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Dispersion Relation for Second Sound in Solids

R. A. GUYER AND J. A. KRUMHANSL*
Laboratory of Atomic and Solid State Physics, Cornell University, Ithaca, New York
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In this paper the dispersion relation for second sound in solids is derived. The starting point of the analysis is a Boltzmann equation for a phonon gas undergoing a temperature perturbation $\delta T_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$; the Callaway approximation to the collision term is employed. We obtain a dispersion relation which explicitly exhibits the need for a "window" in the relaxation time spectrum. Further, the dispersion relation shows that measurement of the attenuation of second sound as a function of frequency is a direct measurement of the normal process and umklapp process relaxation times. We derive macroscopic equations for energy density and energy flux and show their relation to the macroscopic equations with which Chester has treated second sound.

I. INTRODUCTION

IN a recent paper Chester has used macroscopic equations for energy conservation and heat current to discuss the possibility of second sound in solids.[1] In the context of the present discussion the term second sound will be used to describe temperature oscillations of a phonon gas; the situation is analogous to first sound in a particle gas. It is the purpose of this paper to derive the dispersion relation for second sound in a solid from a microscopic solution of an appropriate equation of motion for the distribution function of a phonon gas. In the derivation the role played by phenomenological parameters characterizing the phonon gas is emphasized; the need for a "window" in the relaxation time spectrum is explicitly demonstrated.[2] With the aid of the microscopic solution of the phonon equation employed herein we can arrive at macroscopic equations similar to those with which Chester treated the problem. Our macroscopic equations differ from those of Chester in an important and essential way.

The equation of motion for the phonon distribution function employed in this paper is essentially the Callaway equation.[3] This equation permits a simple separation of normal and umklapp collision processes which can be used to advantage. In Sec. II we remark on certain aspects of the Callaway equation. An exact solution of the equation for a phonon gas subject to temperature perturbation $\delta T_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega_0 t)}$ is obtained in Sec. III. The technique of solution is essentially that of the Chamber's solution of the Boltzmann equation and, parenthetically, it is also an outline of a general proof of the Chamber's solution.[4] The relationship between the phenomenological parameters which define the region of second-sound propagation are introduced in Sec. IV. The energy density and energy flux are calculated from the solution of the Callaway equation and made subject to the condition for space-time energy conservation; the dispersion relation results. In Sec. V we consider the use of macroscopic equations for the discussion of second-sound propagation; the macroscopic equations of Chester are put in perspective. A brief discussion of our results is given in Sec. VI. Several points which might obscure the argument in the main body of the text are cared for in the appendixes.

The physical picture of the initiation of the second sound process which leads us to consideration of the motion of the phonon distribution function is this: we are interested in the behavior of a phonon gas in a local region of space when the local region is subject to a temperature perturbation of the form

$$T_0'(\mathbf{x},t)-T_0=\delta T(\mathbf{x},t)=\delta T_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega_0 t)}. \qquad (1)$$

The temperature perturbation may be regarded as induced by the remainder of the phonon gas or as the result of the application of an external reservoir. In either case the local region of the phonon gas is to be driven at the temperature $T_0'(\mathbf{x},t)$ for a reasonable length of time. The phonon gas is then to be looked at with a Boltzmann equation having the Callaway expression for the collision term, i.e., with the equation

$$\frac{\partial N}{\partial t}+\mathbf{v}\cdot\nabla_{\mathbf{x}}N=-\frac{N-N_\lambda}{\tau_N}-\frac{N-N_0}{\tau_0}. \qquad (2)$$

This equation will be discussed in detail in Sec. II.

II. THE EQUATION OF MOTION FOR THE PHONON DISTRIBUTION FUNCTION

Before proceeding to the solution of the equation of motion for the phonon distribution function, we examine certain aspects of the approximate form of the collision operator first suggested by Callaway and to be employed here.

The Boltzmann equation we have written assumes in the usual way that a "phonon-distribution" function which is space-$\mathbf{x}$, wave number-$\mathbf{q}$, and time-(t) dependent, may be used as a good first approximation to the complete many phonon density matrix, which would in principle be necessary for the description of the problem. This, of course, is all that has been done to date in the treatment of phonon energy transport in solids and

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[1] M. Chester, Phys. Rev. **131**, 2013 (1963).
[2] E. Prohofsky, thesis, Cornell University, 1962 (unpublished).
[3] J. Callaway, Phys. Rev. **113**, 1046 (1959).
[4] R. G. Chambers, Proc. Phys. Soc. (London) **A65**, 428 (1952).

