Incoherent Scattering of Radiation by Plasmas. I. Quantum Mechanical Calculation of Scattering Cross Sections

D. F. DuBois
Hughes Research Laboratories, Malibu, California

AND

V. Gilinsky
The RAND Corporation, Santa Monica, California

(Received 14 October 1963)

The quantum electrodynamics of the scattering of radiation from an interacting, fully-ionized plasma is treated. The theory includes the effects of quantum statistics and is valid to all orders in many-particle perturbation theory. The scattering cross section is related to the partial conductivities of the interacting system. This reduces the problem of the inclusion of close Coulomb collisions, which lie outside of previous random-phase approximation calculations, to the calculation of collision corrected conductivities. The theory is applicable to degenerate quantum plasmas as well as to classical plasmas.

I. INTRODUCTION

Some interest has recently arisen in the incoherent scattering of electromagnetic radiation from plasmas. Experiments have been carried out and more are under way in which high-powered radar beams are backscattered from the ionosphere.[1–3] With the advent of high-intensity laser beams the possibility arises for measuring the incoherent scattering of light from laboratory plasmas. If the frequency of the radiation is such that the material is essentially transparent and, in addition, the wavelength is much greater than the Debye length of the plasma, the theory shows that the line shape of the incoherent scattering is an excellent measure of the spectrum of collective modes in the plasma.

The classical theory of incoherent scattering from plasmas has been given in the random-phase approximation (RPA) Dougherty and Farley,[4] Salpeter,[5] Rostoker and Rosenbluth,[6] and others. The theory which includes the effects of wave mechanics and quantum statistics has not been given previously to our knowledge, nor has a treatment of the effect of close Coulomb collisions.

The purpose of this paper is to present the (nonrelativistic) quantum electrodynamics of scattering of radiation from a plasma, quantum or classical. Using modern techniques, we develop here a theory which includes quantum effects and the effects of close collisions which are not included in previous RPA calculations. In a second paper we apply these results to the classical plasma and calculate the effects of short-range collisions on the scattered line shape. Later, we hope to study the applicability of this phenomena to determine the spectrum of collective excitations in solid-state systems.

We employ a form of many-particle perturbation theory introduced in Ref. 7 in which various interaction processes are represented by Feynman diagrams with an accompanying set of rules for evaluating the contribution of each diagram. These techniques, which are very similar to the Feynman techniques of vacuum electrodynamics, were used in Ref. 7 to compute the high-frequency conductivity of a plasma. They have the advantage of expressing in a very explicit way the basic microscopic processes involved. In the present paper, the coupling of the radiation to the ions via the excitation by the electrons of the acoustic plasma resonance in the screening cloud of the ion is very explicitly shown in the method of calculation.

In Sec. II we present the formalism and perform the calculation of the line shape in the RPA in order to show how the standard results are obtained by our method. In Sec. III a general formula for the scattering rate is developed for a two-component plasma and an electron gas with uniform background. Finally, in Sec. IV we briefly comment on the applicability of these results to experimental situations.

Previous classical calculations relate the scattering cross section to the electron density correlation function (see the Appendix) and then proceed to calculate this quantity directly from the plasma hierarchy[6] or first relate the correlation function to the external conductivity via the fluctuation-dissipation theorem and then calculate the conductivity from the Vlasov equation.[4,5] Our procedure uses the techniques of quantum electrodynamics to derive from the microscopic theory what is essentially the fluctuation-dissipation theorem.

[1] K. W. Bowles, Phys. Rev. Letters **1**, 454 (1958). The experiments were originally suggested by W. E. Gordon, Proc. Inst. Radio Engrs. **46**, 1824 (1955).

[2] V. C. Pineo, L. G. Craft, and H. W. Briscoe, J. Geophys. Res. **65**, 2629 (1960).

[3] H. Carru and M. Petit, Note Technique Projet C.D.S. 2, Centre National d'Etudes des Télécommunications, Issy-les-Moulineaux, 1962 (unpublished).

[4] J. P. Dougherty and D. T. Farley, Proc. Roy. Soc. (London) **A259**, 79 (1960).

[5] E. E. Salpeter, Phys. Rev. **120**, 1528 (1960).

[6] M. N. Rostoker and N. Rosenbluth, Phys. Fluids **5**, 776 (1962).

[7] D. F. DuBois, V. Gilinsky, and M. G. Kivelson, Phys. Rev. **129**, 2376 (1963).

For cases in which the scattered frequency $\omega=\omega_b-\omega_a$ is large enough so the ions cannot respond, and for a fixed value of the wave-number shift $|\mathbf{k}|=|\mathbf{k}_b-\mathbf{k}_a|$ the cross section can be related to the *total* local conductivity. For ω small enough for the ions to respond (as is the case near the acoustic ion plasma wave frequency) to evaluate the cross section, it is necessary to know separately the response of the electrons keeping the ions in equilibrium and vice versa. This effect arises because the ions scatter the radiation only m/M times as effectively as the electrons.

Since the cross section is exactly related to complete conductivities, the problem of including the effect of close collisions is reduced to a conductivity calculation. In a second paper we make use of some recently derived expressions for the collisional conductivities in a classical plasma as an application of the formulas in the present paper.

II. FORMULATION

The description of individual electromagnetic processes in terms of Feynman diagrams is at once the clearest, most elegant, and the most convenient for the purpose of calculation. We shall utilize this method for the problem at hand: the incoherent scattering of photons from a many-electron system in thermodynamic equilibrium.

The total scattering rate Γ_{tot} is given by the familiar Golden Rule

$$\hbar\Gamma\ \text{total}=\tfrac{1}{2}\sum_{e_b,e_a}\int\frac{d^3k_b}{(2\pi)^3}\sum_f\sum_i\rho_i\frac{\hbar^2c^4}{2\omega_b 2\omega_a}\times|\langle f;\mathbf{k}_b,\hat{e}_b|M|i;\mathbf{k}_a,\hat{e}_a\rangle|\times(2\pi\hbar)^3\delta^3(\mathbf{P}_f-\mathbf{P}_i+\hbar\mathbf{k}_b-\hbar\mathbf{k}_a)\times 2\pi\delta(E_f-E_i+\hbar\omega_b-\hbar\omega_a)\,. \quad (2.1)$$

The subscripts i and f refer to the initial and final states of the matter. The subscripts a and b refer to the initial and final states of the radiation ($\mathbf{k}_a$ is the initial wave number, $\hat{e}_a$ is the initial polarization). The photons are normalized in a unit volume. In Eq. (2.1) we sum over all final states and average over all initial states. The correct average over the initial particles states is obtained by using a suitable Gibbs factor,

$$\rho_i=e^{\beta\Omega}e^{-\beta(E_i-\mu N_i)}\,, \quad (2.2)$$

where $\beta^{-1}=kT$, μ is the chemical potential, N_i is the number of particles in the state i, and $\exp(\beta\Omega)$ is the normalizing factor fixed by $\text{Tr}\rho_i=1$. We are considering a unit volume so that we can denote by n the number density and the total particle number.

