Photoproduction of Neutral Pions from Complex Nuclei*

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The photoproduction of neutral pions from complex nuclei is expressed in terms of the production amplitudes from single nucleons, as well as certain properties of the nuclear ground state. Methods are developed for the evaluation of the nuclear matrix elements. Correlations between nucleons lead to suppression of the incoherent cross section for small momentum transfers. The coherent nuclear production is strongly peaked at an angle $\theta \approx 2/kR$. Pions can also be produced by the coupling of the incident photon with the nuclear Coulomb field. The cross section for this process is peaked at an angle $\theta \approx m_\pi^2/2k^2$. Final-state interactions of the produced pion are included by means of the Fernbach-Serber-Taylor model. This leads to attenuation of the nuclear production and to a change in shape of the Coulomb production. The theoretical predictions are compared with experimental measurements at 250 and 900 MeV.

I. INTRODUCTION

APART from the neutral sigma particle, the π^0 is the only known elementary particle whose decay involves neither leptons nor a change in strangeness and is, therefore, probably caused by the strong and the electromagnetic interactions only. For ten years after the existence of the neutral pion was established, attempts to measure its lifetime directly yielded only upper limits. Since strong couplings must be involved in the decay process, theoretical predictions were also lacking, except for a lowest order perturbation theory calculation[1] which yielded a mean lifetime of $(4\pi)^5 M^2/g^2e^4m^3 \approx 5\times10^{-17}$ sec but which nobody seriously believed.

Some attention was therefore directed to a possible way of measuring the lifetime indirectly using a suggestion by Primakoff.[2] Because the π^0 decays into two real photons, it should also be possible for a real photon to interact with a virtual photon of the Coulomb field of the nucleus to produce a neutral pion. Since the transition amplitudes for the two processes are very similar, it should, according to Primakoff, be possible to determine the lifetime for decay by measuring the cross section for photoproduction of a neutral pion in the Coulomb field of a heavy nucleus.

In 1959 a beginning was made at the California Institute of Technology electron synchrotron with an experiment[3] to measure photoproduction of neutral pions from complex nuclei. Although direct measurements[4] of the π^0 lifetime have since yielded a value of $(2.1\pm0.4)\times10^{-16}$ sec, the photoproduction from complex nuclei is still important as a possible way of obtaining an independent lifetime measurement, as well as a way of studying the structure of complex nuclei and the pion-nucleon interaction.

In addition to the relatively simple photoproduction by inverse decay, neutral pions can also be produced by direct nuclear interactions not involving virtual photons. The main purpose of this work is to provide a theoretical estimate[5] of the most important production modes from complex nuclei. This will be based on a model of direct interaction between the incident photon and the individual nucleons, coupled with the impulse approximation. The nuclear cross section can then be expressed in terms of single-nucleon photoproduction amplitudes, properties of the nuclear ground state, and the interactions of pions in nuclei. If these three quantities are known, as they are in principle, the nuclear cross section is completely determined since there are no other parameters which can be adjusted to fit the experiments. The emphasis will be placed on differential cross sections at small angles, where coherent nuclear and Coulomb production can compete with incoherent nuclear production.

In Sec. II the production from single nucleons is discussed briefly, mainly to establish certain facts, conventions, and notations to be used later. In Sec. III, the cross section from complex nuclei is expressed in terms of expectation values of single-nucleon operators in the nuclear ground state. This follows from an application of the impulse approximation and a closure approximation which was developed by various authors.

The matrix elements of the single-nucleon operators are considered in Sec. IV and a general separation theorem is derived [Eq. (4.7)] for closed-shell nuclei.

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[1] R. J. Finkelstein, Phys. Rev. **72**, 415 (1947). See also dispersion theory calculation of M. Goldberger and S. Treiman, Nuovo Cimento **9**, 451 (1958).

[2] H. Primakoff, Phys. Rev. **81**, 899 (1951).

[3] A. V. Tollestrup, S. Berman, R. Gomez, and H. Ruderman, Proc. Ann. Intern. Conf. High Energy Phys. Rochester, 10 (1960), p. 27; H. Ruderman, S. Berman, R. Gomez, A. V. Tollestrup, and R. Talman, Bull. Am. Phys. Soc. **5**, 508 (1960); H. Ruderman, Ph.D. thesis, California Institute of Technology, 1962 (unpublished).

[4] R. F. Blackie, A. Engler, and J. H. Mulvey, Phys. Rev. Letters **5**, 384 (1960); R. G. Glasser, N. Seeman, and B. Stiller, Phys. Rev. **123**, 1014 (1961); H. Shwe, F. M. Smith, and W. H. Barkas, Phys. Rev. **125**, 1024 (1962); J. Tietge and W. Püschel, Phys. Rev. **127**, 1324 (1962); E. L. Koller, S. Taylor, and T. Huetter, Nuovo Cimento **27**, 1405 (1963).

[5] C. A. Engelbrecht, Ph.D. thesis, California Institute of Technology, December 1960 (unpublished).

The spatial parts of the matrix elements are calculated for an independent particle model of the nucleus in Sec. V. The coherent part of the cross section is proportional to the square of the form factor $F(p)$ while the incoherent part is suppressed by a factor $[1-G(p)]$ because of the exclusion principle and the nuclear forces. The Coulomb production is discussed in Sec. VI.

The final-state interactions of the produced pion are incorporated into the theory by a method equivalent to the Fernbach-Serber-Taylor model for neutron interactions in nuclei. This is described in Sec. VII. The main effects of the pion absorption are a reduction of the nuclear production and a change in shape of the Coulomb differential cross section. A final discussion of the results and shortcomings of the general approach is presented in Sec. VIII.

II. PHOTOPRODUCTION FROM A SINGLE NUCLEON

The photoproduction of a neutral pion from a single nucleon can be described by the transition amplitude

$$\langle f,\mathbf{q}|\mathrm{t}|i,\mathbf{k}\rangle. \qquad (2.1)$$

The initial state contains a nucleon in state i and a photon in a plane-wave state with momentum $\mathbf{k}$. The final state contains a nucleon in state f and a neutral pion in a plane-wave state with momentum $\mathbf{q}$. The photon and the pion are described in terms of quantized fields. The nucleon, however, is treated as a particle according to nonrelativistic quantum mechanics. The transition operator t for the complete system contains the photon and pion field operators A_μ and ϕ. The inner product (2.1) with respect to the photon and pion variables is easily found if the field operators are expanded in terms of a complete set of states which will in this case be a set of plane-wave states. For example,

$$\phi=\sum_q N_{\mathbf{q}}(a_{\mathbf{q}}e^{i\mathbf{q}\cdot\mathbf{x}}+a_{\mathbf{q}}^\dagger e^{-i\mathbf{q}\cdot\mathbf{x}})\,, \qquad (2.2)$$

where $a_{\mathbf{q}}^\dagger$ and $a_{\mathbf{q}}$ are creation and destruction operators for a pion in a plane-wave state with momentum $\mathbf{q}$ while $N_{\mathbf{q}}$ is a normalization constant. The result of such an expansion is that the transition amplitude becomes

$$\langle f|e^{-i\mathbf{q}\cdot\mathbf{x}}te^{i\mathbf{k}\cdot\mathbf{x}}|i\rangle\,, \qquad (2.3)$$

where $\mathbf{x}$ is the position operator for the nucleon while t is now a transition operator for the nucleon only. It no longer operates on the photon or the pion fields but does depend on their quantum numbers such as the polarization of the photon and the angle θ between the pion and photon momenta. The exponential factors guarantee the translational invariance of the transition amplitude. They play an important role in the coherent photoproduction from complex nuclei and also in the suppression of the incoherent production. For production from single nucleons, however, they can be omitted if at the same time $|i\rangle$ and $|f\rangle$ are taken to refer to the nucleon's spin-isospin states only, since all effects of the structure of the nucleon are contained in the single-nucleon transition operator t.

If all plane-wave states are normalized to unit density, the total cross section for photoproduction of a neutral pion from a nucleon in spin-isospin state $|i\rangle$ and with momentum $\mathbf{p}_i$, is given by

$$\sigma=\frac{2\pi}{v_i}\sum_f\int\frac{d^3q}{(2\pi)^3}|\langle f|t|i\rangle|^2\delta(E_i-E_f)\,, \qquad (2.4)$$

where $v_i=1-(\mathbf{k}\cdot\mathbf{p}_i)/[k(M^2+p_i^2)^{1/2}]$ is the incident flux while $E_i=k+(p_i^2/2M)$ and $E_f=\omega+(p_f^2/2M)$ are the values of the total energy in the initial and final states. We shall always be using units in which $\hbar=c=1$. Since the final nucleon momentum $\mathbf{p}_f=\mathbf{p}_i+\mathbf{k}-\mathbf{q}$ is independent of the final spin-isospin state, the sum over final spin-isospin states is simply $\sum_f|\langle f|t|i\rangle|^2=\langle i|t^\dagger t|i\rangle$. By further writing $d^3q=q^2dqd\Omega$, we obtain for the differential cross section

$$\frac{d\sigma}{d\Omega}=\frac{1}{v_i}\int\frac{q^2dq}{(2\pi)^2}\langle i|t^\dagger t|i\rangle\delta(E_i-E_f)=\Gamma\langle i|t^\dagger t|i\rangle\,, \qquad (2.5)$$

where all quantities on the right are to be calculated on the energy shell. In the center-of-mass system

$$\Gamma_{\mathrm{c.m.}}=(q\omega/4\pi^2)[1+k/(M^2+k^2)^{1/2}]^{-1}\times[1+\omega/(M^2+q^2)^{1/2}]^{-1}, \qquad (2.6)$$

while in the laboratory system Γ is just 2π times the usual density of states:

$$\Gamma_{\mathrm{lab}}=(q/2\pi)^2[(q/\omega)+(q-k\cos\theta)/M]^{-1}, \qquad (2.7)$$

For small angles θ this is approximately equal to the no-recoil value

$$\Gamma_0=q\omega/4\pi^2. \qquad (2.8)$$

When a neutral pion is produced, the most general form that the single nucleon transition operator can assume is

$$t=K+\mathbf{L}\cdot\boldsymbol{\sigma}+M\tau_3+\mathbf{N}\cdot\boldsymbol{\sigma}\tau_3\,, \qquad (2.9)$$

where $\boldsymbol{\sigma}$ is the nucleon spin operator, τ_3 is the 3-component of the nucleon isospin operator, while K, $\mathbf{L}$, M, and $\mathbf{N}$ may be functions of the nucleon momentum operator. The differential cross sections (2.5) in the center-of-mass system for photoproduction from free protons and free neutrons then become

$$\sigma_p^{\mathrm{c.m.}}=\Gamma_{\mathrm{c.m.}}(|K+M|^2+|\mathbf{L}+\mathbf{N}|^2)\,,$$
$$\sigma_n^{\mathrm{c.m.}}=\Gamma_{\mathrm{c.m.}}(|K-M|^2+|\mathbf{L}-\mathbf{N}|^2)\,. \qquad (2.10)$$

Because of the transversality of the photon and the conservation of angular momentum, it follows in general that K and M (as well as L_z and N_z) must always be proportional to $\sin\theta_{\mathrm{c.m.}}$. For this reason it will also prove convenient to define the amplitudes

$$f_p^{\mathrm{c.m.}}=(\Gamma_{\mathrm{c.m.}})^{1/2}(K+M)(\sin\theta_{\mathrm{c.m.}})^{-1}\,,$$
$$f_n^{\mathrm{c.m.}}=(\Gamma_{\mathrm{c.m.}})^{1/2}(K-M)(\sin\theta_{\mathrm{c.m.}})^{-1}. \qquad (2.11)$$

The differential cross sections in the laboratory system can be expressed in terms of those in the center-of-mass system by

$$\sigma_p^{\rm lab}(\theta_{\rm lab})=\left(\frac{d\Omega_{\rm c.m.}}{d\Omega_{\rm lab}}\right)\sigma_p^{\rm c.m.}(\theta_{\rm c.m.}), \qquad (2.12)$$

and similarly for σ_n. In particular,

$$(f_p^{\rm lab})^2\sin^2\theta_{\rm lab}=\left(\frac{d\Omega_{\rm c.m.}}{d\Omega_{\rm lab}}\right)(f_p^{\rm c.m.})^2\sin^2\theta_{\rm c.m.},$$

so that the transformation coefficients for the amplitudes of (2.11) contain an additional factor[6]

$$(f_p^{\rm lab})^2=\left(\frac{d\Omega_{\rm c.m.}}{d\Omega_{\rm lab}}\right)\left(\frac{\sin^2\theta_{\rm c.m.}}{\sin^2\theta_{\rm lab}}\right)(f_p^{\rm c.m.})^2. \qquad (2.13)$$

For small angles, both transformation coefficients $(d\Omega_{\rm c.m.}/d\Omega_{\rm lab})$ and $(\sin^2\theta_{\rm c.m.}/\sin^2\theta_{\rm lab})$ tend to the value

$$J=(1+u/\beta)^2/(1-u^2), \qquad (2.14)$$

where β is the pion velocity in the center-of-mass system while u is the center-of-mass velocity in the laboratory system. At 250 and 900 MeV, J has the values 1.699 and 2.980, respectively.

