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Repulsive Bardeen-Cooper-Schrieffer Pair-Interaction

K. SAWADA AND C. WARKE*
Department of Physics, University of California, San Diego, La Jolla, California
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With the mere use of the sum rules, the effect of interaction on the energy and one-particle momentum distribution of a system of Fermi particles, interacting with the repulsive BCS-type pair interaction, is investigated with the assumption that the coupling constant is of order one, together with the case when the coupling constant is of the order $1/\Omega$ (Ω being the volume of the quantization box) which was treated earlier by Van Hove in connection with the question of sharpness of Fermi surface. It is observed that in some region of density, this type of interaction has no effect on the energy and sharpness of the Fermi sphere, and above critical density the system will go over a phase change, but still the Fermi sphere remains rigid.

INTRODUCTION

IN the general approach to a solution of the many-body problem, one usually starts with the unperturbed energy and a step-function unperturbed single-particle momentum distribution, and then finds out the effect of switching on the interaction by a systematic application of the perturbation theory. However, we know some examples where the perturbation theory is divergent and is incapable of providing a solution to the problem, as in the case of Bardeen-Cooper-Schrieffer theory. In order to explore the behavior of the exact ground-state energy and the true single-particle momentum distribution, one often chooses a model which provides exact solution of the problem. It looks tentatively that, if the interaction is strong, the single-particle momentum distribution will be smeared out irrespective of the sign of the potential. In the following we have treated a model, which, though not important from a physical point of view, still has its value in the study of true single-particle momentum distribution of the strongly interacting many-body system. Contrary to our tentative guess, this model predicts that, for an infinite system, the single-particle momentum distribution remains step-function distribution up to the uncertainty of the $O(1/N)$ [$O(p)$ hereafter means order of p], where N is the total number of particles in the system.

The effect of the repulsive BCS-type pair interaction was once analyzed by Van Hove[1] in connection with the question of the existence of a sharp Fermi surface. In this case, the interaction was

$$H_{\mathrm{int}}=\gamma \sum_{k<k_{\max}} C_{\mathbf{k}\uparrow}{}^{*}C_{-\mathbf{k}\downarrow}{}^{*} \sum_{l<k_{\max}} C_{-\mathbf{l}\downarrow}C_{\mathbf{l}\uparrow}\,, \qquad (1)$$

where γ is the ratio of the repulsive coupling constant λ, to the volume of the quantization box Ω. We show in Appendix A, rather easily, that this interaction produces a change only of the $O(1)$ in the total energy of the system, and the one-particle momentum distribution changes by a quantity only of the order $1/N$.[2] The case of γ negative is also discussed in Appendix B. The main question we want to investigate here is how the above changes in the total energy of the system and the one-particle momentum distribution are affected if we take γ in (1) to be of the order unity, i.e., $\gamma=\lambda>0$. We also investigate the possibility of a phase transition as we change the number of particles N. It can easily be seen that the perturbation expansion calculation of the ground-state energy diverges badly. By using the sum rules, we shall show in the following that the above interaction with γ positive and of the order unity produces no effect on the single-particle momentum distribution up to $O(1/N)$, i.e., the true momentum distribution differs from the unperturbed Fermi distribution only by a quantity of the order $1/N$ over the entire region of N. And the true ground-state energy assumes the value equal to $(H_0)_{\text{free Fermi}}$ in $N<\sum_{k<k_{\max}}1$ and to $(H_0+H_{\mathrm{int}})_{\text{plane wave HF}}$ in $N>2\sum_{k<k_{\max}}1$ with the uncertainty of the $O(1)$. In the region $2\sum_{k<k_{\max}}1\geqslant N\geqslant\sum_{k<k_{\max}}1$ the ground-state energy takes the value equal to

$$(H_0)_{\text{free Fermi}}+\gamma(N-\textstyle\sum_{k<k_{\max}}1)+O(1)$$

which adds the linear dependence of N as shown in Fig. 3.