We are, however, interested not in the total transition rate but in the rate to all particle states with the final photon state $(\mathbf{k}_b,\omega_b)$ fixed. We denote this by $\Gamma(\mathbf{k},\omega)$. The experimentally observable partial cross section $d\sigma(k\omega)/d\omega_b d\Omega_b$ which is independent of the normalization volume is obtained by dividing $\Gamma(\mathbf{k}\omega)$ by the photon speed c. From Eq. (2.1) one has

$$\frac{d\sigma(\mathbf{k},\omega)}{d\omega_b d\Omega_b}=\tfrac{1}{2}\sum_{e_b,e_a}\sum_f\sum_i\rho_i\frac{\hbar}{4(2\pi)^3}\frac{\omega_b}{\omega_a}\times|\langle f;\mathbf{k}_b,\hat{e}_b|M|i;\mathbf{k}_a\hat{e}_a\rangle|^2(2\pi\hbar)^3\delta^3(\mathbf{P}_i-\mathbf{P}_f+\hbar\mathbf{k})\times 2\pi\delta(E_f-E_i-\hbar\omega)\,. \quad (2.3)$$

To obtain the amplitude M we simply draw all possible modes for scattering a photon in state $(\mathbf{k}_a,e_a)$ to a state $(\mathbf{k}_b,e_b)$. We can make a number of approximations: First, we will consider only the coupling of the incoming and outgoing radiation to matter arising from the $(e^2n/mc^2)A^2$ term in the nonrelativistic Hamiltonian which is represented by the double photon vertex in Fig. 1. In the Appendix this approximation is seen to lead to the well-known expression relating the cross section to the density-density correlation function derived classically as, for example, by Rostoker and Rosenbluth.[6] The single photon vertex, arising from the $p\cdot A$ term in the coupling Hamiltonian, contributes in second-order perturbation theory. These terms can be neglected when $k_a\ll k_D$ (k_D the Debye momentum) and ω_a is much greater than the collision frequencies of the system as can be seen from the following argument. The optical theorem states that the total cross section for scattering and absorbing a photon $(\mathbf{k}_a,e_a)$ is proportional to the imaginary part of the forward scattering amplitude. The imaginary part of the forward scattering amplitude is easily seen to be proportional to the *real* part of the polarization self-energy part for transverse photons discussed in Ref. 7. In the notation of that reference the total cross section is then $\omega_p^2c^{-2}-\text{Re}Q_T(\omega_a,k_a)c^{-2}$. The term $\omega_p^2c^{-2}=nr_0^2$ is the sole contribution from the A^2 interaction of the radiation with matter while $\text{Re}Q_T$ contains all the terms arising from the $p\cdot A$ coupling. The leading term in Q_T arising from the RPA gives a contribution $k^2v_e^2/3c^2=k^2\omega_p^2/(3k_D^2c^2)$ and can be neglected if $k\ll k_D$. The term arising from electron-ion collision corrections, however, is of order $(\omega_p^2/c^2)(\Gamma_{ei}/\omega)$ where Γ_{ei} is of the order of the electron-ion collision frequency. Likewise, from electron-electron collisions there is a contribution of order $(\omega_p^2/c^2)(k^2/k_D^2)(\omega_p^2\Gamma_{ee}/\omega^3)$ where Γ_{ee} is of the order of the electron-electron collision frequency. These estimates are somewhat crude and a more complete discussion will be given elsewhere. The result is that if $k_a\ll k_D$ and $\omega_a\gg\Gamma_{ei},\Gamma_{ee}$, the contributions to the total cross section from the $p\cdot A$ coupling can be neglected relative to the A^2 coupling. This implies that the same is true for the partial scattering cross sections provided the A^2 coupling produces a cross section which is not vanishingly small at any angle which we shall see is the case at least near the important resonances. Furthermore, we shall neglect terms of order $\alpha^2=m/M$, the ratio of the electron and ion masses, so it is not necessary to consider the interaction of the *radiation*

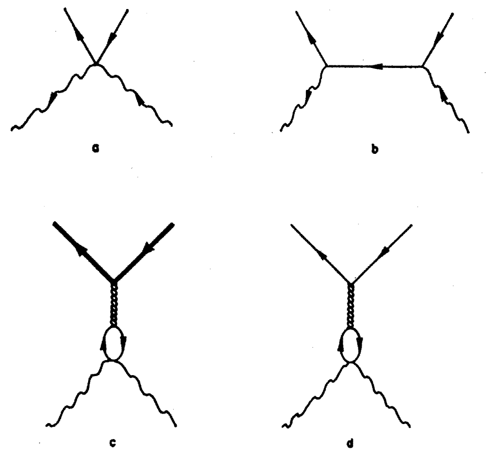

FIG. 1. Lowest order scattering process. The heavy line denotes ions.

with the ions. In Fig. 1(a) we show the simplest interaction of interest here. Other possible processes in a plasma are shown in Figs. 1, 3, and 4. The braided line represents a screened Coulomb interaction.

The amplitude for any process is readily computed from the corresponding diagram by use of the calculating rules presented in Ref. 7. For example, the amplitude for the double photon-electron vertex shown in Fig. 1(a) is given in Ref. 7 to be $i(4\pi e^2/mc^2)\hat{e}_b\cdot\hat{e}_a$. The rate for this process is then

$$\frac{d\sigma(\mathbf{k},\omega)}{d\omega_b d\Omega_b}=\frac{1}{2}\sum_{e_b,e_a}\int\frac{d^3p_2}{(2\pi\hbar)^3}\int\frac{d^3p_1}{(2\pi\hbar)^3}f(\xi_{p_1})\frac{\hbar}{4(2\pi)^3}\frac{\omega_b}{\omega_a}\times\left|\frac{4\pi e^2}{mc^2}\hat{e}_b\cdot\hat{e}_a\right|^2(2\pi\hbar)^3\delta^3(\mathbf{p}_2-\mathbf{p}_1+\hbar\mathbf{k})\times 2\pi\delta\left(\frac{1}{2m}p_2^2-\frac{1}{2m}p_1^2+\hbar\omega\right), \quad (2.4)$$

where $\mathbf{k}=\mathbf{k}_b-\mathbf{k}_a$, $\omega=\omega_a-\omega_b$, and $f(\xi_p)$ is the one particle distribution function with $\xi_p=p^2/2m-\mu$, μ the chemical potential.