At photon energies below 500 MeV, the production of neutral pions is very strongly dominated by the $T=\frac{3}{2}$, $J=\frac{3}{2}^+$ resonance at 320 MeV, which is excited by magnetic dipole radiation.[7] This enables one to write down the angular dependence of $\sigma_p^{\rm c.m.}$ and $f_p^{\rm c.m.}$

$$\sigma_p^{\rm c.m.}=(5-3\cos^2\theta)m_1^2,$$
$$f_p^{\rm c.m.}=4m_1^2, \qquad (2.15)$$

where $m_1=-(\Gamma_{\rm c.m.}/8\pi)^{1/2}M_{13}$ is a resonant-energy-dependent factor. At 250 MeV, for example, $m_1^2=2.35\ \mu$b/sr. A further consequence which follows from the fact that it is the isovector component of the electromagnetic interaction which dominates,[7] is that M and N may be ignored so that $\sigma_n^{\rm c.m.}=\sigma_p^{\rm c.m.}$ and $f_n^{\rm c.m.}=f_p^{\rm c.m.}$.

At higher energies the measured production cross section from protons[8] exhibits peaks at photon energies of 750 and 1050 MeV. These are presumably due to resonances in the $T=\frac{1}{2}$ isospin state. The lower one (second πN resonance) almost certainly has $J=\frac{3}{2}^-$ and could therefore be excited by an E_{13} or an M_{23} amplitude. If the third resonance does have $J=\frac{5}{2}^+$, it can be excited by an E_{25} or an M_{35} amplitude. At high energies there is as yet no theory which is as convincing as the Chew-Low theory[7] for low-energy phenomena, so that it is necessary to rely almost entirely on phenomenological analysis. Fortunately, it appears that the measured angular distributions[8] can be fitted fairly well by using only an E_{13} and an E_{25} amplitude for the second and third resonances, respectively.[9] Defining $e_1=(\Gamma/32\pi)^{1/2}E_{13}$ and $e_2=(3\Gamma/32\pi)^{1/2}E_{25}$, one obtains a reasonable fit to the 950-MeV angular distribution using phase differences $\varphi_{12}=60°$, $\varphi_{23}=30°$, $\varphi_{13}=90°$, and values 0.120, 0.018, and 0.160 μb/sr for m_1^2, e_1^2, and e_2^2, respectively. The corresponding values at 900 MeV are 0.140, 0.038, and 0.096 μb/sr, which leads to values

$$\sigma_p^{\rm c.m.}\approx 0.22\ \mu{\rm b/sr},$$
$$(f_p^{\rm c.m.})^2\approx 0.56\ \mu{\rm b/sr}, \qquad (2.16)$$

in the forward direction. If the parity of the third resonance is not positive but negative, the E_{25} amplitude should be replaced by an M_{25} amplitude. This would increase the value of $(f_p^{\rm c.m.})^2$ in the forward direction to approximately 4.0 μb/sr. It should be emphasized that these fits are not unique and that the experimental results could also be fitted with other combinations of electric and magnetic multipoles.

Although phenomenological analysis enables one to obtain some information about the production amplitudes from protons, the absence of a dynamical theory makes it difficult to say much about the production from neutrons. This derives from the fact that the photons have both an isoscalar and an isovector coupling. The isovector coupling leads to $M=0$, $N=0$, and hence $f_n^{\rm c.m.}=f_p^{\rm c.m.}$. The isoscalar coupling leads to $K=0$, $L=0$, and hence $f_n^{\rm c.m.}=-f_p^{\rm c.m.}$. In both cases $\sigma_n^{\rm c.m.}=\sigma_p^{\rm c.m.}$. When both couplings are present, however, there is no relation between $f_n^{\rm c.m.}$ and $f_p^{\rm c.m.}$ or between $\sigma_n^{\rm c.m.}$ and $\sigma_p^{\rm c.m.}$. Given $f_p^{\rm c.m.}$, for example, $f_n^{\rm c.m.}$ could be made arbitrarily large.

Furthermore, at high energies the exchange of vector mesons will begin to make an important contribution to the photoproduction of neutral pions. This will, in the first place, drastically modify the analysis of the production from protons discussed above since the exchange term would essentially contain spherical harmonics of all orders, unlike the multipole terms. In addition, the exchange of vector mesons of definite isospin would lead to relations between the production from neutrons and protons. The exchange of a $T=0$ meson (like the ω) would lead to $f_n^{\rm c.m.}=f_p^{\rm c.m.}$ and the exchange of a $T=1$ (like the ρ) to $f_n^{\rm c.m.}=-f_p^{\rm c.m.}$.

Although the properties of the production amplitudes at high energies are thus still far from known, it is in principle possible to determine these properties completely by the careful study of angular distributions

[6] This fact was overlooked in Ref. 5 and later pointed out to the author by Dr. A. V. Tollestrup.

[7] K. A. Brueckner, Phys. Rev. **86**, 106 (1952); M. Gell-Mann and K. M. Watson, Ann. Rev. Nucl. Sci. **4**, 267 (1954); G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570, 1579 (1956).

[8] J. I. Vette, Phys. Rev. **111**, 622 (1958); K. Berkelman and J. A. Waggoner, Phys. Rev. **117**, 1364 (1960); R. M. Talman, C. R. Clinesmith, R. Gomez, and A. V. Tollestrup, Bull. Am. Phys. Soc. **7**, 265 (1962); Phys. Rev. Letters **9**, 177 (1962).

[9] R. F. Peierls, Phys. Rev. **118**, 325 (1960), first suggested that the second πN resonance is excited by an electric dipole amplitude. A more recent phenomenological analysis by the present author indicated that the best fits to the angular distributions near the second and third resonances are obtained assuming E_{13} and E_{25} (or M_{25} if the parity is negative) amplitudes, respectively.

and polarizations involved in the photoproduction of neutral pions from single nucleons. For the purpose of the present work, these quantities will be assumed to be known, and the photoproduction from nuclei will be expressed in terms of them, and the expectation values of certain operators in the nuclear ground state.

III. PHOTOPRODUCTION FROM A COMPLEX NUCLEUS

The total cross section for photoproduction of a neutral pion from a complex nucleus can also be written in the form (2.4), except that the single-nucleon transition operator t must be replaced by a transition operator T for the whole nucleus. All possible internal states of the final nucleus are explicitly summed over, while the integral over pion momenta is equivalent to a summation over all relative states of the final pion-nucleus system which are energetically allowed. Unlike the single nucleon case, the total energy E_f in the final state is no longer independent of the internal states f of the final nucleus. The delta function must, therefore, also be included in the closure relation when we sum over f. The result is that the differential cross section in the laboratory system ($v_i=1$) becomes

$$\frac{d\sigma}{d\Omega}=\int \frac{q^2dq}{(2\pi)^2}\langle 0|T^\dagger\delta(E_i-H_0)T|0\rangle\,. \qquad (3.1)$$

The argument of the delta function can be written as

$$E_i-H_0=k+W_0-H_A-\omega\,, \qquad (3.2)$$

where k and ω are the photon and pion energies, H_A is the nuclear Hamiltonian, while W_0 is the energy of the nuclear ground state $|0\rangle$.

This expression for the nuclear cross section is still completely general and exact. It is, however, so general that it contains practically no specific information about the process which we consider; more or less it merely defines an operator T. The main purpose of the present work is to see if it may not be possible to express the nuclear cross section in terms of simpler quantities which are, at least in principle, already known. Such a possibility exists at energies where the wavelength of the incident photon becomes comparable to the distances between nucleons. In this energy region the direct interaction model of Serber,[10] according to which the interaction of the photon with the complex nucleus is expressed in terms of its interactions with the individual nucleons, can be used. The nuclear transition operator is written as a sum of single-nucleon operators

$$T=\sum_{n=1}^{A} e^{i\mathbf{p}\cdot\mathbf{x}_n}t_n\,, \qquad (3.3)$$

where $\mathbf{x}_n$ is the spatial coordinate of nucleon n while $\mathbf{p}$ is the momentum transfer $\mathbf{k}-\mathbf{q}$. The exponential factor, which guarantees translational invariance, was discussed in Sec. II.

The transition operator t for a single bound nucleon which occurs in (3.3), is not identical with the transition operator t_0 for a single free nucleon which was discussed in the previous section. When the binding potential U_B experienced by a single nucleon is small, t can, however, be expressed in terms of a series

$$t=t_0+t_0a_0^{-1}U_Ba_0^{-1}t_0+\text{higher order terms},$$

where $a_0=\epsilon-h+i\eta$, h being the total kinetic Hamiltonian, ϵ the total kinetic energy, and η the usual infinitesimal which determines the boundary conditions. When the expectation value of U_B ($\approx$40 MeV) is small compared to ϵ, the difference between t and t_0 becomes negligible.[11] We shall henceforth assume the validity of the impulse approximation[11] according to which the transition operator for a bound nucleon is identical to that for a free nucleon. The nuclear transition operator T is thus expressed in terms of quantities which can be measured by the study of single nucleons.

After substituting (3.3) into (3.1), the differential cross section can be separated[12] into two parts,

$$\frac{d\sigma}{d\Omega}=\left(\frac{d\sigma}{d\Omega}\right)_D+\left(\frac{d\sigma}{d\Omega}\right)_{ND} \qquad (3.4)$$

which are, respectively, diagonal and nondiagonal in the nucleon indices:

$$\left(\frac{d\sigma}{d\Omega}\right)_D=\int\frac{q^2dq}{4\pi^2}\sum_{n=1}^{A}\langle 0|t_n^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}_n}\delta(E_i-H_0)\times e^{i\mathbf{p}\cdot\mathbf{x}_n}t_n|0\rangle, \qquad (3.5)$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{ND}=\int\frac{q^2dq}{4\pi^2}\sum\sum_{n\neq m}\langle 0|t_m^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}_m}\delta(E_i-H_0)\times e^{i\mathbf{p}\cdot\mathbf{x}_n}t_n|0\rangle. \qquad (3.6)$$

The sum over final nuclear states contained in (3.1) was performed by exact closure. In order to be able to calculate (3.5) and (3.6), the delta functions have to be removed from inside the matrix elements. This will be done by means of a closure approximation developed by Placzek and Wick[13] in their treatments of the scattering of neutrons by molecular systems and later used by Fowler and Watson[14] in connection with the scattering of elementary particles by nuclei.

In the diagonal contribution (3.5) the aim is to commute the delta function through the factor $e^{i\mathbf{p}\cdot\mathbf{x}_n}t_n$ to the right so that it can operate directly on the ground state $|0\rangle$ while the two exponentials just cancel to give

[10] R. Serber, Phys. Rev. **72**, 1114 (1947).

[11] G. F. Chew, Phys. Rev. **80**, 196 (1950); G. F. Chew and M. L. Goldberger, *ibid.* **87**, 778 (1952).

[12] M. Lax and H. Feshbach, Phys. Rev. **81**, 189 (1951).

[13] G. Placzek, Phys. Rev. **86**, 377 (1952); G. C. Wick, *ibid.* **94**, 1228 (1954).

[14] T. K. Fowler and K. M. Watson, Nucl. Phys. **13**, 549 (1959).

unity. The only operator contained in the delta function is the nuclear Hamiltonian H_A. After commuting, the delta function will therefore contain additional terms due to the fact that H_A does not commute with $e^{i\mathbf{p}\cdot\mathbf{x}_n}t_n$. Terms depending on the nuclear binding will be generated by the momentum, spin, and isospin dependence of t_n and by the space exchange and momentum-dependent potentials in H_A. The magnitudes of these contributions have been estimated[14] and found to be small. The only contribution which will be kept, is the one which arises from commutation of the kinetic energy operator with the exponential:

$$H_A e^{i\mathbf{p}\cdot\mathbf{x}_n}=e^{i\mathbf{p}\cdot\mathbf{x}_n}[H_A+(p^2/2M)+\mathbf{p}\cdot\nabla_n/iM].$$

After Fourier-analyzing the nuclear ground state in terms of an initial single nucleon momentum p_i, one can now carry out the operations implied in this expression (for details see Refs. 5 and 14) and write the diagonal part of the differential cross section in terms of the momentum distribution $P(p_i)$ in the ground state:

$$\left(\frac{d\sigma}{d\Omega}\right)_D=A\int\frac{d^3p_i}{(2\pi)^3}P(p_i)\int\frac{q^2dq}{(2\pi)^2}\times\langle 0|t_1^\dagger t_1|0\rangle\delta\left(k+\frac{p_i^2}{2M}-\omega-\frac{p_f^2}{2M}\right). \tag{3.7}$$

Here $\mathbf{p}_f=\mathbf{p}_i+\mathbf{p}$ is the final momentum of the nucleon.

When (3.7) is compared to (2.5) it becomes clear that the diagonal cross section is, apart from the flux factor in (2.5), just A times the free nucleon cross section, averaged over the spin and momentum states which are encountered in the nuclear ground state. As far as the one-particle contribution is concerned, each nucleon therefore behaves like a free independent particle, the only effect of the other nucleons appearing in the specification of its initial state. The consequences of the exclusion principle, as well as all other coherence effects, are still contained in the nondiagonal or two-particle contribution (3.6).

The approximation which is made in the nondiagonal cross section, is to neglect the correction terms arising from the momentum dependence of the transition operators and to replace H_A in the delta function by W_0. The validity of the first approximation is discussed in Ref. 14 while in Ref. 5, it is shown that the second approximation is exact as far as the noncorrelated part of the cross section is concerned while for the part which depends on correlation between the motion of the nucleons, it is a good approximation. We can carry out the sum over m and n and express the nondiagonal cross section (3.6) in terms of the transition operators for nucleons 1 and 2

$$\left(\frac{d\sigma}{d\Omega}\right)_{ND}=A(A-1)\int\frac{q^2dq}{(2\pi)^2}\langle 0|e^{i\mathbf{p}\cdot(\mathbf{x}_1-\mathbf{x}_2)}\times t_2^\dagger t_1|0\rangle\delta(k-\omega). \tag{3.8}$$

The matrix element which remains, will be calculated in the next two sections.