PROOF FROM THE SUM RULES

To proceed the proof, we first write down our Hamiltonian

$$H=H_0+H_{\mathrm{int}}\,,$$

where

$$H_0=\sum \epsilon_k(C_{\mathbf{k}\uparrow}{}^{*}C_{\mathbf{k}\uparrow}+C_{-\mathbf{k}\downarrow}{}^{*}C_{-\mathbf{k}\downarrow}),\quad \epsilon_k=k^2/2m\,. \qquad (2)$$

The sum rule we are going to use here is the following; if we represent the true ground-state Schrödinger function by $|\rangle_0$, then

$$\underset{0}{\langle}|F^{*}[H,F]|\underset{0}{\rangle}\geqslant 0 \qquad (3)$$

for any operator F. More generally, if we write

$$\tilde{H}=H-\mu N\,, \qquad (4)$$

* On study leave from the Tata Institute of Fundamental Research, Bombay, India.

[1] Leon Van Hove, Physica 26, s 200 (1960).

[2] Same conclusion was reached by W. Kohn and J. M. Luttinger (private communications).

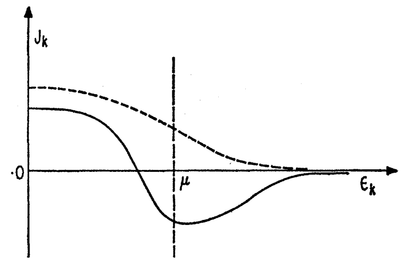

FIG. 1. The solid line is the sketch of J_k as a function of ϵ_k and the dashed line is γN_k, the upper bound of J_k.

where μ is the chemical potential,

$$ {}_0\langle|F^*[\tilde{H},F]|\rangle_0 \geqslant 0 \tag{5}$$

is the sum rule in case F does not conserve the number of particles. For completeness, the proof of (3) and (5) is sketched below. By writing the commutator product in terms of the complete set of the Schrödinger functions of H, and using the fact that E_0 is the ground-state energy, we get for (3)

$$\sum_\beta |{}_\beta\langle F\rangle_0|^2(E_\beta - E_0)\geqslant 0.$$

Moreover, since N commutes with H, eigenfunctions of H are also eigenfunctions of N and therefore (5) becomes

$$\sum_{\beta N'} |{}_\beta^{N'}\langle F\rangle_0^N|^2(E_\beta^{N'}-\mu N'+\mu N-E_0^N)$$
$$=\sum_{\beta N'} |{}_\beta^{N'}\langle F\rangle_0^N|^2(E_\beta^{N}-E_0^N)\geqslant 0.$$

Since $\mu=\partial E_0^N/\partial N$ and the excitation energy $E_\beta^{N'}-E_0^{N'}$ can be approximated by $E_\beta^N-E_0^N$.[2a]

In the following we apply (3) and (5) in the sequence to our system (1), and draw conclusions from these inequalities. Taking $F=C_{-\mathbf{k}\downarrow}C_{\mathbf{k}\uparrow}$, we get from (5)

$$ {}_0\langle|C_{\mathbf{k}\uparrow}^*C_{-\mathbf{k}\downarrow}^*[\tilde{H},C_{-\mathbf{k}\downarrow}C_{\mathbf{k}\uparrow}]|\rangle_0$$
$$=-2(\epsilon_k-\mu)\,{}_0\langle|C_{\mathbf{k}\uparrow}^*C_{-\mathbf{k}\downarrow}^*C_{-\mathbf{k}\downarrow}C_{\mathbf{k}\uparrow}|\rangle_0$$
$$-\gamma\,{}_0\langle|C_{\mathbf{k}\uparrow}^*C_{-\mathbf{k}\downarrow}^*\sum_{l<k_{\max}}C_{-\mathbf{l}\downarrow}C_{\mathbf{l}\uparrow}|\rangle_0$$
$$=-2(\epsilon_k-\mu)N_k-J_k\geqslant 0, \tag{6}$$

where

$$N_k={}_0\langle|C_{\mathbf{k}\uparrow}^*C_{-\mathbf{k}\downarrow}^*C_{-\mathbf{k}\downarrow}C_{\mathbf{k}\uparrow}|\rangle_0$$