We shall deal only with unpolarized radiation, so we sum and average over the polarizations.

$$\frac{1}{2}\sum_{e_b,e_a}(\hat{e}_b\cdot\hat{e}_a)^2=\frac{1}{2}(1+\cos^2\theta),$$

where $\mathbf{k}_b\cdot\mathbf{k}_a=k_bk_a\cos\theta$. The integration in Eq. (2.4) is simple and yields the expression

$$\frac{d\sigma}{d\omega d\Omega}=2\pi r_0^2\tfrac{1}{2}(1+\cos^2\theta)\frac{\omega_b}{\omega_a}\frac{m}{k}\frac{1}{(2\pi\hbar)^3}\times\int_{|m\omega/k+\hbar k/2|}^{\infty}dp\,pf(\xi_p), \quad (2.5)$$

where $r_0=e^2/mc^2$ is the classical electron radius.

In the limit of Boltzmann statistics this becomes (with $f(\xi_p)=n(2\pi\beta/m)^{3/2}\hbar^3\exp\{-p^2/2m\}$)

$$\frac{d\sigma}{d\omega d\Omega}=\frac{nr_0^2}{(2\pi)^{1/2}}\frac{1}{k}\cdot\tfrac{1}{2}(1+\cos^2\theta)\frac{\omega_b}{\omega_a}e^{-(1/2)(\omega/k+\hbar k/2)^2}, \quad (2.5a)$$

where we have gone over to plasma units[7] $\mathbf{k}=(\mathbf{k}_b-\mathbf{k}_a)/k_D$ and $\omega=(\omega_b-\omega_a)/\omega_p$ with $k_b^2=4\pi e^2n\beta$, $\omega_p^2=4\pi e^2$ nm^{-1}. In these units $\hbar$ is in units of $\beta\omega_p$. For most cases of interest in classical plasmas we can neglect the term proportional to $\hbar$ in the exponential.

The scattering due to the process in Fig. 1(a) produces a line of width proportional to the electron rms velocity $v_e=\omega_p/k_D$, as we see from Eq. (2.5a). This is in disagreement with the experiments of Bowles[1] who found a linewidth proportional to the *ion* rms velocity. We look to diagrams involving the interparticle interactions to correct this discrepancy.

Likewise, in the case of degenerate electrons at $T=0$ where the distribution is just a step function, $f(\xi_p)=\eta(p_F-p)$ (p_F is the Fermi momentum), we find

$$\frac{d\sigma}{d\omega d\Omega}=nr_0^2\frac{(1+\cos^2\theta)}{2}\frac{\omega_b}{\omega_a}\frac{3}{4\pi}\frac{m}{kp_F}\frac{p_F^2-|m\omega/k+\hbar k/2|^2}{2p_F^2},$$
$$\text{if}\quad p_F\geq|m\omega/k+\hbar k/2|; \quad (2.5b)$$
$$=0, \text{ otherwise}.$$

The width of the scattered line is of the order kp_F/m or again $k\langle v_e\rangle$.

A narrow line would result if we could replace the electrons by ions in Fig. 1(a), but since the direct interaction of the photons with ions is of order $\alpha^2=m/M$ smaller, this would not account for the observed cross section. However, there is a way of coupling ω, k into the ion motion as shown in Fig. 1(c). In this diagram the ω, k are transmitted to an ion via the dynamically screened interaction (braided line). That is, an electron in the polarization cloud which accompanies each ion (heavy line) scatters the radiation and then interacts with the ion via the screened interaction transferring ω and k directly to the ion. The amplitude for this process (using the notation of Ref. 7) is given by (amplitude for photon-electron vertex)$\times$(amplitude for electron density fluctuation bubble)$\times$(amplitude for screened interaction), or,

$$M_{\mathrm{RPA}}{}^i=\left(i\frac{4\pi e^2}{mc^2}\hat{e}_b\cdot\hat{e}_a\right)(-Q_e{}^0(\mathbf{k},\omega))\left(\frac{-i}{k^2+Q(k,\omega)}\right)=-\frac{(4\pi e^2)}{mc^2}\frac{Q_e{}^0}{k^2+Q^0}\hat{e}_b\cdot\hat{e}_a. \quad (2.6)$$

The function $Q(\mathbf{k},\omega)$ is the proper polarization part[7] which in the lowest order or RPA is the sum of simple electron plus ion loops

$$Q(\mathbf{k},\omega)\simeq Q^0(\omega/k)=Q_e^0(\omega/k)+Q_i^0(\omega/k)\,, \tag{2.7}$$

where

$$Q_i^0(\omega/k)=Q_e^0(\omega/\alpha k),\quad \alpha^2=m/M\,. \tag{2.8}$$

In the classical limit of Boltzmann statistics with $\hbar=0$ this function is

$$Q_e^0\left(\frac{\omega}{k}\right)=1-\frac{\omega}{k}e^{-1/2(\omega/k)^2}\int_0^{\omega/k}dt\,e^{1/2t^2}+i\left(\frac{\pi}{2}\right)^{1/2}\frac{\omega}{k}e^{-1/2(\omega/k)^2}. \tag{2.9}$$

The general expressions for $Q^0(\omega/k)$ can be found elsewhere.[7]

The cross section is then given by substitution of $M_{\mathrm{RPA}}{}^i$ into Eq. (2.3)

$$\left[\frac{d\sigma^i}{d\omega d\Omega}\right]_{\mathrm{RPA}}=\tfrac{1}{2}(1+\cos^2\theta)\int\frac{d^3p_1}{(2\pi\hbar)^3}n\hbar^3\left(\frac{2\pi\beta}{M}\right)^{3/2}e^{-\beta p_1{}^2/2M}\frac{\hbar}{4(2\pi)^2}\frac{\omega_b}{\omega_a}$$

$$\times\left|\frac{4\pi e^2}{mc^2}\frac{Q_e^0}{k^2+Q^0}\right|^2(2\pi\hbar)^3\delta^3(\mathbf{p}_2-\mathbf{p}_1+\hbar\mathbf{k})(2\pi)\delta\left(\frac{1}{2M}p_2{}^2-\frac{1}{2M}p_1{}^2+\hbar\omega\right) \tag{2.10}$$

$$=nr_0{}^2\frac{|Q_e^0|^2}{|k^2+Q^0|^2}\frac{1}{\sqrt{2\pi}\alpha k}e^{-1/2(\omega/\alpha k)^2}\tfrac{1}{2}(1+\cos^2\theta)\frac{\omega_b}{\omega_a}. \tag{2.11}$$

We have taken the limit of Boltzmann statistics with $\hbar=0$. Here we have used the units of Eq. (2.5a).