The kinematics for the diagonal cross section (3.7) is the same as that for photoproduction from a free nucleon, except for a flux factor and the fact that the cross section should be averaged over the momentum distribution in the nucleus. In the noncorrelated part of the nondiagonal cross section, the nucleus remains in its ground state and recoils as a whole. Only for the correlated part of the nondiagonal cross section the kinematical relations are not precisely defined by this approximation. In order to ensure the proper cancellation properties due to the exclusion principle, it seems very likely that the correlated cross section is subject to the same kinematical relations as the diagonal cross section, at least for small momentum transfers.

The momentum transfer to the nucleus is given by

$$p^2=(k-q)^2+2kq(1-\cos\theta)\approx(\Delta+\epsilon)^2+k^2\theta^2, \tag{3.9}$$

where θ is the angle between k and q and where the second expression holds approximately when $\theta\ll 1$. Δ is the nuclear excitation energy. When k is large compared to the pion mass m, ϵ is to a good approximation equal to $m^2/2k$. For $k=250$ and 900 MeV, ϵ takes on the values 40 and 11 MeV, respectively. The value of the momentum transfer is not very sensitive to the nuclear excitation energy. From now on, no distinction will be made between the kinematics pertaining to the different parts of the cross section. The transition amplitudes will be calculated at the energies prescribed in the previous paragraph. Where p occurs in the nuclear form factors, the value given in (3.9) with $\Delta=0$ will be used. The cross sections (3.7) and (3.8) will meanwhile be written

$$\left(\frac{d\sigma}{d\Omega}\right)_D=A\Gamma\langle 0|t_1^\dagger t_1|0\rangle, \tag{3.10}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{ND}=A(A-1)\Gamma\langle 0|e^{i\mathbf{p}\cdot(\mathbf{x}_1-\mathbf{x}_2)}t_2^\dagger t_1|0\rangle. \tag{3.11}$$

Towards the end of Sec. V they will be expressed in terms of quantities σ_p, σ_n, f_p, and f_n which are very closely related to the single-nucleon cross sections and amplitudes in the laboratory system defined in Eqs. (2.12) and (2.13). The amplitudes f_p and f_n differ from f_p^{lab} and f_n^{lab} by a factor $(\Gamma_0/\Gamma_{\text{lab}})^{1/2}$ which reflects the kinematics of the coherent nuclear cross section. In the case of σ_p and σ_n, on the other hand, the free-nucleon differential cross sections σ_p^{lab} and σ_n^{lab} should be averaged over the nucleon-momentum distribution in accordance with (3.7). This averaging process which is required, causes an attenuation of the inelastic nuclear cross section near the threshold for pion production.[12] For the purposes of the present work, neither of the distinctions is very important, however, so that the approximate expressions

$$\sigma_p = J\sigma_p^{\text{c.m.}}, \quad (3.12)$$

$$f_p = Jf_p^{\text{c.m.}}, \quad (3.13)$$

(and similarly for σ_n and f_n) will be used to relate the σ's and f's to the experimentally determined single-nucleon cross sections.

IV. MATRIX ELEMENTS OF TWO-NUCLEON OPERATORS

In this section we shall discuss the evaluation of matrix elements of the form $\langle\Psi|R_{12}O_{12}|\Psi\rangle$, where R_{12} contains the position operators of nucleons 1 and 2 only, O_{12} contains the spin and isospin operators of nucleons 1 and 2 only, while Ψ is a definite state of the nucleus. By means of the permutation operators

$$P_s = \tfrac{1}{2}(1+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2),$$
$$P_t = \tfrac{1}{2}(1+\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2), \quad (4.1)$$

we can define a complete set of orthogonal projections

$$D_1 = D_{ss} = \tfrac{1}{4}(1+P_s)(1-P_t),$$
$$D_2 = D_{sa} = \tfrac{1}{4}(1-P_s)(1+P_t),$$
$$D_3 = D_{as} = \tfrac{1}{4}(1+P_s)(1+P_t), \quad (4.2)$$
$$D_4 = D_{aa} = \tfrac{1}{4}(1-P_s)(1-P_t).$$

The indices denote the space and spin symmetry. For example D_{sa} projects any state of the nucleus onto a state which is symmetric with respect to interchange of the spatial coordinates of nucleons 1 and 2 but antisymmetric with respect to spin-exchange (and hence also symmetric with respect to isospin-exchange). Since these projection operators satisfy the usual relations

$$\textstyle\sum_\lambda D_\lambda = 1, \qquad D_\lambda D_\mu = \delta_{\lambda\mu} D_\lambda, \quad (4.3)$$

the matrix element under consideration can be expressed as

$$\langle\Psi|R_{12}O_{12}|\Psi\rangle = \textstyle\sum_\lambda \langle\Psi|R_{12}D_\lambda D_\lambda O_{12}|\Psi\rangle. \quad (4.4)$$

For purposes of the present discussion we are not interested in aspects of nuclear structure which depend sensitively on specific details of the nuclear model. For this reason we shall only consider nuclei which can be described by a single Slater determinant

$$\Psi = (A!)^{-1/2} \textstyle\sum_P (-1)^P \phi_i(1)\cdots\phi_k(A), \quad (4.5)$$

rather than a sum of such determinants. The set of single nucleon wave functions $\{\phi_1,\cdots\phi_A\}$ which are occupied, must therefore be unique and each spatial state must be occupied by four nucleons corresponding to the four possible combinations of spin and isospin projections. Nuclei which satisfy these requirements will be referred to as closed-shell nuclei. They obviously have $S=L=J=T=0$.

By inserting a complete set of states between the two projection operators in (4.4), one obtains

$$\langle\Psi|R_{12}O_{12}|\Psi\rangle = \textstyle\sum_\lambda\sum_n \langle\Psi|R_{12}D_\lambda|\Phi_n\rangle \times\langle\Phi_n|D_\lambda O_{12}|\Psi\rangle. \quad (4.6)$$

At first sight it may seem as if the proper set of orthonormal states Φ_n to use is the set of all possible nuclear wave functions which are completely antisymmetric with respect to all A nucleons. For the restrictive set of Ψ which we consider, this would then imply that $\langle\Phi_n|D_\lambda O_{12}|\Phi_n\rangle$ vanishes unless $\Phi_n=\Psi$. Hence

$$\langle\Psi|R_{12}O_{12}|\Psi\rangle = \textstyle\sum_\lambda \langle\Psi|R_{12}D_\lambda|\Psi\rangle\langle\Psi|D_\lambda O_{12}|\Psi\rangle. \quad (4.7a)$$

This relation looks very plausible and is, as a matter of fact, implicity contained in the work of Fowler and Watson.[14] One can easily show, however, that it must actually be completely wrong for all nuclei with mass number greater than 4.

If we choose $R_{12}=O_{12}=1$, (4.7a) becomes

$$\langle\Psi|\Psi\rangle = \textstyle\sum_\lambda \langle\Psi|D_\lambda|\Psi\rangle\langle\Psi|D_\lambda|\Psi\rangle,$$

which is inconsistent with (4.3) unless Ψ is a simultaneous eigenstate of all the D_λ.

The source of the trouble is the fact that the set of Φ_n chosen is in fact not complete. Instead of states which are antisymmetric with respect to all A nucleons, one should use states which are antisymmetric with respect to nucleons 1 and 2 and completely antisymmetric with respect to the other A-2 nucleons, but which have no definite symmetry between nucleon 1, say, and any of the A-2. It is convenient to construct these states in such a way that they are eigenstates of the projection operators D_λ.

In order to evaluate the matrix element (4.6), it is helpful to expand Ψ in terms of these Φ_n also. By means of simple but somewhat laborious arguments (see Ref. 5) it can eventually be shown that

$$\langle\Psi|R_{12}O_{12}|\Psi\rangle = \sum_\lambda \frac{\langle\Psi|R_{12}D_\lambda|\Psi\rangle\langle\Psi|D_\lambda O_{12}|\Psi\rangle}{\langle\Psi|D_\lambda|\Psi\rangle}, \quad (4.7)$$

which will, in general, differ from (4.7a). Equation (4.7) is exact for closed-shell nuclei. For other nuclei the correction terms are of order n/A where n is the number of nucleons outside the closed shells.

In the previous section, a matrix element of the form (4.7) was encountered with $R_{12}=e^{i\mathbf{p}\cdot(\mathbf{x}_1-\mathbf{x}_2)}$ and $O_{12}=t_2{}^\dagger t_1$. The spatial elements $\langle 0|R_{12}D_\lambda|0\rangle$ will be evaluated in the next section. For the present we shall confine our attention to the remaining quotients

$$t_\lambda{}^2 \equiv \langle 0|D_\lambda t_2{}^\dagger t_1|0\rangle / \langle 0|D_\lambda|0\rangle. \quad (4.8)$$

Because of the restriction to states with vanishing total angular momentum and isospin, only operators which transform like scalars in ordinary space as well as in isospace, can have nonvanishing expectation values. This follows directly from the Wigner-Eckart theorem. The projection operators D_λ already transform like scalars to that we need to consider the scalar parts of $t_2^\dagger t_1$ only. Recalling that $t_n = K + \mathbf{L}\cdot\boldsymbol{\sigma}_n + M\tau_{3n} + \tau_{3n}\mathbf{L}\cdot\boldsymbol{\sigma}_n$, one can without much trouble decompose $t_2^\dagger t_1$ into parts which transform like irreducible tensors of rank 0, 1, and 2, both in ordinary space and in isospace. The scalar part is given by

$$|K|^2+\tfrac{1}{3}|L|^2(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)+\tfrac{1}{3}|M|^2(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)+\tfrac{1}{9}|N|^2 \times(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\,. \quad (4.9)$$

By using (4.1), (4.2), and (4.9), one can now easily write down the values of (4.8) corresponding to the four different λ:

$$\begin{aligned} t_{ss}^2 &= |K|^2+\tfrac{1}{3}|L|^2-\ |M|^2-\tfrac{1}{3}|N|^2,\\ t_{sa}^2 &= |K|^2-\ |L|^2+\tfrac{1}{3}|M|^2-\tfrac{1}{3}|N|^2,\\ t_{as}^2 &= |K|^2+\tfrac{1}{3}|L|^2+\tfrac{1}{3}|M|^2+\tfrac{1}{9}|N|^2,\\ t_{aa}^2 &= |K|^2-\ |L|^2-\ |M|^2+\ |N|^2. \end{aligned} \quad (4.10)$$

The total two-particle matrix element which appeared in Eq. (3.11), is finally given by

$$\langle 0|R_{12}t_2^\dagger t_1|0\rangle = \sum_\lambda t_\lambda^2\langle 0|R_{12}D_\lambda|0\rangle\,. \quad (4.11)$$

In Eq. (3.10) we also encountered a one-particle matrix element $\langle 0|t_1^\dagger t_1|0\rangle$. The scalar part of $t_1^\dagger t_1$ is given by

$$|K|^2+|L|^2+|M|^2+|N|^2\,.$$

This is the only part which has a nonvanishing expectation value in the closed-shell states so that we can write

$$\langle 0|t_1^\dagger t_1|0\rangle = |K|^2+|L|^2+|M|^2+|N|^2. \quad (4.12)$$

By combining the spin-isospin matrix elements (4.10) with the spatial matrix elements (5.7) calculated in the next section, we shall obtain a simple relationship between the one-particle and two-particle contributions to the cross section. In the limit of vanishing momentum transfer p, the radial operator R_{12} becomes unity and this simple relation can be expressed as

$$\begin{aligned} A(A-1)\langle 0|t_2^\dagger t_1|0\rangle &= A(A-1)\sum_\lambda t_\lambda^2\langle 0|D_\lambda|0\rangle\\ &= A^2|K|^2 - A\{|K|^2+|L|^2+|M|^2+|N|^2\}\\ &= A^2|K|^2 - A\langle 0|t_1^\dagger t_1|0\rangle\,. \end{aligned} \quad (4.13)$$

For nuclei which do not satisfy the restrictive demands which we imposed, the separation (4.7) of spatial and spin-isospin matrix elements is not generally valid. In the limiting case where the radial operator becomes unity, however, a relation analogous to (4.13) can be derived for a general nucleus in terms of the partition quantum numbers T, S, Y (also denoted by P, P', P'') introduced by Wigner[15] to characterize the symmetry properties of the nucleus. In the cases of interest $2T$ can be identified with the eigenvalue of the operator $(-\sum_i \tau_3^i)$, usually called the neutron excess $I=A-2Z$. The more general relation is

$$A(A-1)\langle 0|t_2^\dagger t_1|0\rangle = |AK-IM|^2+\mathfrak{M}_2+\mathfrak{M}_1 - A\langle 0|t_1^\dagger t_1|0\rangle\,, \quad (4.14)$$

where

$$A\langle 0|t_1^\dagger t_1|0\rangle = A\{|K|^2+|L|^2+|M|^2+|N|^2\} - 2I\,\mathcal{R}(K^*M+\mathbf{L}^*\cdot\mathbf{N}), \quad (4.15)$$

$$\mathfrak{M}_1 = 2S\{|\mathbf{L}-\mathbf{N}|^2-|L_z-N_z|^2\}+4(S-Y)\times(\mathbf{L}^*\cdot\mathbf{N}-L_z^*N_z), \quad (4.16)$$

$$\mathfrak{M}_2 = |2SL_z-2YN_z|^2. \quad (4.17)$$

Here $\mathcal{R}(z)$ denotes the real part of z. When the operators $(\sum_i\sigma_z^i)$ and $(-\sum_i\sigma_z^i\tau_3^i)$ are diagonal, they may usually be identified with $2S$ and $2Y$, respectively. The quantities $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are strongly structure-dependent. To a good approximation we can assume that they vanish for even-even nuclei while for odd-A nuclei $\mathfrak{M}_1+\mathfrak{M}_2=|\mathbf{L}\pm\mathbf{N}|^2$ depending on whether Z or N is odd. Unless we specify otherwise, the discussion will henceforth be limited to the closed-shell nuclei for which (4.13) holds.