seems to be a good approximation. Presumably, the technique of thermodynamic Green's functions for the phonon system could be used to formalize a basis for this description. The form of Eq. (2) is

$$\partial N/\partial t+\mathbf{v}\cdot\nabla N=-\partial N/\partial t|_{\mathrm{coll}}, \tag{3}$$

where N is the one phonon number density,

$$N=N(\mathbf{q},\mathbf{x},t).$$

In general, the collision term is a complicated operator resulting from the interactions between the phonons and each other or between the phonons and defects in the solid. Usually it is an extremely good approximation to linearize $\partial N/\partial t|_{\mathrm{coll}}$ in the deviation of N from its equilibrium value; even with this simplification one has a linear integral operator for the collision term. More precisely, the collision operator is nonlocal (integral) in momentum space, though it is assumed to be local in $\mathbf{x}$ space and t. It is expedient to pick simpler forms of this collision operator. The "relaxation-time" approximation has been widely employed for this purpose.

Considerable attention has been given to refinements of the relaxation time approximation including $\mathbf{q}$ and energy dependent variations. Quite often one particularly essential feature of the physics is lost in this approximation. The phonon transport problem has served to sharpen the need for properly accounting for this feature in collisions; we refer to the essential difference between "N processes," which cannot relax a uniformly drifting phonon gas, and "U process," or impurity scattering, which can do so. Actually the fundamentally different nature of the two types of collision processes appear in transport processes occurring in many systems other than solids. The N processes are a special case of those collisions arising from terms in the interaction Hamiltonian which are invariant under uniform translation (in crystals the translation may be restricted to discrete values) such as are the elastic collisions in a particle gas. The "U processes," or impurity scattering arising from other interactions, can transfer momentum from the phonon gas and need not conserve all quantities which are invariant under "N processes."

Thus there is a rather general basis for introducing a relaxation time approximation as Callaway has, recognizing the difference between N processes and others. We have

$$-\frac{\partial N}{\partial t}\bigg|_{\mathrm{coll}}=-\frac{N-N_\lambda}{\tau_N}-\frac{N-N_0}{\tau_0}, \tag{4}$$

where N, N_λ, and N_0 will be functions of $\mathbf{x}$, $\mathbf{q}$, and t. N, N_λ, and N_0 are, respectively, the distribution function of the phonon system, the distribution function of a uniformly drifting phonon gas, and the local equilibrium distribution function. Here, τ_N is the relaxation time for the normal processes; the normal processes relax N to N_λ, *a* distribution function which carries the same momentum current as N. τ_0 is the relaxation time for other processes, i.e., momentum nonconserving processes.

The distribution function N_0 is the Bose distribution at local temperature $T_0'(\mathbf{x},t)$; the prime differentiates this temperature from the equilibrium temperature T_0 which characterizes the system in the absence of the perturbation whose consequences we study.

The distribution function N_λ is that characterizing a uniformly drifting phonon gas having local temperature T_0'; that is,

$$N_\lambda=N_0[(\hbar\omega_0+\boldsymbol{\lambda}\cdot\hbar\mathbf{q})/T_0']. \tag{5}$$

Using for the group velocity the relation $\mathbf{v}_G=1/\hbar\nabla_{\mathbf{q}}\mathcal{E}$, where $\mathcal{E}$ is the energy of the excitations, and interpreting $\omega_0+\boldsymbol{\lambda}\cdot\mathbf{q}$ as the Doppler shifted frequency, it follows that $\boldsymbol{\lambda}$ is the drift velocity. Such a distribution function is not affected by normal (N type) collisions; for $N=N_\lambda$ there is no relaxation or scattering arising from the first term of Eq. (4), as indeed the exact collision operator would have. The second term in (4) will tend to relax any distribution not having the form of a local equilibrium distribution with temperature $T_0'(\mathbf{x},t)$.

Although the approximation (4) to the collision operator has been discussed considerably in the context of steady state, uniform temperature gradient thermal conduction by phonons in solids, we wish to use it in the broader context of space- and time-dependent systems.[5] In so doing we must be careful about conserving quantities which are constants of the exact collision operator.[6] Relaxation time approximations cannot substitute for the proper collision rate with complete adequacy. The inadequacy may trace from the form of the collision operator or from the particular form of the distribution functions which are being considered.

In the present application to second sound we must demonstrate that the "N process" relaxation time approximation used conserves both momentum, $\langle\hbar\mathbf{q}\rangle$, and energy, $\langle\mathcal{E}(\mathbf{q})\rangle$, as is done by the exact collision operator. Whether this will be is conditional not only on the form of $\tau_N(\mathbf{q})$ but also on the functional form of $N(\mathbf{q},\mathbf{x},t)$. Since the latter is known only after Eq. (2) is solved for N, the argument must be one of self-consistency. We thus defer further examination to a later section. The discussion of Abrikosov and Khalatnikov of the Fermi liquid illustrate such considerations.[7]

We do not proceed in generality beyond this point, however, we believe that the important points of principle concerning second sound can be obtained from this suitably simplified model which is mathematically

[5] Treatment of the Callaway collision term can be found in Ref. (3) or in the review article by P. Carruthers, Rev. Mod. Phys. 33, 92 (1961).