This is the part of the well-known formula[4–6] in the RPA which is significant near the center of the scattered line. In distinction to Eq. (2.5a) the exponential factor here produces a narrow line with a width proportional to the *ion* thermal velocity $v_i=\alpha v_e=(m/M)^{1/2}(\omega_p/k_D)$. The same width but a magnitude reduced by a factor of α^2 arises from the direct coupling of the ion to the radiation. The intermediate coupling of the screening cloud causes an additional enhancement of the scattering rate due to the denominator $|k^2+Q^0|^2$ which is proportional to the absolute square of the longitudinal dielectric function ϵ_L of the plasma (see Sec. II). This becomes small at the collective resonances of the system. Since the exponential in Eq. (2.11) cuts off frequencies $\omega>\alpha k$, the electron plasma resonance at $\omega=1$ does not contribute to this formula. However, the very broad ion acoustic resonance (for equal electron and ion temperatures) contributes two humps in the curves shown in Fig. 2.

In the model case of a degenerate electron gas with uniform positive background there is, strictly speaking, no such contribution. In solid-state cases the ions should be replaced by lattice phonons and impurity ions. In addition, the screened interaction should also contain the interaction via phonons. The formulas in this case will have additional resonances for given k corresponding to possible phonon modes which can be excited. We will not treat these cases explicitly in the present paper, however, it is clear that in all these cases the line given by Eq. (2.11) or its analogs is considerably narrower than in the Doppler broadened scattering from free electrons [Eq. (2.5a)].

It is clear that an analogous process is possible in which the radiation interacts with the screening cloud surrounding an electron transferring ω and k to the electron as in Fig. 1(d). The amplitude for this process should be added to the amplitude for the direct scattering by an electron since both processes have the same final state. The total amplitude in this case is

$$M_{\mathrm{RPA}}{}^e=i\frac{4\pi^2}{mc^2}\hat{e}_b\cdot\hat{e}_a\left[1-\frac{Q_e^0}{k^2+Q^0}\right] \tag{2.12}$$

and the corresponding partial cross section in the

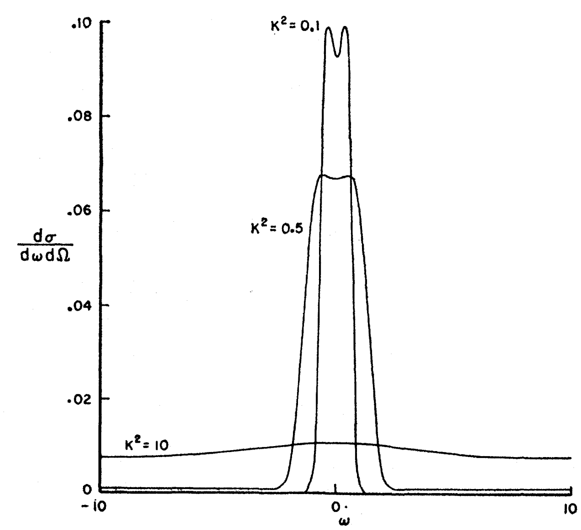

FIG. 2. Line shape of scattered radiation in the random-phase approximation for several values of k. The scale is arbitrary. The diagram does not extend far enough to show the high-frequency electron resonances on both sides.

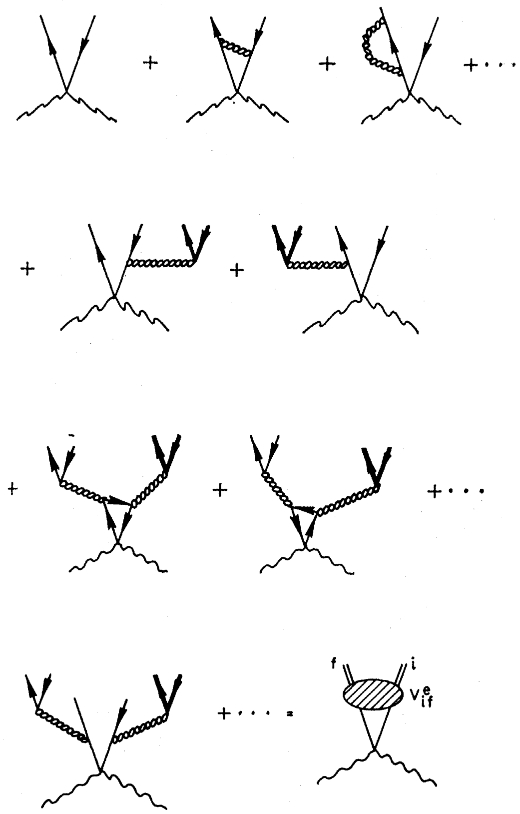

FIG. 3. Nonresonant scattering processes.

classical limit:

$$\left[\frac{d\sigma^e}{d\omega d\Omega}\right]_{\mathrm{RPA}}=nr_0^2\left|1-\frac{Q_e^0}{k^2+Q^0}\right|^2\frac{1}{\sqrt{2}\pi k}e^{-1/2(\omega/k)^2}\times\tfrac{1}{2}(1+\cos^2\theta)\frac{\omega_b}{\omega_a}. \quad (2.13)$$

This contribution has a width [like Eq. (2.5a)] proportional to kv_e but is of order $\alpha=(m/M)^{1/2}$ smaller than $\Gamma_{\mathrm{RPA}}{}^i$ at $\omega=0$ and so is unimportant near the central line. However, because of the broad exponential the sharp plasma resonance at $\omega\approx 1$ can contribute through this term giving the so-called plasma satellite lines.

The cross section $d\sigma^e/d\omega d\Omega$ as given by Eq. (2.13) is the only RPA contribution to the scattering in an electron gas. The case of degenerate electrons can be found by noting that the factor $(2\pi)^{-1/2}e^{-1/2(\omega/k)^2}$ in Eq. (2.13) is replaced by $(3m/8\pi p_F{}^3)[p_F{}^2-(m\omega/k+\hbar k/2)^2]$ as was the case in Eqs. (2.5a) and (2.5b). In addition, the appropriate expression for $Q(k,\omega)$ in the quantum limit must be used.

The total RPA scattering rate

$$\frac{d\sigma}{d\omega d\Omega_{\mathrm{RPA}}}=\frac{d\sigma^i}{d\omega d\Omega_{\mathrm{RPA}}}+\frac{d\sigma^e}{d\omega d\Omega_{\mathrm{RPA}}} \quad (2.14)$$

is plotted in Fig. 2 for various values of k^2 for a classical electron-ion plasma. The curves do not extend far enough to show the satellite plasma resonances. The expression for this total rate is easily shown to be exactly that derived by previous authors by other methods.

III. GENERAL FORMULAS

We will now generalize the arguments of Sec. III to derive a general formula for $d\sigma/d\omega d\Omega$. We will carry through the discussion for the electron-ion plasma and comment at the end on the modifications necessary in treating other systems. Near the resonances, this expression is determined by the local conductivity of the plasma which is already known in certain limiting cases.