and

$$J_k=\gamma\,{}_0\langle|C_{\mathbf{k}\uparrow}^*C_{-\mathbf{k}\downarrow}^*\sum_{l<k_{\max}}C_{-\mathbf{l}\downarrow}C_{\mathbf{l}\uparrow}|\rangle_0 \tag{7}$$

for $k<k_{\max}$. Hence,

(I) $\quad J_k\leqslant 0,\quad$ for $\quad \epsilon_k>\mu.$

In order to obtain further information about N_k, and J_k, we choose $F=C_{\mathbf{k}\uparrow}^*C_{\mathbf{k}\uparrow}$ in (5).

$$ {}_0\langle|C_{\mathbf{k}\uparrow}^*C_{\mathbf{k}\uparrow}[\tilde{H},C_{\mathbf{k}\uparrow}^*C_{\mathbf{k}\uparrow}]|\rangle_0$$
$$=\gamma\,{}_0\langle|C_{\mathbf{k}\uparrow}^*C_{-\mathbf{k}\downarrow}^*C_{-\mathbf{k}\downarrow}C_{\mathbf{k}\uparrow}|\rangle_0$$
$$-\gamma\,{}_0\langle|C_{\mathbf{k}\uparrow}^*C_{-\mathbf{k}\downarrow}^*\sum_{l<k_{\max}}C_{-\mathbf{l}\downarrow}C_{\mathbf{l}\uparrow}|\rangle_0$$
$$=\gamma N_k-J_k\geqslant 0 \tag{8}$$

and from the definition (7)

$$\sum_{k<k_{\max}}J_k=\gamma\,{}_0\langle|\sum_{k<k_{\max}}C_{\mathbf{k}\uparrow}^*C_{-\mathbf{k}\downarrow}^*\sum_{l<k_{\max}}C_{-\mathbf{l}\downarrow}C_{\mathbf{l}\uparrow}|\rangle_0$$
$$={}_0\langle|H_{\text{int}}|\rangle_0\geqslant 0. \tag{9}$$

From (9) and (I), we can conclude that

(II) $J_k\geqslant 0$ in some part of the region defined by $\epsilon_k<\mu$. It follows from (8) that the upper bound of J_k is γN_k and along with (II) we get

(III) $\gamma N_k\geqslant J_k\geqslant 0$ in some part of the region $\epsilon_k<\mu$.

From (I) and (III), the behavior of J_k as a function of ϵ_k is sketched in Fig. 1, assuming that J_k is a continuous function of ϵ_k. Performing summation over k in (8) and using (9), we obtain the following inequality.

(IV) $$\frac{\gamma N}{2}\geqslant\gamma\sum_{k<k_{\max}}N_k\geqslant\sum_{k<k_{\max}}J_k$$
$$={}_0\langle|H_{\text{int}}|\rangle_0=\sum_{J_k>0}J_k+\sum_{J_k<0}J_k\geqslant 0.$$

Since the value of J_k is bounded in the region $J_k>0$ (III) and the sum of J_k over the entire region is also bounded (IV), J_k can take values only of the $O(1)$ in the continuous region where $J_k<0$ except in the few isolated small regions of the $O(1/N)$, where it can become of the $O(N)$. Therefore we conclude that

(V) $\quad J_k=O(1)$

in the region where J_k is negative, except at few isolated points where it can be of the $O(N)$. Actually, if we use Schwarz inequality, we get the following bound on an absolute value of J_k.