[6] We are indebted to G. V. Chester for directing out attention to these matters.

[7] Khalatnikov and Abrikosov, Rept. Progr. Phys. 22, 329 (1969).

tractable but has sufficient generality in form to be physically meaningful. The questions which have been raised in this section regarding the relaxation time approximation and conservation laws are investigated for the model of the phonon system discussed herein in Appendixes A and B. Here we simply state the results: the relaxation time approximation can be applied to space- and time-dependent systems, maintaining the conservation conditions for at least a widely useful class of nonequilibrium distributions; the model of the phonon system which we employ is of this class. In Appendix B we show that endowing the model with considerable generality does not substantially change our conclusions regarding second sound.

III. SOLUTION OF THE EQUATION OF MOTION FOR THE PHONON DISTRIBUTION FUNCTION

The equation of motion for the phonon distribution function is Eq. (2). When the right-hand side is linearized using (1) and (4) one obtains

$$\frac{\partial N}{\partial t}+L_0N=-\frac{N-N_0-\Delta n_0(1-i\beta\mathbf{k}\cdot\mathbf{v})}{\tau_N}-\frac{N-N_0-\Delta n_0}{\tau_0}, \quad (6)$$

where $\mathbf{v}$ is the velocity of a phonon, $L_0=\mathbf{v}\cdot\nabla_{\mathbf{x}}$, $\Delta n_0=(\partial N_0/\partial T)\delta T(\mathbf{x},t)$ and N_λ has been written in the form

$$N_\lambda=N_0+\Delta n_0[1-\beta(\mathbf{x},t)i\mathbf{k}\cdot\mathbf{v}]. \quad (7)$$

$\boldsymbol{\lambda}\cdot\mathbf{q}/kT_0=i\mathbf{k}\cdot\mathbf{v}\beta(\mathbf{x},t)\delta T(\mathbf{x},t)$, the possibility that the drift of the phonon gas may depend on position and time has been introduced in the expression for β. Note there is already $\mathbf{x}$ and t dependence in the drift term through the factor $\delta T(\mathbf{x},t)$. $\beta(\mathbf{x},t)$ has the dimensions of a time; the condition that the normal processes do not contribute to a change in the momentum current gives the necessary condition for determining $\beta(\mathbf{x},t)$.

Equation (6), subject to the condition on the normal process collision term, specifies the problem. In the remainder of this section the trajectory integral method is applied to obtain the solution of the problem. Some detail is reproduced because a simple generalization of the method of solution constitutes a proof of the Chambers' integral solution of the Boltzmann equation.[8]

We write (6) in the form

$$\frac{\partial N}{\partial t}+L_0N+\frac{N}{\tau_c}=\frac{N_0}{\tau_c}+\Delta n_0(\mathbf{x},t)\left[\frac{1}{\tau_c}-i\frac{\beta(\mathbf{x},t)}{\tau_N}\mathbf{k}\cdot\mathbf{v}\right], \quad (8)$$

where $\tau_c^{-1}=\tau_N^{-1}+\tau_0^{-1}$. Multiplying (8) by $e^{t/\tau_c}e^{L_0t}$ produces the exact differential $(\partial/\partial t)(e^{t/\tau_c}e^{L_0t}N)$ on the left-hand side of (8); the resulting equation can be integrated and rearranged to yield

$$N(t)=N_0[T_0(\mathbf{x},t)]-\int_0^t e^{t'-t/\tau_c}e^{L_0(t'-t)}\times\left[1+\frac{\beta(\mathbf{x},t')}{\tau_N}\right]i\mathbf{k}\cdot\mathbf{v}\Delta n_0(\mathbf{x},t')dt'. \quad (9)$$