It is convenient to separate the diagrams into two classes as in Figs. 3 and 4. The first class are those diagrams in which the ω and k transferred to the matter at the scattering vertex is immediately dissipated along several routes as in the diagrams in Fig. 3. These terms correspond to direct scattering of the radiation from the *electrons* in the plasma. (The corresponding terms from direct ion scattering are, of course, m/M times smaller because of the mass dependence of the scattering vertex.)

The second class of diagrams are those in which ω and k are carried by a single screened interaction line to a second vertex at which ω and k are dissipated along several routes as in the diagrams in Fig. 4. These diagrams correspond to the (virtual) excitation of a col-

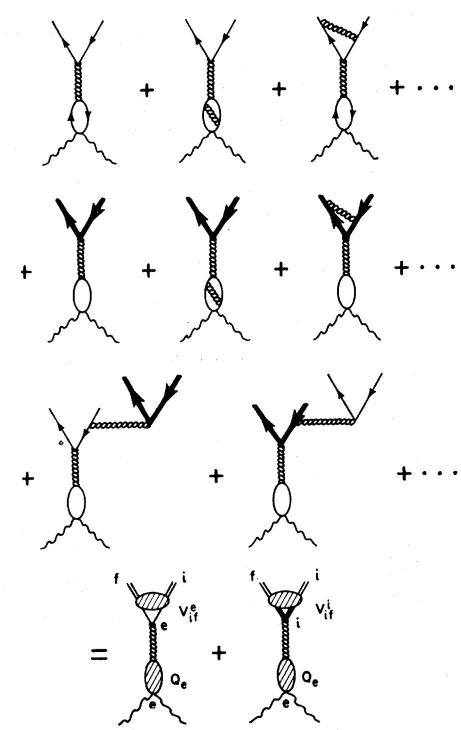

FIG. 4. Resonant scattering processes.

lective longitudinal field (or wave) by the scattering radiation which ultimately is damped by Landau damping and collision damping.

The original scattering vertex factor is proportional to $m_s^{-1}n_s$, where m_s is the mass and n_s the density operator for species s. Thus, ions do not contribute appreciably to the scattering vertex. However, the vertex which damps the collective wave is proportional to the total density operator n and ions can contribute appreciably to this vertex (at low ω).

$$\langle f,\mathbf{k}_b,\hat{e}_b|M|i,\mathbf{k}_a,\hat{e}_a\rangle$$
$$=i\frac{4\pi e^2\hat{e}_a\cdot\hat{e}_b}{mc^2}\left\{V_{if}{}^e+\frac{m}{M}V_{if}{}^i-\left[Q_e(\mathbf{k},\omega)+\frac{m}{M}Q_i(\mathbf{k},\omega)\right]\times V_s(\mathbf{k},\omega)[V_{if}{}^e+V_{if}{}^i]\right\},\quad(3.1)$$

where $V_{if}{}^e(V_{if}{}^i)$ is the sum of the amplitudes for all diagrams leading from a given initial state i to a given final state f via an *electron* (ion) density fluctuation vertex.[8] The factor $V_s(\mathbf{k},\omega)$ is as usual the complete propagator for the screened interaction. Thus, $Q_e(\mathbf{k},\omega)$ $(Q_i(\mathbf{k},\omega))$ is determined to be the sum of all amplitudes of all proper polarization diagrams leading from an electron (ion) vertex and ending in an arbitrary (electron or ion) vertex. The complete polarization part $Q(\mathbf{k},\omega)$ is the sum

$$Q(\mathbf{k},\omega)=Q_e(\mathbf{k},\omega)+Q_i(\mathbf{k},\omega).\quad(3.2)$$

However, we see that a different combination arises in Eq. (3.1) because of the mass dependence of the scattering vertex.

The amplitudes $V_{if}{}^i$ and $V_{if}{}^e$ which are represented in Fig. 4 are intimately related to the dissipative part of the local longitudinal conductivity by the formula

$$4\pi\,\mathrm{Im}\sigma_L(\mathbf{k},\omega)$$
$$=\frac{\omega}{k^2}\mathrm{Im}Q(\mathbf{k},\omega)=\frac{1}{2}\frac{\omega}{k^2}(4\pi e^2)\sum_{if}\rho_i|V_{if}{}^i+V_{if}{}^e|^2\times(1-e^{-\beta\hbar\omega})(2\pi)\delta(\hbar\omega+E_i-E_f)(2\pi\hbar)^3\times\delta^3(\hbar\mathbf{k}+\mathbf{P}_i-\mathbf{P}_f).\quad(3.3)$$

This expression follows directly from the arguments in Ref. 7 and we will not discuss it further here, except to say that it makes use of the gauge invariance of the theory to relate diagrams with a density vertex to those with a current vertex which arise in the general expression for the conductivity. In the case of ω much greater than the collision frequencies of the system, only a finite number of the diagrams in the series for $V_{if}{}^e$ need be included. For ω much less than the collision frequencies, an infinite series of terms must be summed, since it can be shown that the expansion involves powers of λ/ω^2. In any case, the formal statements which we derive are valid to infinite order in perturbation theory.

If we drop terms proportional to m/M and substitute Eq. (3.1) into Eq. (2.3) we have

$$\frac{d\sigma(\mathbf{k},\omega)}{d\omega_b d\Omega_b}=\frac{r_0{}^2\hbar}{(2\pi)}\tfrac{1}{2}(1+\cos^2\theta)\frac{\omega_b}{\omega_a}\times\sum_{if}\rho_i\left|V_{if}{}^e\left(1-\frac{Q_e(\mathbf{k},\omega)}{k^2\epsilon_L(\mathbf{k},\omega)}\right)-V_{if}{}^i\frac{Q_e(\mathbf{k},\omega)}{k^2\epsilon_L(\mathbf{k},\omega)}\right|^2\times(2\pi\hbar)^3\delta^3(\hbar\mathbf{k}+\mathbf{P}_i-\mathbf{P}_f)(2\pi)\delta(\hbar\omega+E_i-E_f).\quad(3.4)$$

This expression simplifies in several cases:

i. *Near resonance.* In this case ϵ_L is nearly zero if the resonances are sharp so that the terms proportional to $\epsilon_L{}^{-1}$ dominate the amplitude. Using Eq. (3.3) we can write

$$\frac{d\sigma(\mathbf{k},\omega)}{d\omega_b d\Omega_b}=\frac{nr_0{}^2\hbar}{\pi m\omega_p{}^2}\tfrac{1}{2}(1+\cos^2\theta)\times\frac{\omega_b}{\omega_a}\left|\frac{Q_e(\mathbf{k},\omega)}{k^2\epsilon_L(\mathbf{k},\omega)}\right|^2\frac{k^2}{\omega}\frac{4\pi\,\mathrm{Im}\sigma_L(\mathbf{k},\omega)}{(1-e^{-\beta\hbar\omega})}.\quad(3.5)$$