[2a] *Note added in proof.* Because of the nature of the interaction, energy difference $2(E_0^{N+1}-E_0^N)$ is different from $E_0^{N+2}-E_0^N$ in general. In the following, we take F as CC or C^*C^*, hence our definition of μ should be more correctly $2\mu=E_0^{N+2}-E_0^N=E_0^N-E_0^{N-2}$. The authors would like to thank Professor John M. Blatt for pointing out this difference.

$$|J_k|^2 = {}_0\langle|\gamma C_{\mathbf{k}\uparrow}{}^*C_{-\mathbf{k}\downarrow}{}^* \sum_{l<k_{\max}} C_{-\mathbf{l}\downarrow}C_{\mathbf{l}\uparrow}|\rangle_0 {}_0\langle|\gamma \sum_{l<k_{\max}} C_{\mathbf{l}\uparrow}{}^*C_{-\mathbf{l}\downarrow}{}^*C_{-\mathbf{k}\downarrow}C_{\mathbf{k}\uparrow}|\rangle_0$$

$$\leqslant \gamma^2 {}_0\langle|C_{\mathbf{k}\uparrow}{}^*C_{-\mathbf{k}\downarrow}{}^*C_{-\mathbf{k}\downarrow}C_{\mathbf{k}\uparrow}|\rangle_0 {}_0\langle| \sum_{l<k_{\max}} C_{\mathbf{l}\uparrow}{}^*C_{-\mathbf{l}\downarrow}{}^* \sum_{p<k_{\max}} C_{-\mathbf{p}\downarrow}C_{\mathbf{p}\uparrow}|\rangle_0$$

$$= \gamma N_k {}_0\langle|H_{\text{int}}|\rangle_0;$$

therefore

$$|J_k| \leqslant N_k{}^{1/2}\gamma^{1/2} {}_0\langle|H_{\text{int}}|\rangle_0{}^{1/2}.$$

In the case of $\gamma = O(1)$, ${}_0\langle H_{\text{int}}\rangle_0{}^{1/2} \leqslant O(N^{1/2})$ from (IV); hence in the region $J_k<0$ absolute value of J_k has a bound of the $O(N^{1/2})$. For $\gamma=\lambda/\Omega$, $|J_k| \leqslant N_k{}^{1/2}[(\lambda/\Omega)O(N)]^{1/2} = O(1)$, the bound of $|J_k|$ is of the $O(1)$.

Again from (5) with $F=\sum_{k<k_{\max}} C_{-\mathbf{k}\downarrow}C_{\mathbf{k}\uparrow}$, we get

$$\begin{aligned}
&{}_0\langle| \sum_{l<k_{\max}} C_{\mathbf{l}\uparrow}{}^*C_{-\mathbf{l}\downarrow}{}^*[\tilde{H}, \sum_{k<k_{\max}} C_{-\mathbf{k}\downarrow}C_{\mathbf{k}\uparrow}]|\rangle_0 \\
&\quad = -{}_0\langle| \sum_{l<k_{\max}} C_{\mathbf{l}\uparrow}{}^*C_{-\mathbf{l}\downarrow}{}^* \sum_{k<k_{\max}} 2(\epsilon_k-\mu)C_{-\mathbf{k}\downarrow}C_{\mathbf{k}\uparrow}|\rangle_0 - \gamma(\sum_{k<k_{\max}} 1 - N_{\text{in}}+2) {}_0\langle| \sum_{l<k_{\max}} C_{\mathbf{l}\uparrow}{}^*C_{-\mathbf{l}\downarrow}{}^* \sum_{k<k_{\max}} C_{-\mathbf{k}\downarrow}C_{\mathbf{k}\uparrow}|\rangle_0 \\
&\quad = -2\sum_{k<k_{\max}} (\epsilon_k-\mu)J_k - (\sum_{k<k_{\max}} 1 - N_{\text{in}}+2) {}_0\langle|H_{\text{int}}|\rangle_0 \geqslant 0,
\end{aligned} \tag{10}$$

where N_{in} is the number of particles inside $k<k_{\max}$. Since

$$[H, \sum_{k<k_{\max}} (C_{\mathbf{k}\uparrow}{}^*C_{\mathbf{k}\uparrow}+C_{-\mathbf{k}\downarrow}{}^*C_{-\mathbf{k}\downarrow})]=0,$$

the number of particles inside $k<k_{\max}$ is a constant of the motion. From (V) it follows that the first term on the right-hand side of (10) is at most of the $O(N)$ and positive. Hence, we deduce from (10) that if

