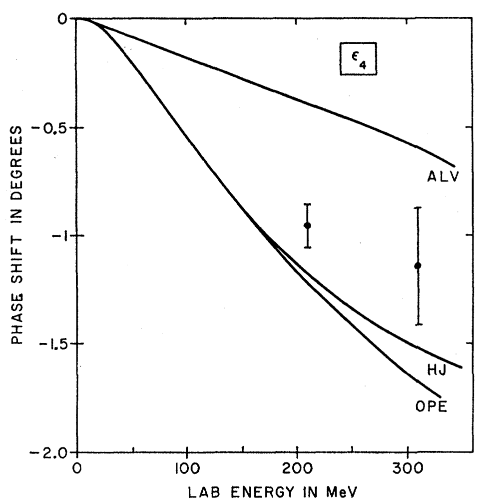

FIG. 11. The ϵ_4 phases from the various models. (●) = modified phase analysis.

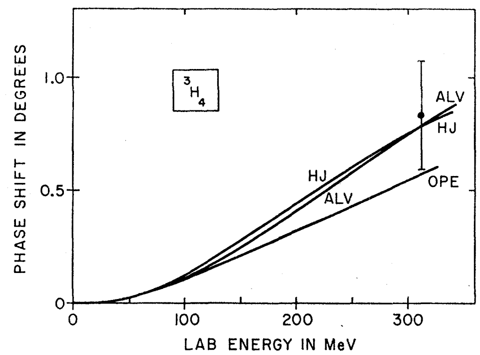

FIG. 12. The $^3H_4$ phases from the various models. (●) = modified phase analysis.

It is encouraging, apart from the troubles at 96.5 MeV, that the low-L phases predicted by ALV+ data mostly agree within their standard deviations with the phases from the modified phase-shift analyses. The exceptions, $^1D_2$ at two energies, $^3P_1$ and $^3F_2$ at one energy, do not present a gross dispersion. Thus one has an indication of the extent to which knowledge of the low-L phases is independent of accurate knowledge of multipion exchange effects.

Although the data below 200 MeV are insufficient to decide between the usual $^3F_2$ energy dependence and that of ALV, the ALV $^3F_2$ phase is clearly inconsistent with estimates from the data at 210 MeV (Fig. 8). The similarly strange energy dependence of the ALV ϵ_4 (Fig. 11) does not seem to be substantiated, although the evidence is weaker. Certainly the numerical calculation of these two phases by the ALV method should be carefully checked.

Finally, the ALV phases provide considerably poorer fits to the data than do the modified phase analyses except at the lowest energy. On the other hand, they are a considerable improvement over the purely OPE phases, but with only weak evidence of improvement for $J \geq 3$.

ACKNOWLEDGMENTS

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[6] D. P. Saylor, R. A. Bryan, and R. E. Marshak, Phys. Rev. Letters **5**, 266 (1960).

[7] Note the OPE(●) point at 142 MeV in Fig. 8. The equivalent point at 96.5 MeV, from OPE(7), is $^3F_2 = 0.41 \pm 0.40$, which again embraces both OPE and ALV values.