Since $\epsilon_L=1+(4\pi\sigma_L/\omega)$ we have expressed everything except Q_e in terms of the local conductivity. We will return to Q_e below.

ii. $\omega/k\gg v_e=(kT/m)^{1/2}$. In this case $V_{if}{}^i\ll V_{if}{}^e$. This is seen as follows. Each term in the expansion $V_{if}{}^s$ has an energy denominator

$$\frac{1}{\omega-\mathbf{k}\cdot\mathbf{p}/M_s+i\epsilon}=\frac{1}{\omega}+\frac{\mathbf{k}\cdot\mathbf{p}}{\omega^2M_s}+\cdots\quad(3.6)$$

arising from the initial vertex. The terms proportional to $1/\omega$ can be shown to vanish by gauge invariance so that the leading term is $(m/M)^{1/2}$ times smaller for ions than electrons. By the same argument $Q_i(\mathbf{k},\omega)\ll Q_e(\mathbf{k},\omega)$ so we have $Q(\mathbf{k},\omega)\simeq Q_e(\mathbf{k},\omega)$. Thus we can write

$$\frac{d\sigma(\mathbf{k},\omega)}{d\omega_b d\Omega_b}=\frac{nr_0{}^2\hbar}{\pi m\omega_p{}^2}\cdot\tfrac{1}{2}(1+\cos^2\theta)\frac{\omega_b}{\omega_a}\left|1-\frac{Q(\mathbf{k},\omega)}{k^2\epsilon_L(\mathbf{k},\omega)}\right|^2\frac{k^2}{\omega}\frac{4\pi\,\mathrm{Im}\sigma_L(\mathbf{k},\omega)}{1-e^{-\beta\hbar\omega}}\quad(3.7)$$

[8] Throughout the body of this paper the functions Q, Q_e, Q_i, V_s, and ϵ_L are *retarded* propagators as defined for example in Ref. 7. In the Appendix the superscripts + and − are used explicitly to denote retarded and advanced functions.

or

$$\frac{d\sigma(\mathbf{k},\omega)}{d\omega_b d\Omega_b}=\frac{nr_0^2\hbar}{\pi m\omega_p^2}\cdot\tfrac{1}{2}(1+\cos^2\theta)\frac{k^2}{\omega}\frac{\omega_b}{\omega_a}\left|1+\frac{4\pi\sigma_L(\mathbf{k},\omega)}{\omega}\right|^{-2}\frac{4\pi\,\mathrm{Im}\sigma_L(\mathbf{k},\omega)}{1-e^{-\beta\hbar\omega}} \tag{3.8}$$

and

$$\frac{d\sigma(\mathbf{k},\omega)}{d\omega_b d\Omega_b}=\frac{nr_0^2\hbar}{\pi m\omega_p^2}\tfrac{1}{2}(1+\cos^2\theta)\frac{\omega_b}{\omega_a}\frac{k^2}{\omega^2}\frac{\mathrm{Im}\epsilon_L^{-1}(\mathbf{k},\omega)}{1-e^{-\beta\hbar\omega}}. \tag{3.9}$$

Thus, we can relate the cross section directly to the complete local conductivity in this case. This is because in this limit the conductivity is determined only by the electrons (to order m/M) so that Q_e can be replaced by Q. This formula is *not* valid if $\omega/k\ll v_e$ which is the case near the ion resonance. However, Eq. (3.5) is valid here. The high-frequency plasma resonance satisfies $\omega/k\gg v_e$ and the two formulas, Eqs. (3.9) and (3.5), give identical results here. (Note that $Q_e\cong Q=-k^2$ at this resonance.) For the ion acoustic resonance (see following paper) it is found that

$$Q_e\cong -Q_i\,,$$

so that Eq. (3.5), which is correct, predicts a cross section $1/k^2$ times larger than Eq. (3.7).

The quantity $Q_e(\mathbf{k},\omega)$ measures the fluctuating polarization induced by the scattered radiation. The *total* polarization $Q(\mathbf{k},\omega)$ is related to the total local conductivity (which is proportional to the total current) by

$$4\pi\sigma_L(\mathbf{k},\omega)=(\omega/k^2)Q(\mathbf{k},\omega)\,, \tag{3.10}$$

which merely expresses the charge conservation relation between the induced current and the induced fluctuating charge. We can discuss the separate electron and ion currents and their related conductivities

$$\sigma_L=\sigma_L^{(e)}+\sigma_L^{(i)}\,,$$

and by conservation of electrons it follows that

$$4\pi\sigma_L^{(e)}=(\omega/k^2)Q_e. \tag{3.11}$$

The quantity $\sigma_L^{(e)}$ can be interpreted as the *total* longitudinal current (divided by E_L) induced when *only* the *electrons* are perturbed by the field E_L. Thus $\sigma_L^{(e)}$ (and therefore Q_e) can be easily obtained from any calculation of the complete conductivity.

The general expression can be reduced to a form involving conductivities by observing that σ_L has the following structure:

$$\sigma_L(\mathbf{k},\omega)=\sigma_L^{ee}+\sigma_L^{ie}+\sigma_L^{ei}+\sigma_L^{ii}\,, \tag{3.12}$$

where we define $(s,\,s'=i,\,e)$

$$\begin{aligned}4\pi\,\mathrm{Im}\sigma_L^{ss'}(\mathbf{k},\omega)&\\ =-\frac{1}{2}\frac{\omega}{k^2}(4\pi e^2)&\sum_{if}\rho_i(V_{if}^{s})^*(V_{if}^{s'})\\ &\times(1-e^{-\beta\hbar\omega})(2\pi)\delta(\hbar\omega+E_i-E_f)\\ &\times(2\pi\hbar)^3\delta^3(\hbar\mathbf{k}+\mathbf{P}_i-\mathbf{P}_f)\,.\end{aligned} \tag{3.13}$$

In physical terms $\sigma_L^{ss'}$ is proportional to the current of species s induced if only species s' interacts with the perturbing field. It follows from this definition that[9]

$$\mathrm{Im}\sigma^{ie}(\mathbf{k},\omega)=[\mathrm{Im}\sigma^{ei}(\mathbf{k},\omega)]^*. \tag{3.14}$$

In this notation $\sigma_L^{(e)}$ and $\sigma_L^{(i)}$ defined above are given by

$$\begin{aligned}\sigma_L^{(e)}&=\sigma_L^{ee}+\sigma_L^{ie}\,,\\ \sigma_L^{(i)}&=\sigma_L^{ii}+\sigma_L^{ei}.\end{aligned} \tag{3.15}$$