$$\text{(VI)} \qquad \begin{aligned} &\sum_{k<k_{\max}} 1 - N_{\text{in}}+2 = O(N) > 0, \\ &{}_0\langle|H_{\text{int}}|\rangle_0 = O(1) \end{aligned}$$

meaning that it cannot become of the $O(N)$. The above condition, for the ground state, can always be satisfied for the Fermi momentum $k_f \leqslant k_{\max}/2^{2/3}$, since for the ground state $N_{\text{in}}=N$ as we shall see later. The weaker dependence of this condition on $k_{\max}$ makes this case more important if this model has any physical significance. Finally with $F=\sum_{k<k_{\max}} C_{\mathbf{k}\uparrow}{}^*C_{-\mathbf{k}\downarrow}{}^*$, (5) becomes

$$\begin{aligned}
&{}_0\langle| \sum_{p<k_{\max}} C_{-\mathbf{p}\downarrow}C_{\mathbf{p}\uparrow}[\tilde{H}, \sum_{k<k_{\max}} C_{\mathbf{k}\uparrow}{}^*C_{-\mathbf{k}\downarrow}{}^*]|\rangle_0 \\
&\quad = {}_0\langle| \sum_{p<k_{\max}} C_{-\mathbf{p}\downarrow}C_{\mathbf{p}\uparrow} \sum_{k<k_{\max}} 2(\epsilon_k-\mu)C_{\mathbf{k}\uparrow}{}^*C_{-\mathbf{k}\downarrow}{}^*|\rangle_0 + (\sum_{k<k_{\max}} 1 - N_{\text{in}})\gamma {}_0\langle| \sum_{p<k_{\max}} C_{-\mathbf{p}\downarrow}C_{\mathbf{p}\uparrow} \sum_{k<k_{\max}} C_{\mathbf{k}\uparrow}{}^*C_{-\mathbf{k}\downarrow}{}^*|\rangle_0 \\
&\quad = \sum_{k<k_{\max}} 2(\epsilon_k-\mu) {}_0\langle|1-C_{\mathbf{k}\uparrow}{}^*C_{\mathbf{k}\uparrow}-C_{-\mathbf{k}\downarrow}{}^*C_{-\mathbf{k}\downarrow}|\rangle_0 + \sum_{k<k_{\max}} 2(\epsilon_k-\mu)J_k \\
&\qquad + (\sum_{k<k_{\max}} 1 - N_{\text{in}})[\gamma(\sum_{k<k_{\max}} 1 - N_{\text{in}}) + {}_0\langle|H_{\text{in}}|\rangle_0] \geqslant 0.
\end{aligned}$$

In the above inequality the first two terms are at most of the order N, and hence, if

$$\sum_{k<k_{\max}} 1 - N_{\text{in}} = O(N) < 0,$$

$$\gamma(\sum_{k<k_{\max}} 1 - N_{\text{in}}) = {}_0\langle|H_{\text{int}}|\rangle_0 = O(1) > 0,$$

because

$$\gamma(\sum_{k<k_{\max}} 1 - N_{\text{in}}) + {}_0\langle|H_{\text{int}}|\rangle_0 = {}_0\langle|\gamma \sum_{k<k_{\max}} C_{-\mathbf{p}\downarrow}C_{\mathbf{p}\uparrow} \sum_{k<k_{\max}} C_{\mathbf{k}\uparrow}{}^*C_{-\mathbf{k}\downarrow}{}^*|\rangle_0 > 0,$$

from which we get

$${}_0\langle|H_{\text{int}}|\rangle_0 = \gamma(N_{\text{in}} - \sum_{k<k_{\max}} 1) + O(1). \tag{11}$$

However, for the ground state, this above condition is fulfilled for $k_f > k_{\max}/2^{2/3}$, which follows from the same reasoning we used earlier in the first possibility. (We shall explain this possibility later in detail.) The stronger dependence on the artificial cutoff momentum $k_{\max}$ makes this case less important, even if this model has any physical significance.