V. NUCLEAR FORM FACTORS

For the calculation of the spatial matrix elements it is convenient to introduce two-particle densities

$$\rho_\lambda(\mathbf{x}_1,\mathbf{x}_2) = \sum_s \int \Psi^*(1\cdots A)D_\lambda\Psi(1\cdots A)d^3x_3\cdots d^3x_A\,,$$
$$\Psi(1\cdots A) = \langle \mathbf{x}_1\cdots\mathbf{x}_A|0\rangle\,, \quad (5.1)$$

where $\sum_s$ indicates that the inner product with respect to all spin-isospin variables must be taken. With this definition

$$\langle 0|e^{i\mathbf{p}\cdot(\mathbf{x}_1-\mathbf{x}_2)}D_\lambda|0\rangle = \int \rho_\lambda(\mathbf{x}_1,\mathbf{x}_2)e^{i\mathbf{p}\cdot(\mathbf{x}_1-\mathbf{x}_2)}d^3x_1d^3x_2. \quad (5.2)$$

If we once again confine our attention to the closed-shell nuclei described in Sec. IV, the two-particle densities can be expressed in terms of the "mixed density"

$$d(\mathbf{x}_1,\mathbf{x}_2) = \sum_{j=1}^{A} \varphi_j(\mathbf{x}_1)\varphi_j^*(\mathbf{x}_2)\,. \quad (5.3)$$

Each single-nucleon spatial wave function appears in the sum four times. The single-nucleon density is given by

$$\rho(\mathbf{x}) = A^{-1}d(\mathbf{x},\mathbf{x})\,. \quad (5.4)$$

We also define a correlation function

$$h(\mathbf{x}_1,\mathbf{x}_2) = \frac{1}{4}\frac{d(\mathbf{x}_1,\mathbf{x}_2)d(\mathbf{x}_2,\mathbf{x}_1)}{d(\mathbf{x}_1,\mathbf{x}_1)d(\mathbf{x}_2,\mathbf{x}_2)}\,, \quad (5.5)$$

which is a manifestation of the way in which the position

[15] E. P. Wigner, Phys. Rev. 51, 106 (1937).

of nucleon 1 is influenced by the position of nucleon 2 (and vice versa) due to the exclusion principle and the nuclear forces. If the nucleons had been completely independent, the correlation function would have had the constant value $-1/A$.

With these definitions, the two-particle densities become

$$\rho_\lambda(\mathbf{x}_1,\mathbf{x}_2)=[1-(1/A)]^{-1}p_\lambda\rho(\mathbf{x}_1)\rho(\mathbf{x}_2)[1\pm 4h(\mathbf{x}_1,\mathbf{x}_2)], \quad (5.6)$$

where $p_{ss}=p_{sa}=\frac{3}{16}$, $p_{as}=\frac{9}{16}$, and $p_{aa}=\frac{1}{16}$. The upper sign holds for ρ_{ss} and ρ_{sa}, and the lower sign for ρ_{as} and ρ_{aa}. Substituting back into (5.2), one obtains the following equation:

$$A(A-1)\langle 0|e^{i\mathbf{p}\cdot(\mathbf{x}_1-\mathbf{x}_2)}D_\lambda|0\rangle = p_\lambda\{A^2|F(p)|^2\pm 4AG(p)\}, \quad (5.7)$$

with the same assignment of signs as before. The form factors

$$F(p)=\int \rho(\mathbf{x})e^{i\mathbf{p}\cdot\mathbf{x}}d^3x, \quad (5.8)$$

$$G(p)=A\int \rho(\mathbf{x}_1)\rho(\mathbf{x}_2)h(\mathbf{x}_1,\mathbf{x}_2)e^{i\mathbf{p}\cdot(\mathbf{x}_1-\mathbf{x}_2)}d^3x_1d^3x_2, \quad (5.9)$$

tend to zero when p tends to infinity and become unity when p vanishes.

Combining (3.11), (4.10), (4.11), and (5.7), one obtains for the two-particle contribution to the photoproduction cross section the value

$$\left(\frac{d\sigma}{d\Omega}\right)_{ND}=\left(\frac{d\sigma}{d\Omega}\right)_{NC}-\left(\frac{d\sigma}{d\Omega}\right)_{C}, \quad (5.10)$$

where the noncorrelated part is given by

$$\left(\frac{d\sigma}{d\Omega}\right)_{NC}=A^2\Gamma|F(p)|^2|K|^2, \quad (5.11)$$

while the correlated part is given by

$$\left(\frac{d\sigma}{d\Omega}\right)_{C}=A\Gamma G(p)(|K|^2+|L|^2+|M|^2+|N|^2). \quad (5.12)$$

The differential cross section for elastic photoproduction, where the nucleus remains in its ground state, is given by

$$\left(\frac{d\sigma}{d\Omega}\right)=\Gamma|\langle 0|\ \textstyle\sum_n t_n e^{i\mathbf{p}\cdot\mathbf{x}_n}|0\rangle|^2. \quad (5.13)$$

It is easy to see that this is exactly equal to (5.11) so that the noncorrelated cross section is simply the elastic cross section where the nucleus remains in its ground state and where the production amplitudes from different nucleons are added coherently. In Sec. II we saw that K must always contain a factor $\sin\theta$. On the other hand, $F(p)$ decreases with p and hence, according to (3.9), with θ. Thus, the coherent cross section should exhibit a strong peak. This occurs at an angle

$$\theta_{\text{peak}}\approx 2/kR, \quad (5.14)$$

where R is the nuclear radius.

If the distribution of nucleons in the nucleus is assumed to be uniform inside a sphere of radius R, and zero outside

$$\begin{aligned}\rho(r)&=3/4\pi R^3 \qquad r<R,\\ &=0 \qquad\qquad r>R;\end{aligned} \quad (5.15)$$

the form factor $F(p)$ is given by

$$F(p)=\mathcal{L}(pR), \quad (5.16)$$

$$\mathcal{L}(x)=3[\sin x-x\cos x]/x^3=3j_1(x)/x. \quad (5.17)$$

The function $\mathcal{L}(x)$ begins like $1-(x^2/10)$ for small x and then oscillates with decreasing amplitude, the first zero occurring at $x=4.49$. Of course (5.15) is not a very good approximation to the nuclear density. When a distribution with a diffuse edge is used, significant deviations from (5.16) do not appear, however, until far beyond the peak (5.14) and shortly before the first zero. Since the final-state interactions will smear out the minima in the diffraction pattern anyway, it is unnecessary to use a more realistic density than (5.15).

The second form factor $G(p)$ will first be calculated using the Fermi gas model for infinite nuclear matter. The single-nucleon wave functions are plane waves $(1/V^{1/2})e^{i\mathbf{k}\cdot\mathbf{x}}$, where V is the normalization volume. If A is the number of nucleons contained in V, the Fermi momentum p_F is given by

$$p_F^3=3\pi^2A/2V, \quad (5.18)$$

The mixed density and the correlation function become

$$\begin{aligned}d(\mathbf{x}_1,\mathbf{x}_2)&=(A/V)\mathcal{L}(p_Fr),\\ h(\mathbf{x}_1,\mathbf{x}_2)&=\tfrac{1}{4}\mathcal{L}^2(p_Fr),\end{aligned} \quad (5.19)$$

and the form factor

$$\begin{aligned}G(p)&=1-\tfrac{3}{4}(p/p_F)+\tfrac{1}{16}(p/p_F)^3 \qquad p<2p_F,\\ &=0 \qquad\qquad p>2p_F.\end{aligned} \quad (5.20)$$

By combining Eqs. (3.10), (4.12), and (5.12), one obtains the interesting result that

$$\left(\frac{d\sigma}{d\Omega}\right)_{D}-\left(\frac{d\sigma}{d\Omega}\right)_{C}=A\Gamma[1-G(p)]\times(|K|^2+|L|^2+|M|^2+|N|^2) \quad (5.21)$$

since the two terms on the left have the same dependence on the transition amplitude. The correlated cross section thus has the effect of partially canceling the diagonal cross section by means of the suppression factor

$$\begin{aligned} 1-G(p)&=\tfrac{3}{4}(p/p_F)-\tfrac{1}{16}(p/p_F)^3 \qquad p<2p_F\,,\\ &=1 \qquad p>2p_F\,. \end{aligned}\tag{5.22}$$

In the Fermi gas model all the nuclear correlation is due to the exclusion principle so that this must also be the source of the attenuation. By allowing only the part of the Fermi sphere which lies outside another Fermi sphere whose center is displaced by a distance p, one indeed obtains exactly the suppression factor (5.22). From now on the combined cross section (5.21) will be called the incoherent cross section while (5.11) will be referred to as the coherent cross section.

The suppression factor for finite nuclei presumably differs from that for nuclear matter. The place where this difference should be the most significant, is in the light nuclei. In order to calculate $G(p)$, one has to know the single-particle wave functions. As an example, we shall calculate the form factor for O^{16}, using an independent-particle model with harmonic oscillator wave functions. The mixed density is

$$d(\mathbf{x}_1,\mathbf{x}_2)=4(\alpha/\pi^{1/2})^3[1+2\alpha^2\mathbf{x}_1\cdot\mathbf{x}_2]\times\exp[-\tfrac{1}{2}\alpha^2(x_1^2+x_2^2)]\,,\tag{5.23}$$

where the parameter α is related to the distance E between energy levels and the radius R of the equivalent uniform nucleus (5.15) by

$$\alpha^2=ME=15/(4R^2)\,,\tag{5.24}$$

M being the nucleon mass. The form factor becomes

$$G(p)=[1+(p/2\alpha)^4]\exp[-p^2/2\alpha^2]\,.\tag{5.25}$$

For small values of p the suppression factor is

$$1-G(p)\approx 0.133\ (pR)^2\approx 1.9(p/p_F)^2\,.\tag{5.26}$$

For small p, the finite nucleus suppression factor is much smaller than that for nuclear matter, which implies that the exclusion principle is more effective in a finite nucleus. The ratio of (5.26) to (5.22) rises almost linearly until $p\approx 120$ MeV/c and then remains within 5% of unity. If one considers He^4 instead of O^{16}, the same general behavior is found except that the nuclear matter suppression factor is not approached until $p\approx 200$ MeV/c which confirms the conjecture that the agreement of the actual suppression factor with (5.22) should improve with increasing A. Based on these two results only, one can perhaps use the following very crude estimate of the ratio between actual suppression factor and (5.22):

$$\begin{aligned} \frac{[1-G(p)]_A}{[1-G(p)]_\infty}&\approx\frac{1}{2}pR \qquad pR<2\,,\\ &\approx 1 \qquad pR>2\,. \end{aligned}\tag{5.27}$$

Primakoff[16] has recently used the following representation for the correlation function

$$\begin{aligned} 4h(\mathbf{x}_1,\mathbf{x}_2)&=1 \qquad |\mathbf{x}_1-\mathbf{x}_2|<d\,,\\ &=0 \qquad |\mathbf{x}_1-\mathbf{x}_2|>d\,, \end{aligned}\tag{5.28}$$

in connection with his theory of muon capture, employing a correlation distance given by $d^3=3.2r_0^3$ ($r_0^3=R^3/A$). Actually, d is determined uniquely by the normalization of $G(p)$, namely $d^3=4r_0^3$. The form factor becomes

$$G(p)=\mathfrak{L}(pd)\,,\tag{5.29}$$

which implies that $[1-G(p)]$ does not reach the nuclear matter value until $p\approx 400$ MeV/c. Thus, the correlation function (5.28) gives very poor agreement with the form factors which follow from an independent-particle model. Of course, one cannot completely exclude the possibility that residual hard-core interactions enhance the Pauli suppression to the extent indicated by (5.28).

The coherent and incoherent cross sections are given by (5.11) and (5.21) in the case of the closed-shell nuclei. We shall now for a moment return to the more general nuclei discussed briefly at the end of Sec. IV. Although the rigorous derivation which holds for closed-shell nuclei can no longer be carried out, there are strong grounds to conjecture that, for nonclosed-shell nuclei

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm coh}=\Gamma|F(p)|^2\{|AK-IM|^2+\mathfrak{M}_2\}\,,\tag{5.30}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm inc}=\Gamma[1-G(p)]\{A(|K|^2+|L|^2+|M|^2+|N|^2)-2I\mathfrak{R}(K^*M+\mathbf{L}^*\cdot\mathbf{N})\}+\Gamma G(p)\mathfrak{M}_1\,.\tag{5.31}$$

Since K, M, L_z, and N_z must all contain $\sin\theta$ as a factor, the coherent cross section still vanishes in the forward direction. Because $G(0)=1$, the first part of the incoherent cross section is also very small in the forward direction. The part depending on $\mathfrak{M}_1$ is not suppressed by the Pauli principle, however. In nuclei where $\mathfrak{M}_1$ does not vanish, pions can be produced in the forward direction by flipping the nuclear spin, just as in the case of single nucleons.