Using these definitions we can write the completely general Eq. (3.4) in the form

$$\begin{aligned}\frac{d\sigma(\mathbf{k},\omega)}{d\omega_b d\Omega_b}=\frac{nr_0^2\hbar}{\pi m\omega_p^2}\tfrac{1}{2}(1+\cos^2\theta)\frac{\omega_b}{\omega_a}\frac{k^2}{\omega}(1-e^{-\beta\hbar\omega})^{-1}4\pi\Bigg\{\mathrm{Im}\sigma_L^{ee}(\mathbf{k},\omega)\left|1-\frac{Q_e(k,\omega)}{k^2\epsilon_L(k,\omega)}\right|^2+\mathrm{Im}\sigma_L^{ii}(\mathbf{k},\omega)\left|\frac{Q_e(\mathbf{k},\omega)}{k^2\epsilon_L(\mathbf{k},\omega)}\right|^2\\ -\mathrm{Im}\sigma_L^{ei}(\mathbf{k},\omega)\left[\left(1-\frac{Q_e(\mathbf{k},\omega)}{k^2\epsilon_L(\mathbf{k},\omega)}\right)^*\frac{Q_e(\mathbf{k},\omega)}{k^2\epsilon_L(\mathbf{k},\omega)}\right]-\mathrm{c.c.}\Bigg\}.\end{aligned} \tag{3.16}$$

In applying this derivation to an electron gas in a uniform background one simply sets $V_{if}^{i}=0$ throughout. Then it follows that Eqs. (3.7) and (3.9) are correct for all frequencies for this model.

For electrons interacting with phonons or fixed impurities, Eqs. (3.7) to (3.9) are still valid for high-frequency shifts near the electron plasma frequency where only electrons can respond appreciably. The low-fre-

[9] Note that $\mathrm{Im}\sigma^{ie}$ and $\mathrm{Im}\sigma^{ei}$ are not necessarily real functions but that $\mathrm{Im}(\sigma^{ie}+\sigma^{ei})$ is real.

quency portions of the scattered line must be handled separately in each case. The results should be essentially as described in Sec. II, except that Q is replaced by the complete electron polarization part. A more detailed discussion of these cases is outside of the main area of the present paper.

Near the resonances, at least if they are sharp, Eq. (3.5) is dominated by the small denominator $|1+4\pi\sigma_L/\omega|^2=|\epsilon_L(\mathbf{k},\omega)|^2$. If the resonance is at frequency ω_L and the damping rate γ_L is small, we can approximately write the dispersion relation for ω_L

$$1+\frac{4\pi\,\mathrm{Re}\sigma_L(\mathbf{k},\omega_L)}{\omega_L}=0. \tag{3.17}$$

Then, for ω near ω_L we have

$$\frac{1}{\epsilon_L(\mathbf{k},\omega)}=\tfrac{1}{2}\omega_L\frac{Z_L}{\omega-\omega_L+\frac{1}{2}i\gamma_L}+O\left(\frac{\gamma_L}{\omega_L}\right), \tag{3.18}$$

where (see Ref. 7)

$$Z_L^{-1}=\frac{\omega_L}{2}\frac{\partial}{\partial\omega}\left[\frac{4\pi\,\mathrm{Re}\sigma_L(\mathbf{k},\omega)}{\omega}\right]_{\omega=\omega_L} \tag{3.19}$$

and the damping factor is given by

$$\gamma_L(k)=Z_L4\pi\,\mathrm{Im}\sigma_L(\mathbf{k},\omega_L). \tag{3.20}$$

Combining these equations, we have near resonance

$$\frac{d\sigma(\mathbf{k},\omega)}{d\omega_b d\Omega_b}=\frac{nr_0^2\hbar}{4\pi m\omega_p^2}\frac{1}{2}\frac{(1+\cos^2\theta)}{(1-e^{-\beta\hbar\omega_L})}\frac{\omega_b}{\omega_a}\frac{|Q_e(\mathbf{k},\omega_L)|^2}{k^2}\times\frac{\omega_L Z_L\gamma_L}{(\omega-\omega_L)^2+\frac{1}{4}\gamma_L^2}+O(\gamma_L^2/\omega_L^2). \tag{3.21}$$

If $\gamma_L/\omega_L\ll 1$, the resonance is very sharp. The area under the resonance as obtained from this expression is

$$\frac{nr_0^2\hbar\omega_L}{2\omega_p^2}Z_L(k)\frac{1}{2}\frac{(1+\cos^2\theta)}{(1-e^{-\beta\hbar\omega_L})}\frac{\omega_b}{\omega_a}\frac{|Q_e(\mathbf{k},\omega_L)|^2}{k^2}, \tag{3.22}$$

which is independent of γ_L.

IV. REMARKS

These formulas are, of course, examples of the general fluctuation-dissipation theorem. They relate the spectrum of electron density fluctuations to the dissipative part of the longitudinal conductivity. This is discussed more fully in the Appendix where we present a more formal derivation of these results. The treatment given above has the advantage of exhibiting very clearly the basic microscopic processes involved. Namely, the scattering of the radiation by an electron in the screening cloud of an ion or another electron which excites a collective oscillation of this cloud which is ultimately damped by various processes.

The expressions derived in this paper are applicable to both classical and quantum plasmas. In the following paper an application of these results to classical plasmas will be made. These results have a bearing on the experiments under way of scattering of radar beams from the ionosphere.

A perhaps more interesting application would be to the scattering of intense laser light from dense semiconductor plasmas. Because of the intrinsic smallness of the (Thompson) cross section per electron, it appears necessary to use degenerate or nearly degenerate semiconductors with a high density of carriers. A detailed study of the feasibility of such experiments is now under way. For such experiments to be feasible it is necessary, of course, to find materials which are highly transparent at existing laser frequencies.

APPENDIX: GENERAL FORMULA FOR THE SCATTERING RATE

The particle-field coupling Hamiltonian which is significant for this problem is

$$H_1(t)=\frac{e^2}{2c^2}\sum_s\frac{z_s}{m_s}\int d^3x n_s(x,t)\mathbf{A}(\mathbf{x},t)\cdot\mathbf{A}(\mathbf{x},t). \tag{A1}$$

The electromagnetic field is described by the (time-dependent) operator $\mathbf{A}(x,t)$, and $n_s(\mathbf{x},t)$ is the number density operator for particles of species s with charge ez_s and mass m_s.