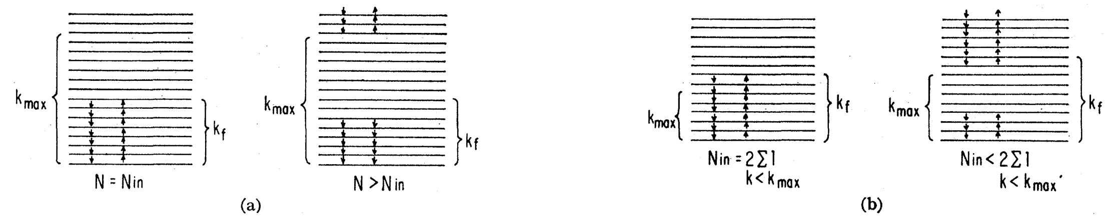

FIG. 2. (a) $N=N_{\text{in}}$ shows the possible ground-state configuration for cases (*a*) and (*c*). (b) $N_{\text{in}}=2\sum_{k<k_{\max}}1$ shows the possible ground state of case (*d*).

To summarize, we have reached the following conclusions: (1) if

$$\sum_{k<k_{\max}} 1-N_{\text{in}}+2=O(N)>0;\quad {}_0\langle|H_{\text{int}}|\rangle_0=O(1);$$

and (2) if

$$\sum_{k<k_{\max}} 1-N_{\text{in}}=O(N)<0;$$

$$ {}_0\langle|H_{\text{int}}|\rangle_0=\gamma(N_{\text{in}}-\sum_{k<k_{\max}} 1)+O(1),$$

where N_{in} ($\leqslant N$) is the eigenvalue of the equation,

$$\sum_{k<k_{\max}} (C_{\mathbf{k}\uparrow}{}^*C_{\mathbf{k}\uparrow}+C_{-\mathbf{k}\downarrow}{}^*C_{-\mathbf{k}\downarrow})|\rangle_0=N_{\text{in}}|\rangle_0.$$

For the ground state, the first condition can always be satisfied for $k_f\leqslant k_{\max}/2^{2/3}$, while the second possibility can occur for $k_f>k_{\max}/2^{2/3}$. Obviously, for sufficiently large cutoff momentum $k_{\max}$, the first possibility is more realistic than the second one.

Let us discuss various possible cases of the ground-state energy of the Hamiltonian (1), which by definition is the minimum energy. First we notice from the variational principle that

$$\begin{aligned}(H_0+H_{\text{int}})_{\text{plane-wave free Fermi}}&=(H_0)_{\text{free Fermi}}+\tfrac{1}{2}\gamma N_{\text{in}}\geqslant E_0(\gamma)={}_0\langle|H_0|\rangle_0+{}_0\langle|H_{\text{int}}|\rangle_0\\&\geqslant {}_0\langle|H_0|\rangle_0\geqslant (H_0)_{\text{free Fermi}}.\end{aligned}$$

From the above inequality we have

$$(H_0)_{\text{free Fermi}}+\tfrac{1}{2}\gamma N_{\text{in}}\geqslant E_0(\gamma)\geqslant (H_0)_{\text{free Fermi}},$$

from which we conclude that

$$\lim_{\gamma\to 0} E_0(\gamma)=(H_0)_{\text{free Fermi}}.$$

Case (*a*): $\sum_{k<k_{\max}}1>N\geqslant N_{\text{in}}$ [see Fig. 2(a)].