In terms of the quantities defined in Eqs. (3.12) and (3.13), the coherent and incoherent cross sections for nuclei with $\mathfrak{M}_1=\mathfrak{M}_2=0$ can be written

$$\begin{aligned} \left(\frac{d\sigma}{d\Omega}\right)_{\rm coh}&=|Zf_p+Nf_n|^2S_{\rm coh}\,,\\ \left(\frac{d\sigma}{d\Omega}\right)_{\rm inc}&=(Z\sigma_p+N\sigma_n)S_{\rm inc}\,, \end{aligned}\tag{5.32}$$

[16] H. Primakoff, Rev. Mod. Phys. **31**, 802 (1959).

where the shape factors are given by

$$S_{\text{coh}}=|F(p)|^2\sin^2\theta\,,$$
$$S_{\text{inc}}=1-G(p)\,. \tag{5.33}$$

VI. INVERSE DECAY OF THE NEUTRAL PION

The two-photon decay of the neutral pion can be schematically represented by the first diagram in Fig. 1. This is not a Feynman diagram but the sum of all possible Feynman diagrams with one external pion line and two external photon lines. The energy-momentum fourvectors of the incident pion and the final photons are q_μ, k_μ, and k_μ'. The polarization fourvectors of the photons are e_μ and e_μ'. On account of the conservation law $q_\mu=k_\mu+k_\mu'$, only four of these five vectors are linearly independent.

Because the pion is pseudoscalar, the form of the vertex operator for this diagram is uniquely defined since only one pseudoscalar can be constructed out of four fourvectors

$$\epsilon_{\kappa\lambda\mu\nu}e_\kappa e_\lambda' k_\mu k_\nu' Y(k_\mu k_\mu', k_\mu k_\mu, k_\mu' k_\mu')\,. \tag{6.1}$$

Here $\epsilon_{\kappa\lambda\mu\nu}$ is the completely antisymmetric unit tensor in four dimensions. The form factor Y may still depend on the three scalars which do not involve the polarization vectors. We shall use the relativistic metric in which $q_\mu q_\mu=\mathbf{q}\cdot\mathbf{q}-\omega^2=-m^2$. The transition rate can now be calculated by the usual Feynman rules namely

$$\frac{(2\pi)^4}{2}\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3k'}{(2\pi)^3}\frac{\delta^4|q_\mu-k_\mu-k_\mu')}{8kk'\omega}\times\sum_e\sum_{e'}|\epsilon_{\kappa\lambda\mu\nu}e_\kappa e_\lambda' k_\mu k_\nu' Y|^2,$$

where $\omega=q_0$ and $k=k_0$ are the pion and photon energies. In the rest system of the pion its mean lifetime τ is found to be given by

$$\frac{1}{\tau}=\frac{m^3}{64\pi}|Y(-\tfrac{1}{2}m^2,0,0)|^2. \tag{6.2}$$

Primakoff[2] first drew attention to the fact that the same vertex can give rise to photoproduction of neutral pions through the second diagram in Fig. 1. Here p_μ and p_μ' represent the initial and final momentum-fourvectors of a complex nucleus and k_μ and q_μ those of the incident photon and the produced pion. The interaction of the virtual photon (momentum t_μ) with the nucleus is represented by a second vertex Γ_μ. The S-matrix element can be written down directly

$$\int\frac{d^4t}{(4k)^{1/2}}\frac{\delta^4(k_\mu+t_\mu-q_\mu)}{it_\mu t_\mu}\epsilon_{\mu\nu\lambda\kappa}\Gamma_\mu e_\nu t_\lambda k_\kappa Y(k_\mu t_\mu, t_\mu t_\mu, k_\mu k_\mu). \tag{6.3}$$

The components of t_μ are given by

$$\mathbf{t}=\mathbf{q}-\mathbf{k}=-\mathbf{p},\qquad t_0=\omega-k=-\Delta\,, \tag{6.4}$$

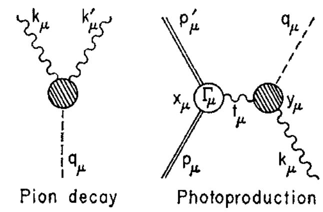

FIG. 1. Diagrams representing (a) decay of neutral pion into two photons, (b) photoproduction of pion from complex nucleus by exchange of virtual photon.

where $\mathbf{p}$ and Δ are the momentum and energy transferred to the nucleus. Assuming that $\Delta=0$ and setting $k_\mu k_\mu=0$ for the real incident photon, one finds that the transition amplitude for photoproduction is proportional to $Y(-\mathbf{k}\cdot\mathbf{p},p^2,0)$. The value of the form factor on which the production depends is therefore, in general, not identical with the value which occurs in expression (6.2) for the lifetime. As long as $p\ll M$, however, these two can be expected to be approximately equal since it is extremely unlikely that the nonlocality of Y has a range which is large compared to the Compton wavelength of the proton. We shall, therefore, from now on, approximate Y by the constant value

$$Y=8(\pi/m^3\tau)^{1/2}e^{i\delta}\,, \tag{6.5}$$

where δ is an unknown phase factor.

The S-matrix element (6.3) still contains the factor

$$\epsilon_{\mu\nu\lambda\kappa}\Gamma_\mu e_\nu t_\lambda k_\kappa=\Gamma_0\hat{e}\cdot(\mathbf{q}\times\mathbf{k})+(\mathbf{p}k-\mathbf{k}\Delta)\cdot(\mathbf{\Gamma}\times\hat{e})\,. \tag{6.6}$$

If the nucleus remains in its ground state, the vertex function Γ_0 is proportional to the form factor

$$\Gamma_0=2\pi Ze\delta(E_1-E_2)H(p)\,,$$
$$H(p)=\int\rho(\mathbf{x})e^{i\mathbf{p}\cdot\mathbf{x}}d^3x\,, \tag{6.7}$$

where $\rho(x)$ is the nuclear-charge distribution ($\int\rho d^3x=1$) and Ze its total charge. If the charge distribution and the nucleon distribution are identical, which we shall for simplicity assume, $H(p)$ becomes identical with $F(p)$, the form factor for nuclear production of pions (5.8). The T-matrix element, defined by $S_{fi}=\delta_{fi}-2\pi i\delta(E_i-E_f)T_{fi}$, is then given by

$$T_{fi}=\frac{4\pi}{p^2}\frac{\hat{e}\cdot(\mathbf{q}\times\mathbf{k})}{(\pi k\omega m^3\tau)^{1/2}}\{ZeF(p)\}e^{i\delta}. \tag{6.8}$$

The spatial part $\mathbf{\Gamma}$ gives rise to a somewhat similar expression:

$$T_{fi}=\frac{4\pi}{p^2}\frac{\hat{e}\times(\mathbf{p}k-\mathbf{k}\Delta)}{(\pi k\omega m^3\tau)^{1/2}}\sum\nolimits_l\{p(\mathbf{E}_l^s+\mathbf{M}_l^s)+\Delta(\mathbf{E}_l^r+\mathbf{M}_l^r)\}e^{i\delta}. \tag{6.9}$$

The E's and M's are related to the multipole amplitudes defined by Blatt and Weisskopf.[17] For example

$$\mathbf{E}_1{}^r = \sum_{s=1}^{A} q_s \langle f|\mathbf{x}_s|i\rangle ,$$

$$\mathbf{M}_1{}^s = \sum_{s=1}^{A} \mu_s \langle f|\boldsymbol{\sigma}_s|i\rangle ,$$

where q_s and μ_s are the electric charge and the magnetic-dipole moment of nucleon s. For the closed-shell nuclei, (6.9) contributes to inelastic photoproduction only. Inelastic contributions ($\Delta\neq 0$) also arise from Γ_0. In Ref. 5 an estimate was made of the following three contributions to the cross section for the production of π^0's by 900-MeV photons:

(1) The total inelastic contribution from Γ_0;
(2) the total spin-magnetic contribution from $\sum_l \mathbf{M}_l{}^s$;
(3) the electric-dipole contribution from $\mathbf{E}_1{}^r$.

The first two were estimated by applying the formalism of Secs. IV and V to the Coulomb production, while the third estimate was based on the dipole sum rule. It was found that all three are very small compared to the contribution from (6.8). Also, they are proportional to Z whereas the elastic cross section is proportional to Z^2. From now on only the elastic production arising from Γ_0 will be considered.

By using (6.8), one finds for the differential cross section

$$\frac{d\sigma}{d\Omega} = Z^2\sigma_0 S_{\mathrm{Coul}} ,$$

$$S_{\mathrm{Coul}} = (q^3k/p^4)|F(p)|^2 \sin^2\theta , \tag{6.10}$$

$$\sigma_0 = 8\alpha/m^3\tau = \left(\frac{2.03\times10^{-16}}{\tau \text{ in sec}}\right)\times 3\times10^{-5}\mu\text{b}. \tag{6.11}$$

Here α is the fine structure constant $e^2/4\pi$. Using expression (3.9) for the momentum transfer p,

$$|F(p)|^2 \approx 1-\tfrac{1}{5}R^2(\epsilon^2+k^2\theta^2) \tag{6.12}$$

for small θ. This slow decrease is completely overshadowed by the rapid variation of the factor

$$(q^3k/p^4)\sin^2\theta \approx k^4\theta^2/(\epsilon^2+k^2\theta^2)^2 \tag{6.13}$$

which starts at zero, reaches a peak value of

$$(k/2\epsilon)^2 \approx (k/m)^4 , \tag{6.14}$$

at an angle given by

$$\theta_{\mathrm{peak}} \approx \epsilon/k \approx m^2/2k^2 , \tag{6.15}$$

and then decreases as $1/\theta^2$.

[17] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952), p. 599.

Unlike the coherent nuclear peak, which occurs around $\theta\approx 2/kR$, the position of the peak in the Coulomb production (6.15) is independent of the nuclear mass number. The Coulomb peak occurs at a smaller angle than the nuclear peak. This angular separation is the basis of recent attempts to distinguish the Coulomb production from the nuclear background experimentally[3] (Primakoff originally suggested that the Coulomb production be distinguished by looking for a Z^2-dependent term in the total cross section). The maximum value of the coherent nuclear cross section (5.32a) is of the order

$$|Zf_p+Nf_n|^2/(kR)^2 ,$$

which is, apart from the energy dependence of f_p and f_n, proportional to $k^{-2}A^{4/3}$. The Coulomb peak value is roughly

$$Z^2\sigma_0(k/m)^4 ,$$

so that the ratio of Coulomb to nuclear is proportional to $k^6A^{2/3}$. Actually, experimental measurements always involve the integral of the differential cross section over some finite angular interval:

$$\int_{\theta_0-\delta}^{\theta_0+\delta} \sigma(\theta)\sin\theta d\theta \approx 2\delta \sin\theta_0 \sigma(\theta_0) ,$$

where δ is the angular resolution of the counter. Thus, the ratio of measured yields at the two peaks is proportional to k^5A (provided that δ is smaller than the peak widths). If we want to compare the total contributions from the two peaks, we have to choose 2δ in each case equal to the full width of half-maximum. Taking this to be roughly equal to the peak angle θ_0, one obtains a ratio proportional to $k^4A^{4/3}$, the nuclear production alone being proportional to $k^{-4}A^{2/3}$. In any case, the ratio of Coulomb to nuclear production increases rapidly with photon energy and more slowly with nuclear mass number.

The coherent nuclear cross section is given by (5.13) so that the transition amplitude can be written as $\langle 0|T^N|0\rangle$ where the nuclear transition operator is

$$T^N = \sum_n \{K_N + \mathbf{L}_N\cdot\boldsymbol{\sigma}_n + M_N\tau_{3n} + \mathbf{L}_N\cdot\boldsymbol{\sigma}_n\tau_{3n}\} e^{i\mathbf{p}\cdot\mathbf{x}_n}. \tag{6.16}$$

The transition amplitude for production by the Coulomb field (6.8) can be written in the same form with

$$T^C = \sum_n \{K_C + M_C\tau_{3n}\} e^{i\mathbf{p}\cdot\mathbf{x}_n} , \tag{6.17}$$

where

$$K_C = M_C = (4\pi/p^2)(\alpha/k\omega m^3\tau)^{1/2}\hat{e}\cdot(\mathbf{q}\times\mathbf{k})e^{i\delta}. \tag{6.18}$$

The coherent nuclear (5.32a) and Coulomb (6.10) cross sections arise from $|\langle 0|T^N|0\rangle|^2$ and $|\langle 0|T^C|0\rangle|^2$, respectively. Actually, the complete coherent cross section will be equal to Γ multiplied by $|\langle 0|T^N+T^C|0\rangle|^2$. In addition to what we called the coherent nuclear and Coulomb cross sections, we should therefore include a contribution due to interference between these production modes, namely,

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{int}}=2\cos(\delta_N-\delta_C)Z\sigma_0^{1/2}|Zf_p+Nf_n|S_{\text{int}}, \quad (6.19)$$

where the shape factor for the interference cross section is

$$S_{\text{int}}=(k/p)^2|F(p)|^2\sin^2\theta, \quad (6.20)$$

which is simply the geometric mean of S_{coh} and S_{Coul}.

Since the electromagnetic coupling is quite weak, the Coulomb amplitude (6.18) is probably real so that the Coulomb phase δ_C is about 0° or 180°. The pions are mostly produced outside the nucleus so that interactions between the pions and nucleons are also unlikely to produce a phase. At low energies, the phase δ_N for nuclear photoproduction is, according to the Fermi-Watson theorem, equal to the pion scattering phase. At high energies the only theorems of this nature which are available, are much weaker and at present the nuclear phase is unknown.

We have now obtained the four most significant contributions to pion photoproduction from complex nuclei, namely the nuclear incoherent and coherent, the Coulomb coherent and the interference cross sections. The solid curves on Figs. 2 and 3 represent the shape factors at 900 MeV for these four contributions without inclusions of final-state interactions, calculated using radii of 3.05 fm and 7.10 fm for carbon and lead, respectively. These are not the actual shapes of the cross sections, of course, since σ_p, σ_n, f_p, and f_n will in general have an additional angular dependence.