The complete amplitude for the transition $i\rightarrow f$ is then proportional to

$$\int_{-\infty}^{\infty}dt\langle f;\mathbf{k}_b,\hat{e}_b|H_1(t)|i;\mathbf{k}_a,\hat{e}_a\rangle$$
$$=\frac{e^2}{2c^2}\frac{4\pi\hbar c^2}{(2\omega_b 2\omega_a)^{1/2}}2(\hat{e}_b\cdot\hat{e}_a)\sum_s\frac{z_s}{m_s}\langle f|n(0,0)|i\rangle\times(2\pi)^3\delta^3(\mathbf{P}_f-\mathbf{P}_i+\hbar\mathbf{k}_b-\hbar\mathbf{k}_a)\times2\pi\delta(E_f-E_i+\hbar\omega_b-\hbar\omega_a). \tag{A2}$$

One obtains for the partial scattering cross section

$$\frac{d\sigma(\mathbf{k},\omega)}{d\omega_b d\Omega_b}=r_0^2\sum_f\sum_i\rho_i\frac{\hbar}{2\pi}\frac{\omega_b}{\omega_a}(\hat{e}_b\cdot\hat{e}_a)^2\left|\sum_s z_s\frac{1}{m_s}\langle f|n(0,0)|i\rangle\right.\times(2\pi\hbar)^3\delta^3(\mathbf{P}_2-\mathbf{P}_1+\hbar\mathbf{k})\times2\pi\delta(E_2-E_1+\hbar\omega). \tag{A3}$$

It is clear that we can neglect the ion contribution because of the mass dependence. On averaging over polarizations, we have then (8)

$$\frac{d\sigma(k,\omega)}{d\omega_b d\Omega_b}=\frac{nr_0^2\hbar}{\pi m\omega_p^2}\frac{\omega_b}{\omega_a}\tfrac{1}{2}(1+\cos^2\theta)\frac{\mathrm{Im}\Pi_e^+(k,\omega)}{1-e^{-\beta\hbar\omega}}, \tag{A4}$$

where

$$\mathrm{Im}\Pi_e^+(\mathbf{k},\omega)=\sum_f\sum_i \rho_1|\langle f|n_e(0,0)|i\rangle|^2 \times(2\pi)^3\delta(\mathbf{P}_f-\mathbf{P}_i+\hbar\mathbf{k})(2\pi)\delta(E_f-E_i+\hbar\omega)\cdot(1-e^{-\beta\hbar\omega}) \quad \text{(A5)}$$

is the imaginary part of

$$\Pi_e^+(\mathbf{k},\omega)=\int_{-\infty}^{\infty}d(t_1-t_2)\int d^3(\mathbf{x}_1-\mathbf{x}_2)\times\Pi_e^+(1,2)e^{i\omega(t_1-t_2)}e^{-i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_2)}, \quad \text{(A6)}$$

where

$$\Pi_e^+(1,2)=(4\pi e^2)\eta(t_1-t_2)\langle[n_e(1),n_e(2)]\rangle \quad \text{(A7)}$$

is the retarded averaged electron density commutator. The combination which occurs in Eq. (A4), namely,

$$\Pi_e^+(\mathbf{k},\omega)/(1-e^{-\beta\hbar\omega}),$$

is easily seen to be the transform of the usual electron density correlation function

$$\langle n_e(1)n_e(2)\rangle$$

thus establishing the relation of our approach to that of Salpeter,[5] and Rostoker and Rosenbluth.[6]

An analysis of functions such as $\Pi_e^+(1,2)$ in the framework of many-particle perturbation theory was carried out in Ref. 7 (There we analyzed the current correlation $\Pi_{ij}^+(1,2.)=\eta(t_1-t_2)\langle[J_i(1),J_j(2)]\rangle$ which has the same formal structure as Π_e^+.) Without repeating the details, it can be shown by the same methods of reordering and summing the elementary perturbation series that Π_e^+ can be represented in the form

$$\Pi_e^+(\mathbf{k},\omega)=Q_{ee}^+(\mathbf{k},\omega)-Q_e^+(\mathbf{k},\omega)\frac{v(\mathbf{k})}{1+v(\mathbf{k})Q^+(\mathbf{k},\omega)}\times Q_e^-(-\mathbf{k},-\omega), \quad \text{(A8)}$$

where Q_{ee}^+ is the sum of *proper* polarizations diagrams which begin and end with electron density vertices and Q_e^+ is defined in Sec. III. Using the isotropy of the system and the time reversal invariance it is easy to show that

$$Q_e^-(-\mathbf{k},-\omega)=Q_e^-(\mathbf{k},-\omega)=[Q_e^+(\mathbf{k},-\omega)]^* \quad \text{(A9)}$$

so

$$\Pi_{ee}^+(\mathbf{k},\omega)=Q_{ee}^+(\mathbf{k},\omega)-\frac{[Q_e^+(\mathbf{k},\omega)Q_e^-(\mathbf{k},-\omega)]v(\mathbf{k})}{1+v(\mathbf{k})Q(\mathbf{k},\omega)}, \quad \text{(A10)}$$

$$\mathrm{Im}\Pi_{ee}^+(\mathbf{k},\omega)=\mathrm{Im}Q_{ee}^+(\mathbf{k},\omega)-\frac{[Q_e^+(\mathbf{k},\omega)Q_e^-(\mathbf{k},-\omega)]\,\mathrm{Im}Q^+(\mathbf{k},\omega)}{|k^2+Q^+(\mathbf{k},\omega)|^2}. \quad \text{(A11)}$$

The resonant part clearly arises from the second term in this expression with the resonance denominator. Substitution into Eq. (A4) gives

$$\frac{d\sigma(\mathbf{k},\omega)}{d\omega_b d\Omega_b}=\frac{nr_0^2\hbar}{\pi m\omega_p^2}\frac{\omega_b}{\omega_a}\frac{1}{2}\frac{(1+\cos^2\theta)}{(1-e^{-\beta\hbar\omega})}\left\{-\mathrm{Im}Q_{ee}^+(\mathbf{k},\omega)+\frac{[Q_e^+(\mathbf{k},\omega)Q_e^-(\mathbf{k},-\omega)]\mathrm{Im}Q^+(\mathbf{k},\omega)}{k^4|\epsilon_L^+(\mathbf{k},\omega|)^2}\right\}. \quad \text{(A12)}$$

In the classical limit $1-e^{-\beta\hbar\omega}\simeq\beta\hbar\omega$ and using $Q_e^+\simeq(4\pi k^2/\omega)\sigma_L(\mathbf{k},\omega)$

$$\frac{d\sigma(\mathbf{k},\omega)}{d\omega_b d\Omega_b}=\frac{nr_0^2}{\pi k_D^2}\frac{\omega_b}{\omega_a}\frac{(1+\cos^2\theta)}{2}\left\{-\frac{\mathrm{Im}Q_{ee}^+(\mathbf{k},\omega)}{\omega}+\frac{[Q_e^+(\mathbf{k},\omega)Q_e^-(\mathbf{k},-\omega)]\,\mathrm{Im}\sigma_L(\mathbf{k},\omega}{k^2\omega^2|\epsilon_L^+(\mathbf{k},\omega)|^2}\right\}. \quad \text{(A13)}$$

The last term in this expression clearly contains the resonant part of Eq. (3.4) as given by Eq. (3.5). Eq. (A13) can in fact be shown to be completely equivalent to Eq. (3.17).