As we have pointed out earlier, this is the most realistic case, since we can always satisfy the restriction on N for any $k_{\max}$ at sufficiently low density, such that $\sum_{k<k_{\max}}1-N_{\text{in}}=O(N)>0$. In this case; if $N-N_{\text{in}}=O(N)>0$, there are $N-N_{\text{in}}$ free particles outside $k<k_{\max}$ and hence $\lim_{\gamma\to 0}E_0(\gamma)$ cannot approach $(H_0)_{\text{free Fermi}}$. This indicates that such a state can not represent the ground state at sufficiently small γ (yet of order one). We are left with the possibility of $N=N_{\text{in}}$ for the ground state. However, since $\partial E_0(\gamma)/\partial\gamma=(1/\gamma)\,{}_0\langle|H_{\text{int}}|\rangle_0=0(1)$ and $\lim_{\gamma\to 0}E_0(\gamma)=(H_0)_{\text{free Fermi}}$, we can conclude that

$$E_0(\gamma)=(H_0)_{\text{free Fermi}}+\text{ of the }O(1),$$

and consequently

$$\begin{aligned}{}_0\langle|C_{\mathbf{k}\uparrow}{}^*C_{\mathbf{k}\uparrow}|\rangle_0&={}_0\langle|C_{-\mathbf{k}\downarrow}{}^*C_{-\mathbf{k}\downarrow}|\rangle_0\\&=(n_{\mathbf{k}})_{\text{free Fermi}}+\text{of the }O(1/N).\end{aligned}$$

This proves the required result that for $\gamma=O(1)$, H_{int} produces no effect on the system in this case.

The following three cases are less important as far as the study of single-particle momentum distribution is concerned. Firstly, because of the artificial construction of the regions of N, and secondly, because of the dependence on the artificial cutoff momentum $k_{\max}$. However, such a division of the region of N allows us to remark about the possible phase change, which in our approach, exists owing to the artificial introduction of the cutoff momentum $k_{\max}$.

Case (*b*): $N>\sum_{k<k_{\max}}1>N_{\text{in}}$.

Such a state cannot represent the ground state for sufficiently small γ (of order one) by the same reasoning as in case (*a*), $N>N_{\text{in}}$.

Case (*c*): $2\sum_{k<k_{\max}}1\geqslant N\geqslant N_{\text{in}}>\sum_{k<k_{\max}}1$.

Again for the ground state we should have $N=N_{\text{in}}$ and in this case we have $N-\sum_{k<k_{\max}}1\leqslant\frac{1}{2}N$. Therefore from (11) the ground-state energy must satisfy

$$\partial E_0(\gamma)/\partial\gamma=(N-\sum_{k<k_{\max}} 1)+\text{of the }O(1)$$

and

$$\lim_{\gamma\to 0} E_0(\gamma)=(H_0)_{\text{free Fermi}}.$$

Hence

$$\begin{aligned}E_0(\gamma)&=(H_0)_{\text{free Fermi}}+\gamma(N_{\text{in}}-\sum_{k<k_{\max}} 1)+O(1)\\&\equiv 2\sum \epsilon_k N_{\mathbf{k}}+{}_0\langle|H_{\text{int}}|\rangle_0,\end{aligned}$$

where $N_{\mathbf{k}}$ is the true single-particle momentum distribution. The comparison of the last two expressions

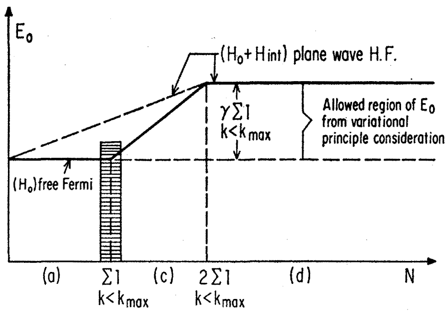

FIG. 3. The solid line indicates the behavior of the ground-state energy as a function of the number of particles N. In the shaded portion, for finite γ, our results do not hold since the condition $\gamma<|k_{\max}^2/2m-k_f^2/am|$ is violated. For convenience the $N^{5/2}$ behavior is sketched by the horizontal lines and the inclined line starting from the horizontal line denotes the superposition of a linear N dependence on that $N^{5/2}$ curve.

of $E_0(\gamma)$ forces us to conclude that the true single-particle momentum distribution

$$N_{\mathbf{k}}=(n_{\mathbf{k}})_{\text{free Fermi}}+O(1/N)\,,$$

since $\underset{0}{\langle}|H_{\rm int}|\underset{0}{\rangle}=\gamma(N_{\rm in}-\sum_{k<k_{\max}}1)+O(1)$ in this case. The above expression of $E_0(\gamma)$ will be the minimum for the variation of $N_{\rm in}$ if $\gamma<(k_{\max}/2m-k_f/2m)$, because, if we reduce $N_{\rm in}$ by 1, the potential energy decreases by γ and the kinetic energy increases approximately by $[(k_{\max}^2/2m)-(k_f^2/2m)]$.