VII. FINAL-STATE INTERACTIONS OF THE PRODUCED PION

If we denote the total nuclear Hamiltonian plus the kinetic energy operators for the photon and pion fields by H_0, the photoproduction interaction by V, and the interaction of the nucleus with the pion field by U, the general photoproduction transition amplitude is given by[18]

$$\langle\phi_f|(U+V)|\psi_i^{(+)}\rangle=\langle\chi_f^{(-)}|V|\psi_i^{(+)}\rangle+\langle\phi_f|U|\chi_i^{(+)}\rangle \quad (7.1)$$
$$=\langle\chi_f^{(-)}|\mathfrak{T}|\phi_i\rangle,$$

where ϕ, χ, and ψ are, respectively, eigenstates of H_0, (H_0+U), and (H_0+U+V), and where we have made use of the fact that $\langle\phi_f|U|\chi_i^{(+)}\rangle=0$ for this process. The usual photoproduction transition operator not including pion-nuclear interactions is denoted by $\mathfrak{T}$ and the $+$ and $-$ superscripts refer to states which have either only incoming or only outgoing spherical waves at infinity. The initial and final states can be written as products of nuclear and either photon or pion wave functions, namely,

$$\phi_i=\xi_i\phi_k, \quad \chi_f^{(-)}=\xi_f\chi_q^{(-)},$$

where, for example, ϕ_k denotes a photon plane wave.

The transition operator T of Eq. (3.2), which has been used until now, is defined by the equation

$$\langle\xi_f|T|\xi_i\rangle=\langle\xi_f\phi_q|\mathfrak{T}|\xi_i\phi_k\rangle$$
$$=\langle\xi_f|\textstyle\sum_n\phi_q(\mathbf{x}_n)t_n\phi_k(\mathbf{x}_n)|\xi_i\rangle \quad (7.2)$$
$$=\langle\xi_f|\textstyle\sum_n e^{-i\mathbf{q}\cdot\mathbf{x}_n}t_n e^{i\mathbf{k}\cdot\mathbf{x}_n}|\xi_i\rangle.$$

As we saw in the beginning of Sec. II, the matrix element with respect to the pion and photon variables can be taken by expanding $\mathfrak{T}$ in terms of creation and destruction operators for the states ϕ_q and ϕ_k with respect to which the matrix element is defined. The corresponding wave functions then simply become factors of the transition operator for nucleons alone. In order to express (7.1) as a matrix element with respect to nuclear states only, we must therefore expand $\mathfrak{T}$ in terms of the interacting states $\chi_q^{(-)}$ instead of the free

[18] M. Gell-Mann and M. L. Goldberger, Phys. Rev. 91, 398 (1953).

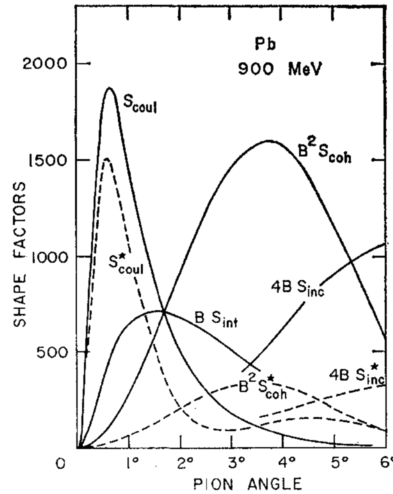

FIG. 3. Same as Fig. 2 except for lead. Normalizing factor $B=1000$.

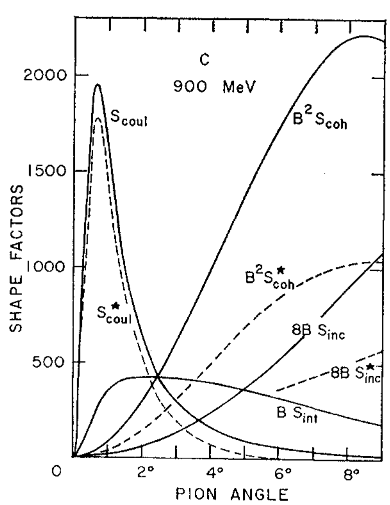

FIG. 2. Shape factors of various contributions to photoproduction differential cross section of carbon at 900 MeV. Solid curves are shape factors without absorption for incoherent (S_{inc}) and coherent (S_{coh}) nuclear production, Coulomb production (S_{Coul}) and interference term (S_{int}). Dashed curves include calculated effect of final-state interactions of pion. Normalizing factor $B=500$.

pion states ϕ_q. It follows directly that the transition operator (3.2) should be replaced by

$$T=\sum_n \chi_q^{(-)}(\mathbf{x}_n)t_n e^{i\mathbf{k}\cdot\mathbf{x}_n}. \quad (7.3)$$

In this section we shall investigate the effects of the inclusion of final-state interactions on the various cross sections by replacing T wherever it occurs by (7.3).

The pion wave function including interactions with the nucleus may be determined from the Schrödinger equation

$$(m^2+q^2)^{1/2}\chi_q(\mathbf{x})-(m^2-\nabla^2)^{1/2}\chi_q(\mathbf{x}) = \int \langle \mathbf{x}|V|\mathbf{x}'\rangle \chi_q(\mathbf{x}')d^3x', \quad (7.4)$$

where $\langle \mathbf{x}|V|\mathbf{x}'\rangle$ is a nonlocal optical-model potential which may be replaced by an energy-dependent local potential. Neglecting certain second-order terms, we can replace (7.4) by the integral equation

$$\chi_q^{(-)}(\mathbf{x})=e^{-i\mathbf{q}\cdot\mathbf{x}}+(2q/v) \times \int G_q^{(-)}(\mathbf{x},\mathbf{x}')V(\mathbf{x}')\chi_q^{(-)}(\mathbf{x}')d^3x', \quad (7.5)$$

where $G_q^{(-)}$ is the Green's function for the scalar Helmholtz equation while q and v are the values of the pion's momentum and velocity outside the range of the interaction.

For infinite nuclear matter $V(\mathbf{x})$ is just a constant complex number

$$V(\mathbf{x})=V_R+iV_I. \quad (7.6)$$

The solution of (7.5) is then

$$\chi_q^{(-)}(x)=\exp(-in\mathbf{q}\cdot\mathbf{x})\exp(\hat{q}\cdot\mathbf{x}/2\lambda), \quad (7.7)$$

where n and λ are the real refractive index and mean free path for pions in nuclear matter. When V_R and V_I are small compared to the total pion energy,

$$n=1-(V_R/qv), \quad (1/\lambda)=-(2/v)V_I. \quad (7.8)$$

In the optical-model inelastic scattering is described as absorption so that λ consists of two parts $\lambda^{-1}=\lambda_s^{-1}+\lambda_a^{-1}$. The refractive index and the mean free path for inelastic scattering can be expressed in terms of the nucleon density ρ and the real and imaginary parts $\mathcal{R}f$ and $\mathcal{I}f$ of the forward pion scattering amplitude:

$$n=1+(2\pi/q^2)\rho\mathcal{R}f, \quad \lambda_s^{-1}=(4\pi/q)\rho\mathcal{I}f. \quad (7.9)$$

Frank, Gammel, and Watson[19] determined $\mathcal{R}f$ and $\mathcal{I}f$ by means of dispersion relations and used the Brueckner-Serber-Watson model to estimate λ_a. It is obvious from (7.9) that the results depend very sensitively on the nuclear radius parameter r_0. The value $r_0=1.4$ fm gives the best fit to pion scattering and absorption measurements.[20] This then leads to the following values[19] for the optical-model parameters:

E(MeV)	n	λ(fm)	V_R(MeV)	V_I(MeV)
240	1.20	1.75	−42.5	−49.0
890	1.00	3.05	+ 3.5	−32.0

E is here the total pion energy outside the nucleus.

We shall use the pion wave function (7.7) in conjunction with the uniform-density model [Eq. (5.15)] of the nucleus. This procedure is completely equivalent to the method applied to neutron interactions in nuclei by Fernbach, Serber, and Taylor[21] (FST). In this method the effects of the final-state interactions can be separated into four independent aspects:

(a) Attenuation of the emerging pion beam due to absorption. Apart from the fact that inelastically scattered pions, which would in fact contribute to the incoherent cross section, are treated as if they are absorbed, the model being used should describe this effect quite well. At 900 MeV, λ_a is much smaller than λ_s anyhow, so that the inelastic effect should be small.

(b) Modification of the pion momentum inside the nucleus with resulting change in diffraction pattern—adequately described by model.

(c) Refraction of the pion beam at the nuclear surface. In a more realistic nucleus, this effect would be spread out through the thickness of the nuclear surface. This would change the phase slightly. At 900 MeV, $n=1$ so that no refraction occurs anyway.

(d) Internal reflection of the pion beam. This is the only aspect in which the abrupt nuclear surface which we use, has an appreciable effect, namely by giving far too much internal reflection. We shall, however, apply the model in such a way that internal reflection is not considered at all.

In the formulation which we have used for cross sections, the wave function of the outgoing pion must be normalized to unit volume in the asymptotic region. Thus at a point $\mathbf{x}_0$ on the nuclear surface, the pion wave function is:

$$\chi_q^{(-)}(\mathbf{x}_0)=\exp(-i\mathbf{q}\cdot\mathbf{x}_0). \quad (7.10)$$

At a position $\mathbf{x}$ inside the nucleus, the wave function (7.7) becomes:

$$\chi_q^{(-)}(\mathbf{x})=N\exp(\hat{q}\cdot\mathbf{x}/2\lambda)\exp(-in\mathbf{q}\cdot\mathbf{x}). \quad (7.11)$$

The normalization constant is determined by the requirement that (7.10) and (7.11) agree on the nuclear surface if $(\mathbf{x}_0-\mathbf{x})$ is parallel to $\mathbf{q}$. This is the essence of the FST model. Taking the center of the nucleus as origin and defining $l(\mathbf{x})$ by:

$$\mathbf{x}_0=\mathbf{x}+l\hat{q}, \quad x_0=R, \quad (7.12)$$

[19] R. M. Frank, J. L. Gammel, and K. M. Watson, Phys. Rev. **101**, 891 (1956); K. M. Watson and C. Zemach, Nuovo Cimento **10**, 452 (1958).

[20] T. A. Fuji, Phys. Rev. **113**, 695 (1959); also, papers by Ferretti and by Ignatenko at CERN Symp. High Energy Accelerators and Pion Phys., Geneva, 1956.

[21] S. Fernbach, R. Serber, and T. B. Taylor, Phys. Rev. **75**, 1352 (1949).

one finds for the pion wave function

$$\chi_q^{(-)}(\mathbf{x}) = \exp[i(n-1)ql] \exp(-l/2\lambda) \exp(-i\mathbf{q}\cdot\mathbf{x}). \tag{7.13}$$

The effect of (7.3) and (7.13) on the various form factors will now be considered.

When final-state interactions are included, the form factor for coherent nuclear production (5.8) becomes

$$F^\star(p) = \int \rho(\mathbf{x})\exp(i\mathbf{p}\cdot\mathbf{x}) \exp[i(n-1)ql - l/2\lambda]d^3x. \tag{7.14}$$

Inserting the value (5.15) for the density, and defining

$$\alpha = kR\cos\theta - qR, \quad \gamma = R/2\lambda,$$
$$\beta = kR\sin\theta, \quad \delta = (n-1)qR,$$

and

$$K[(1-r^2)^{\frac{1}{2}}] = \cos\alpha r - e^{-2\gamma r}\cos[(\alpha-2\delta)r],$$
$$L[(1-r^2)^{\frac{1}{2}}] = \sin\alpha r + e^{-2\gamma r}\sin[(\alpha-2\delta)r],$$

one can reduce this expression for $F^\star(p)$ to the following one-dimensional integral:

$$F^\star(p) = \int_0^1 (3r/2)J_0(\beta r)[K(r)+iL(r)][\gamma+i\alpha-i\delta]^{-1}dr. \tag{7.15}$$

This integral could not be evaluated analytically and its value was therefore calculated numerically on an IBM 709 electronic digital computer.

The boundary condition described in the previous paragraph actually implies that effect (c) of the pion interactions is neglected. To include this effect, one should require (7.10) and (7.11) to agree on the nuclear surface if $(\mathbf{x}_0 - \mathbf{x}_1)$ has the orientation with respect to $\mathbf{q}$ dictated by the refractive index. However, instead of $F(p)$, one then must consider the complete shape factor (5.33a)

$$S_{\text{coh}} = |e^{i\varphi}\sin\theta F(p)|^2,$$

since the refractive index affects also the direction (θ,φ) into which the pion must be produced in order to escape from the nucleus in a specified direction. In Ref. 5 this was achieved in an approximate way, but, since the procedure is somewhat tedious, it will not be reproduced here. The result of the numerical calculations is that the inclusion of the refraction effect (c) leads to enhancement of the shape factor [over the inclusion of (a) and (b) only] by factors 1.1, 1.4, 1.6, and 2.6 in the case of C, Ca, Cu, and Pb at 250 MeV. Because of the approximations involved when (c) is included, the reliability of these numbers is very uncertain. Since $n=1$ at 900 MeV, this effect is fortunately not present at the higher energy.

The incoherent cross section consists of two parts which have to be considered separately. Whereas the density form factor (7.14) involves a threefold integral only, which in the Fermi gas case was easily reduced to a one-dimensional integral (7.15), the correlation form factor $G(p)$ involves a sixfold integral (5.9) over an integrand containing the product of pion wave functions at two different points. In general it will thus be very difficult to carry out the same procedure for $G(p)$ as for $F(p)$. In the specific case which was considered in Sec. V, however, namely with harmonic-oscillator wave functions applied to O^{16}, $G(p)$ can be expressed as the product of two twofold integrals. This was done in Ref. 5 where a further unimportant approximation was made to reduce each twofold integral to the product of two simple integrals which were then again evaluated numerically.