Case (*d*): $N>2\sum_{k<k_{\max}}1\geqslant N_{\rm in}>\sum_{k<k_{\max}}1$ [see Fig. 2(b)].

The ground state is obtained for $N_{\rm in}=2\sum_{k<k_{\max}}1$ and we can derive easily from (11) that

$$E_0(\gamma)=(H_0)_{\text{free Fermi}}+\gamma\sum_{k<k_{\max}}1+O(1)$$
$$=(H_0+H_{\rm int})_{\text{plane-wave HF}}+O(1)\,.$$

With the similar arguments as in case (c), $E_0(\gamma)$ will be minimum if $\gamma<(k_f^2/2m)-(k_{\max}^2/2m)$. In this case also the momentum distribution is the same as the free Fermi distribution up to order $1/N$, as can be seen from the same reasoning as in case (*c*).

CONCLUSION

From the discussion of the various possible cases above, we conclude in the following about the behavior of the ground-state energy and the one-particle momentum distribution, as a function of the total number of particles. As far as γ is of order one and small, in the entire region of N, the one-particle momentum distribution is the same as the free Fermi distribution up to the difference of the $O(1/N)$, except in the region of density where $\gamma\approx|(k_f^2/2m)-(k_{\max}^2/2m)|$. The ground-state energy remains unperturbed, equal to $(H_0)_{\text{free Fermi}}$ in region (a) and equal to $(H_0+H_{\rm int})$ plane-wave HF in region (d), up to $O(1)$, as shown in the Fig. 3. In the region (c), it joins the values in (a) and (b) with an additional linear dependence of N. The behavior of the ground-state energy E_0 as a function of the number of particles N, compels us to conclude that the system described by (1) undergoes a phase change as shown in Fig. 3, without the change in the single-particle momentum distribution. All our results are valid for $\gamma<|(k_{\max}^2/2m)-(k_f^2/2m)|$ and therefore for finite γ, in the neighborhood of $k_f=k_{\max}$, the expression for the ground-state energy is to be obtained by minimizing it with respect to $N_{\rm in}$.

APPENDIX A: $\gamma=\lambda|\Omega>0$

Using (IV) we can conclude that $\underset{0}{\langle}|H_{\rm int}|\underset{0}{\rangle}=0(1)$. Moreover, we have shown that

$$\lim_{\lambda\to 0}E_0(\lambda)=(H_0)_{\text{free Fermi}}\,.$$

From this result and along with $\partial E_0(\lambda)/\partial\lambda=\underset{0}{\langle}|H_{\rm int}|\underset{0}{\rangle}/\lambda=O(1)$, we obtain $E_0=(H_0)_{\text{free Fermi}}+O(1)$.

This compels us to conclude that, in the case which Van Hove[1] treated, $H_{\rm int}$ changes the ground-state energy only by a quantity of the $O(1)$ and consequently the momentum distribution changes by a small number of $O(1/N)$.

APPENDIX B: $\gamma=\lambda|\Omega<0$

From Eq. (8), $\gamma N_k\geqslant J_k$, and when $\gamma=\lambda|\Omega<0$, J_k is always negative. Moreover in this case $\underset{0}{\langle}|H_{\rm int}|\underset{0}{\rangle}=\sum_{k<k_{\max}}J_k<0$. Therefore, J_k can be arbitrarily large and negative, since there does not exist any lower bound on J_k. This clearly indicates the possibility of $\underset{0}{\langle}|H_{\rm int}|\underset{0}{\rangle}=\sum_{k<k_{\max}}J_k=O(N)<0$, which happens actually in the BCS theory.