The leading part of the incoherent cross section arises from the diagonal contribution which is proportional to $\langle 0|t_1^\dagger t_1|0\rangle$. It is easy to see that in this case the effect of absorption is simply to multiply the cross section by a factor

$$\zeta = \int e^{-l(\mathbf{x})/\lambda}\rho(\mathbf{x})d^3x, \tag{7.16}$$

which for the uniform density model simply reduces to $\zeta(2R/\lambda)$ where

$$\zeta(x) = (\tfrac{3}{2}x)\{1-(2/x^2)[1-(1+x)e^{-x}]\}. \tag{7.17}$$

When evaluated using the mean-free-path values λ quoted above and the uniform radii measured by Hofstadter,[22] the values of ζ can be represented quite well by the following empirical expressions:

$$\zeta = 0.69A^{-1/4} \quad \text{at} \quad 250\text{ MeV},$$
$$\zeta = 0.87A^{-1/5} \quad \text{at} \quad 900\text{ MeV}. \tag{7.18}$$

When absorption of the pion is not included, the incoherent shape factor (5.33b) is given by $(1-G(p))$. We shall denote the Fermi gas value, pertaining to nuclear matter, by $(1-G_F)$ and the value calculated in Sec. V for O^{16} using oscillator wave functions, by $(1-G_0)$. As we have seen above, numerical calculations were performed to determine the value of G_0 when absorption is included. The result will be denoted $G_0^\star$. The 1, which comes from the diagonal contribution, is changed to ζ when absorption is included. The total incoherent shape factor for oxygen with inclusion of absorption, thus becomes

$$\zeta - G_0^\star. \tag{7.19}$$

Since $G^\star$ could not be calculated for nuclei other than O^{16}, it is of interest to see how well (7.19) can be approximated by simply multiplying the $(1-G)$ values without absorption, by ζ. At large p values $\zeta(1-G_0)$ and $\zeta(1-G_F)$ both approach (7.19). In the forward direction $\zeta(1-G_0)$ is too small by a factor 2 at 250 MeV and a factor 8 at 900 MeV. On the other hand, at small momentum transfers $\zeta(1-G_F)$ is within 5% of (7.19)

[22] R. Hofstadter, Ann. Rev. Nucl. Sci. 7, 231 (1957).

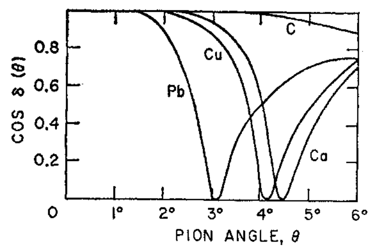

FIG. 4. Phase correction due to pion interactions calculated for lead, copper, calcium, and carbon at 900 MeV.

and it never differs from it by more than 20%. When the incoherent shape factor with absorption is needed for nuclei other than oxygen, we shall therefore use

$$S_{\rm inc}^{\star}=\zeta(1-G_F)\,. \tag{7.20}$$

In Sec. VI the cross section for photoproduction by the Coulomb field of the nucleus was derived by utilizing the Feynman rules in momentum space. It is also possible to apply the Feynman rules in coordinate space to the diagram in Fig. 1. Of course one obtains exactly the same result as in Eq. (6.8) except that the form factor first appears in the form:

$$H(p)=\frac{p^2}{4\pi}\int\frac{\rho(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|}e^{i\mathbf{k}\cdot\mathbf{y}}e^{i\mathbf{q}\cdot\mathbf{y}}d^3x d^3y\,. \tag{7.21}$$

The physical significance of this expression is obvious: the nuclear charge density at **x** causes photoproduction of a pion at **y** by means of the long-range Coulomb potential which decreases inversely with the distance between **x** and **y**. Although (7.21) looks very different from the form factor for nuclear production (5.8), it is easy to show that the two are identical, at least for spherically symmetric density distributions.

When a uniform density is used, it is convenient to divide $H(p)$ into two parts, one arising from production points inside the nucleus ($y<R$) and the other arising from exterior production points ($y>R$):

$$\begin{aligned} H(p)&=H_{\rm in}(p)+H_{\rm ex}(p)\,,\\ H_{\rm in}(p)&=\mathcal{L}(pR)-\cos(pR)\,,\\ H_{\rm ex}(p)&=\cos(pR)\,. \end{aligned} \tag{7.22}$$

When absorption of the pion is included, the calculation of $H_{\rm in}^{\star}(p)$ is very similar to that of $F^{\star}(p)$. In terms of the parameters defined in connection with (7.15), one obtains

$$H_{\rm in}^{\star}(p)=\tfrac{1}{4}(pR)^2\int_0^1 dr\int_{-s}^{s}dz(3r-rz^2-r^3)J_0(\beta r)\times\exp[(r+i\alpha-i\delta)z+(i\delta-\gamma)s]\,, \tag{7.23}$$

where $s=(1-r^2)^{1/2}$. The integral over z is simple but the remaining integral over r was again performed on a computer.

To calculate $H_{\rm ex}^{\star}(p)$, one needs the pion wave function outside the nucleus. According to the FST model the pion wave function is a plane wave everywhere except behind the nucleus where it is attenuated. The expression which one obtains, is

$$H_{\rm ex}^{\star}(p)=H_{\rm ex}(p)+\tfrac{1}{4}(pR)^2\int_0^1 dr(2r)J_0(\beta r)f(r)\times\{\exp[2(i\delta-\gamma)(1-r^2)^{1/2}]-1\}\,, \tag{7.24}$$

where

$$f(r)=\int_1^\infty z^{-1}e^{-i\alpha z}dx\,, \tag{7.25}$$

and $z=(x^2-r^2)^{1/2}$. This last integral diverges logarithmically when α vanishes, i.e., when $\mathbf{p}\cdot\mathbf{q}=0$. This is due to the fact that the FST wave function does not contain diffracted waves. The exact pion wave function, which does include diffracted waves, will essentially not "see" the nucleus beyond a distance qR^2 behind the nucleus. To take this into account, the integral (7.25) was smeared out over such a distance, and (7.24) was then calculated numerically.

The shape factor for the interference term when absorption is included, $S_{\rm int}$, may simply be taken to be the geometric mean of $S_{\rm coh}$ and $S_{\rm Coul}$, provided that the phase difference occurring in (6.19) is changed to

$$(\delta_N-\delta_C)^{\star}=(\delta_N-\delta_C)+\delta(\theta)\,. \tag{7.26}$$

The main contribution to the phase correction $\delta(\theta)$ comes from the difference between $H_{\rm ex}^{\star}(p)$ and $H_{\rm ex}(p)$. The results of the calculations at 900 MeV are shown in Fig. 4.

As one expects, the main effect of the inclusion of pion absorption is attenuation of the cross section. The incoherent cross section is diminished by a factor $\zeta(2R/\lambda)$. Near the peak, the coherent nuclear cross section is attenuated by factor of, roughly, $\zeta^2(R/\lambda)$. The minima in the coherent cross section, on the other hand, are filled in so that the general angular distribution becomes smoother. In the case of the Coulomb production, the effect on the shape of the cross section is much more pronounced. Since pions produced inside the nucleus can in general be expected to be more strongly absorbed than those outside, the ratio of $H_{\rm in}^{\star}$ to $H_{\rm ex}^{\star}$ should become smaller. It then follows directly from Eq. (7.22) that the maxima and minima in the differential cross section would be shifted towards smaller angles. The results of the numerical computations of the shape factors with absorption are indicated by the dashed curves on Figs. 2 and 3. The second hump in the curve for $S_{\rm Coul}$ for lead, is again due to the difference between $H_{\rm ex}^{\star}$ and $H_{\rm ex}$.

VIII. DISCUSSION

The photoproduction of neutral pions from complex nuclei has been discussed in three articles[23–25] which were published after the submission of the thesis[5] on

[23] V. Glaser and R. A. Ferrell, Phys. Rev. **121**, 886 (1961).
[24] C. Chiuderi and G. Morpurgo, Nuovo Cimento **19**, 497 (1961).
[25] S. M. Berman, Nuovo Cimento **21**, 1030 (1961).

which the present work is based (a preprint of Ref. 24 was actually seen by the author before the completion of the thesis). These three articles will be considered briefly and their results compared with those of the present work. The results of the previous sections will then be investigated in the light of the available experimental information.

Glaser and Ferrell[23] primarily considered the coherent Coulomb production which arises from the Γ_0 term in Eq. (6.6). They showed that the contribution from excited final nuclear states may be ignored. For the nuclear coherent and incoherent differential cross sections, the expressions $(A^*)^2\sigma_H \sin^2\theta$ and $(A^*)\sigma_H$ were simply assumed, where σ_H was identified with the value of about 1 μb/sr which Berkelman and Waggoner[8] found for the 30-deg differential cross section for π^0 production from hydrogen at 950 MeV. The effect of absorption was taken into account by using an effective number $A^*=2A^{2/3}$ of nucleons. This is equivalent to a reduction by factors 0.87 and 0.34 for the incoherent and by factors 0.76 and 0.11 for the coherent nuclear cross sections for C and Pb, respectively. (The corresponding reduction factors found in the present work were 0.53 and 0.30 for the incoherent, and 0.47 and 0.21 for the coherent production.) The authors finally concluded that the π^0 mean life should be measurable to ten-percent accuracy by detecting pions produced at angles smaller than $3\epsilon/k$ (about 2°) by 1-GeV photons.

A somewhat better discussion of the nuclear contribution was given by Chiuderi and Morpurgo.[24] They wrote down expression (3.3) for the nuclear transition operator

$$T=\sum_i t_i e^{i\mathbf{p}\cdot\mathbf{x}_i}, \tag{8.1}$$

where the single nucleon operator t has the form given in Eq. (2.9). The coherent cross section was written in a form equivalent to (8.3) with an estimate of 0.5 μb/sr for f^2. For the incoherent cross section in the forward direction, only $\mathbf{L}$ and $\mathbf{N}$ are important. The exponential was expanded and only the first two terms kept:

$$T=\sum_i t_i(1+i\mathbf{p}\cdot\mathbf{x}_i)=T_1+T_2.$$

The neglect of higher terms was partly based on an estimate of less than 30 MeV for the excitation energy of the final nucleus. For the contribution from T_1 they essentially found the $\mathfrak{M}_2$ term [Eq. (4.17)] which we included in the coherent nuclear cross section [Eq. (5.30)]. For the cross section arising from the one-particle terms in $\sum_n|\langle n|T_2|0\rangle|^2$ they found

$$\Gamma A(|\mathbf{L}|^2+|\mathbf{N}|^2)[10^{-2}+10^{-3}A^{2/3}],$$

in our notation. This expression represents the terms neglected in the closure approximation when we went from Eq. (3.5) to Eq. (3.7). It amounts to only about 3% of the diagonal cross section (3.10), which provides added justification for the method. One should remark, however, that the importance of these correction terms is enhanced by the partial cancellation of the diagonal by the correlated contribution (5.12) to the cross section.

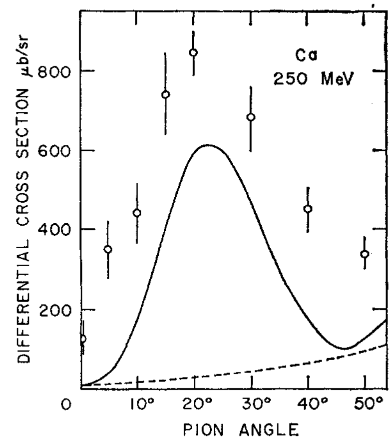

FIG. 5. Photoproduction from calcium at 250 MeV. Data points are from Davidson's thesis (Ref. 26). Solid curve is differential cross section predicted by theory. Dashed curve is incoherent component of theoretical cross section.

The only really systematic investigation of the nuclear coherent and incoherent π^0 photoproduction was given by Berman.[25] He also started with expression (8.1) but represented the single nucleon operator in terms of D, $B=K\pm M$ and $\mathbf{E}$, $\mathbf{C}=\mathbf{L}\pm\mathbf{N}$. As in Sec. III and Ref. 12, the cross section was separated into a diagonal and a nondiagonal contribution. For the diagonal (one-particle) contribution, precisely Eq. (4.15) was found [apart from erroneous factors of 3 which should be omitted from Eq. (4) and from the equations on p. 1023 of Ref. 25]. The two-particle contribution was, however, analyzed in a fashion which differs from that used in the present work and follows Lax and Feshbach[12] instead. In terms of our form factors, the quantities used by Berman (as well as in Ref. 12) are:

$$\begin{aligned}\tfrac{1}{2}(V_s+V_a)&=|F(p)|^2,\\ \tfrac{1}{2}(V_s-V_a)&=4A^{-1}(G(p)-|F(p)|^2).\end{aligned} \tag{8.2}$$

The leading terms thus obtained are completely equivalent to the expression in Eqs. (5.30) and (5.31) of the present work (again when allowance is made for slight discrepancies in Ref. 25). The general formalism which is used, however, tends to obscure the origin of the particle cancellation of the incoherent cross section. In order to obtain quantitative results, Berman used the Primakoff[16] correlation function. If, as discussed in Sec. V above, this does in fact lead to values of $G(p)$ which are too large, the value of the incoherent cross section would be underestimated by this procedure.

Based on the lifetime measurements of Ref. 4, the Coulomb production cross section should reach a maximum value of $3Z^2\times10^{-4}$ μb/sr at 250 MeV, and can therefore be ignored completely. The nuclear amplitudes at this energy are well known and were discussed in Sec. II. The theoretical production cross sections

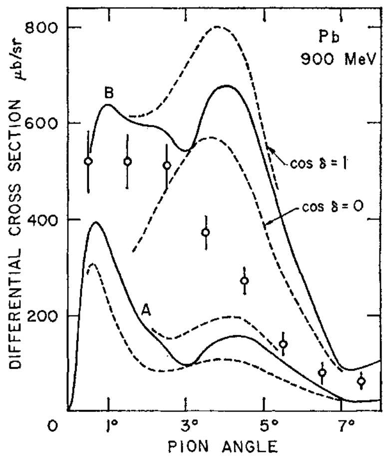

FIG. 6. Photoproduction from lead at 900 MeV. Data points are from Ref. 3. Solid curves A and B are theoretical cross sections using single-nucleon cross sections discussed in text and cos δ from Fig. 4. Dashed curves obtained using cos $\delta=1$ and cos $\delta=0$ instead.

from complex nuclei can therefore be calculated at this energy. The values obtained for calcium are represented by the solid curve on Fig. 5. The dashed curve represents the incoherent contribution, which is relatively unimportant at angles smaller than 40°. The experimental data points on Fig. 5 were taken from Davidson.[26] Near the peak, the theoretical curve lies some 30% below the experimental points.

At high energies the Coulomb and interference terms in the cross section become important at small angles. Unfortunately, the single-nucleon amplitudes are not yet known well. The various contributions to the differential photoproduction cross section can be written

$$\sigma_{\mathrm{coh}}(\theta)=A^2f^2S_{\mathrm{coh}}^{\star}, \tag{8.3}$$

$$\sigma_{\mathrm{Coul}}(\theta)=Z^2\sigma_0S_{\mathrm{Coul}}^{\star}, \tag{8.4}$$

$$\sigma_{\mathrm{int}}(\theta)=2(Af)(Z\sigma_0^{1/2})S_{\mathrm{int}}^{\star}\cos\delta^{\star}, \tag{8.5}$$

$$\sigma_{\mathrm{inc}}(\theta)=A\sigma S_{\mathrm{inc}}^{\star}, \tag{8.6}$$

where $Af=Zf_p+Nf_n$ and $A\sigma=Z\sigma_p+N\sigma_n$. The constant σ_0 has a value of about 3×10^{-5} μb/sr (if the π^0 lifetime is around 2×10^{-16} sec). In Sec. II it was seen that a reasonable fit to the 950-MeV-proton differential cross section leads to a value $(f_p^{\mathrm{c.m.}})^2=0.56$ μb/sr (and hence $f_p^2=4.97$ μb/sr) at 900 MeV. If the third πN resonance has negative parity, this is increased to $(f_p^{\mathrm{c.m.}})^2=4.0$ μb/sr (and $f_p^2=35.5$ μb/sr) in the forward direction. Actually the value of the proton differential cross section at 30° places an upper limit of around this latter value on $f_p^{\mathrm{c.m.}}$, unless there is a pole term due to the exchange of a vector meson. The results of Talman *et al.*,[8] however, indicate that the effects of exchange terms are small below 1 GeV. About f_n nothing is so far known experimentally. Although there is no theoretical justification to do so at high energies, we shall set $f=f_p$ and $\sigma=\sigma_p$ in order to make quantitative estimates.

The two sets of curves on Fig. 6 were calculated for lead at 900 MeV using the shape factors calculated in the present work. For set A the values $(f_p^{\mathrm{c.m.}})^2=0.56$ μb/sr and $\sigma_p^{\mathrm{c.m.}}=0.22$ μb/sr were used, and for set B the values $(f_p^{\mathrm{c.m.}})^2=4.0$ μb/sr and $\sigma_p^{\mathrm{c.m.}}=1.0$ μb/sr, which are roughly the upper limits which can at present be set on the proton cross sections. The solid curves were calculated using for $\delta^{\star}$ simply the values $\delta(\theta)$ of Eq. (7.26) and Fig. 4. The dashed curves, on the other hand, represent the cross sections in the case of maximum constructive interference ($\cos\delta=1$ at all angles) and the case of no interference ($\cos\delta=0$).

Ruderman *et al.*[3] measured the photoproduction of neutral pions from Pb, Cu, Al, and C at 900 MeV. The π^0 yield was measured as a function of a quantity involving an integral over θ, the angular resolution being about 4° in the forward direction and about 2° at large angles. The data points for lead on Fig. 6 were taken from Ruderman's thesis and normalized to the cross sections reported at Rochester[3] (no absolute values were quoted in the thesis). Because of the smearing of such a wide range of angles in the experimental measurements, the data points should not be compared directly with the curves. Instead, the theoretical cross sections should be folded into the angular resolution. The parameters occurring in Eqs. (8.3)–(8.6) can then be varied until a good fit to the measured yields is obtained. This was done by Ruderman, who, however, used the Primakoff correlation function to calculate S_{inc} and a Gaussian form factor:

$$|F(p)|^2=\exp[-(pR)^2/5]$$

to calculate S_{coh} and S_{Coul}. Absorption was included by simply multiplying S_{coh} by the attenuation factor calculated in Ref. 5 and the present work, and $\cos\delta$ was assumed to be constant. The results of the fitting procedure were:

Target material	Al	Cu	Pb
τ (in 10^{-16} sec)	0.7	0.3	1.0
cos δ	1.0	−0.3	1.0
f^2 (in μb/sr)	34	36	32

If f_p is set equal to f, this results in a value of 3.8 μb/sr for $(f_p^{\mathrm{c.m.}})^2$, which at present still falls within the range of values allowed by the experiments. The lifetime of 0.7×10^{-16} sec is about three times smaller than the value obtained by direct methods.[4] The angular resolution should, however, at least be reduced to less than 2° before one would see enough structure to permit a reliable separation between the various components of the cross section.

Since the calculated cross section for Ca at 250 MeV is about 30% lower than the experimental value, it is important to look for possible sources of error. The basic assumptions of the method, namely the direct-

[26] G. Davidson, Ph.D. thesis, MIT, September 1959 (unpublished); R. Barringer, R. Meunier, and L. S. Osborne, CERN Symp. High Energy Accelerators and Pion Phys., Geneva, 1956, Proc. p. 282.

interaction model and impulse approximation, seem to be fairly well established at high energies. Of course, any pions produced by a process in which it is essential that many nucleons must take part, would not be included by this formalism.

In the evaluation of the nuclear matrix elements, the most serious assumption was that of independent particle motion, since it neglects strong correlations due to the residual forces between nucleons. Especially processes where the nucleus is disrupted, may be appreciably enhanced due to the presence of high-momentum components arising from the hard-core repulsion between nucleons. The principal effect, however, would be on the incoherent cross section which is much smaller than the coherent processes at small angles.

A considerable change in the cross sections could be achieved by modifying the interaction parameters used in Sec. VII. In order to fit the 250-MeV cross section, the pion mean free path would have to be increased to 3 fm. Such a long mean free path can be ruled out on experimental grounds.[20] At high energies, however, the interaction parameters are not at all certain yet and may very well have to be changed.

In the case of pure absorption (refractive index $n=1$), the FST model is probably quite adequate for the calculation of the effect of final-state interactions on the coherent nuclear cross section. As stated in Sec. VII, the calculation may be much less reliable when refraction has to be taken into account, since this was done to lowest order in $(n-1)$. At 250 MeV, $n=1.2$ so that this weakness of the method may be largely responsible for the disagreement between theory and experiment at 250 MeV. Another aspect of the FST model which is open to some degree of doubt, occurs in the Coulomb case, where the wave function used in the calculation of H_{ex} could possibly lead to erroneous results.

The optical model treats inelastically scattered particles as if they are absorbed, whereas they may in fact still be detected and included in the incoherent component of the measured cross section. The incoherent cross section in the forward direction is strongly suppressed because of the bias against small-momentum transfers which arises from the exclusion principle. One should therefore also consider pions produced in another direction, where the Pauli suppression is negligible, and scattered into the forward angles. A closely related source of additional pions, arises from the primary production of charged pions, which are then changed into neutral pions by secondary charge-exchange scattering events.

This latter process could also contribute to the dominant coherent cross section, and its magnitude should therefore be estimated. Let us first consider the production chain $\gamma\to\pi^+\to\pi^0$. In order to compare the cross section with the regular direct coherent production, we first have to replace the amplitude Af by Zf_+, where f_+ is the nonspin flip amplitude (divided by $\sin\theta$) for the elementary process $\gamma+p\to n+\pi^+$. This has to be multiplied by the probability amplitude for the charge-exchange scattering $1-\zeta(R/\lambda_+)$, where $(1/\lambda_+)=(N/A)(3/4\pi r_0^3)\sigma(\pi^+n\to\pi^0p)$. Finally there is a nuclear matrix element which will simply be replaced by unity in order to obtain an upper limit. A similar contribution comes from the chain $\gamma\to\pi^-\to\pi^0$. If these two are added, assuming $f_-=f_+$, $\lambda_-=\lambda_+$, and $Z=N$, and reasonable values are used for the various cross sections, one finds that the primary production of charged pions could contribute at the very most 1% (Ca at 250 MeV) or 0.5% (Pb at 900 MeV) to the coherent cross section.

For the additional contributions to the incoherent cross section, both due to primary charged-pion production and to neutral-pion production and rescattering, a slightly different treatment should be used. Firstly there is the primary production process with probability $A\rho(\mathbf{x})\sigma_1(\theta_1)(1-G)$ which includes the Pauli suppression factor $(1-G)$. The produced pion is then propagated from $\mathbf{x}$ to $\mathbf{y}$ where its flux is $(1/4\pi l_1^2)e^{-\kappa_1 l_1}$, where $l_1=|\mathbf{x}-\mathbf{y}|$, while κ_1 is the reciprocal of the pion mean free path. The probability of the secondary scattering process is $A\rho(\mathbf{y})\sigma_2(\theta_2)$ in the case of $\gamma\to\pi^0\to\pi^0$, and approximately $\frac{1}{2}A\rho\sigma_2$ when the intermediate pion is charged. Finally there is a factor $e^{-\kappa_2 l_2}$, the probability that the final pion emerges intact. If only the exactly forward direction is considered, $\theta_1=\theta_2$. In order to make a quantitative estimate possible, the complete expression for the cross section is approximated by a product of an angular integral

$$(\tfrac{1}{2}\text{ or }\tfrac{1}{4})\int A\sigma_1(\theta)(1-G)A\sigma_2(\theta)\sin\theta d\theta\,,$$

and a double volume integral

$$\int(1/4\pi l_1^2)e^{-\kappa_1 l_1}e^{-\kappa_2 l_2}\rho(\mathbf{x})\rho(\mathbf{y})d^3x d^3y\,.$$

Inserting reasonable values, one finds that the process $\gamma\to\pi^0\to\pi^0$ could be expected to contribute about 15 μb/sr (Ca at 250 MeV) and 45 μb/sr (Pb at 900 MeV) at $\theta=0$. The expected contributions from primary charged-pion production are about 7 and 4 μb/sr, respectively. Although both processes are comparable with the direct incoherent cross section, their contributions are still very small compared to the coherent peak.

The 30% discrepancy between theory and experiment at 250 MeV should probably be ascribed to one or both of the following causes: (i) the unreliability of the simple FST model for refractive processes, and (ii) the existence of an unaccounted for constant background to the differential cross section. At 900 MeV both sources of error appear to be absent, (i) because $n=1$, and (ii) because of the small values of the experimentally measured cross section at 8° (see Fig. 6). The main uncertainties at 900

MeV arise from: (a) the pion mean free path to be used in the final-state interaction calculations, and (b) the single nucleon amplitude $Af=Zf_p+Nf_n$. Conversely, this means that the photoproduction of neutral pions from complex nuclei could be used as a means of investigating the interactions of pions with nucleons and nuclear matter. If the production amplitudes are determined by other methods, the mean free path in nuclear matter can be obtained. If, on the other hand, a reliable estimate of the mean free path is obtained from measurements of absorption of positive and negative pions by complex nuclei, the proton and neutron amplitudes f_p and f_n can be calculated from the π^0 photoproduction cross sections of nuclei with different Z-to-N ratios.

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Analysis of (p,2p) Angular Correlation Experiments*

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(p,2p) angular correlations for various p-shell nuclei are analyzed in the distorted-wave Born approximation with simple shell-model assumptions for the struck particle. The object is to define the limits on the information that can be obtained about nuclear structure from present experiments and to suggest how experiments should be improved. It is found that a simple shell-model wave function for the struck particle can be quite well defined by fitting present experimental data but that the choice of optical-model parameters is highly ambiguous. The ambiguity extends to the determination of the effective two-body potential and the necessity for configuration mixing. The rms radius of the proton distribution obtained from (p,2p) curve fitting agrees well with that obtained from electron scattering. The primary need is for better energy and angular resolution.

1. INTRODUCTION

THE first (p,2p) experiments[1] were performed with very high-energy (340-MeV) protons. The distribution of momentum transfer to the residual nucleus was measured by measuring either the angular correlation for a given energy sharing between the emitted protons or the energy distribution at fixed angles. As a first approximation the momentum transfer was regarded as being due only to the motion of the struck particle. The momentum-transfer distribution was equated with the momentum distribution of nucleons in the nucleus.

It was suggested by Eisberg[2] that a measurement of the angular correlation between two medium-energy nucleons emitted in time coincidence in a direct interaction might provide a sensitive test of the validity of the assumption that an incident nucleon collides with a single nucleon in the nuclear surface at intermediate energies.

Angular correlations in (p,2p) experiments were measured by Cohen,[3] and Griffiths and Eisberg.[4] In Cohen's experiment not enough data were obtained on the angular correlation for a significant determination of its characteristics. The major effort was expended on

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