(9) is the formal statement of the Chambers' integral (or trajectory integral) solution of (8).[9] L_0 is the Liouville operator for a free phonon, i.e., a phonon in the absence of collisions. e^{L_0t} is the free phonon time development operator; its effect is to translate the phonon with velocity $\mathbf{v}$. This operator has no effect on $\mathbf{v}$ and $\mathbf{q}$ or functions of $\mathbf{v}$ or $\mathbf{q}$, e.g., $\tau_N(\mathbf{q})$, $\tau_c(\mathbf{q})$. Its operation on $\mathbf{x}$ is

$$e^{L_0t}\mathbf{x}=\mathbf{x}+\mathbf{v}t. \quad (10)$$

Making use of (10) and the remark above regarding $\mathbf{v}$ and $\mathbf{q}$, one can write (9) in the form

$$N(t)=N_0[T(\mathbf{x},t)]-\int_0^t e^{t'-t/\tau_c}\left\{1+\frac{\beta[\mathbf{x}+\mathbf{v}(t'-t),\,t']}{\tau_N}\right\}\times\Delta n_0(\mathbf{x},t)e^{i(\mathbf{k}\cdot\mathbf{v}-\omega_0)(t'-t)}dt'. \quad (11)$$

Performing the t integration over the term not involving the unknown function $\beta(\mathbf{x},t)$ yields

$$N(t)=N_0(T_0)+\Delta n_0(\mathbf{x},t)\times\Bigg[\frac{1/\tau_c+i(\mathbf{k}\cdot\mathbf{v}-\omega_0)\exp\{-[1/\tau_c+i(\mathbf{k}\cdot\mathbf{v}-\omega_0)]t\}}{1/\tau_c+i(\mathbf{k}\cdot\mathbf{v}-\omega_0)}-\frac{i\mathbf{k}\cdot\mathbf{v}}{\tau_N}\int_0^t\exp\{[1/\tau_c+i(\mathbf{k}\cdot\mathbf{v}-\omega_0)](t'-t)\}\times\beta[\mathbf{x}+\mathbf{v}(t'-t);\,t']dt'\Bigg]. \quad (12)$$

Equation (12) is the desired solution of (6).

Equation (6) contains the unknown function $\beta(\mathbf{x},t)$, which is determined by the requirement that the normal process collision term not change the momentum current, i.e., by the requirement

$$\int\hbar\mathbf{q}\frac{N(t)-N_\lambda(t)}{\tau_N}d^3q\equiv0. \quad (13)$$

[8] In the past year a number of investigators have discussed proofs of the Chambers' solution of the Boltzmann equation: J. Budd, Phys. Soc. Japan **18**, 142 (1963); H. Suzuki, Phys. Soc. Japan **17**, 1542 (1962); R. A. Guyer and J. A. Krumhansl, Bull. Am. Phys. Soc. **8**, 256 (1963).

[9] The problem treated in this paper is particularly simple. The Boltzmann equation

$$\partial f/\partial t+Lf=-(f-f_0)/\tau$$

has the solution

$$f(\mathbf{x},\mathbf{p},t)=\int_{-\infty}^t\exp\left[\int_t^{t'}e^{(t''-t)L}(1/\tau)dt''\right]e^{L(t'-t)}\frac{f_0}{\tau}dt',$$

where $L=\mathbf{v}\cdot\nabla_{\mathbf{x}}+\mathbf{F}_0\cdot\nabla_{\mathbf{p}}$. This solution is the Chambers or trajectory integral solution. It is valid in general for $\mathbf{F}$ independent of t.

Using (7) and (12) this is

$$\int \hbar \mathbf{q} \Delta n_0(\mathbf{x},t) \times \Bigg[1-\frac{1/\tau_c+i(\mathbf{k}\cdot\mathbf{v}-\omega_0)\exp\{-[1/\tau_c+i(\mathbf{k}\cdot\mathbf{v}-\omega_0)]t\}}{1/\tau_c+i(\mathbf{k}\cdot\mathbf{v}-\omega_0)} -i\mathbf{k}\cdot\mathbf{v}\bigg\{\beta(\mathbf{x},t)-\int_0^t \exp\{[1/\tau_c+i(\mathbf{k}\cdot\mathbf{v}-\omega_0)](t'-t)\} \times\beta(\mathbf{x}+\mathbf{v}(t'-t),\, t')\frac{dt'}{\tau_N}\bigg\}\Bigg]d^3q=0. \quad (14)$$

Equation (14) is called the β equation. In (14), a factor $\delta T(\mathbf{x},t)$ factors from $\Delta n_0(\mathbf{x},t)$ which contains the explicit $\mathbf{x}$ dependence, so that it is consistent to choose $\beta=\beta(0,t)$. Equations (12) and (14) completely determine the distribution function for the phonon gas.

IV. THE SECOND-SOUND DISPERSION RELATION

From the solution (12) of the equation of motion for the phonon distribution function, one can compute the energy current (heat flux) and energy density in the phonon gas. The energy density is

$$\langle \mathcal{E} \rangle = \int \mathcal{E} N(t) d^3q. \quad (15)$$

The energy current is

$$\langle \mathbf{Q} \rangle = \int \mathbf{Q} N(t) d^3q. \quad (16)$$

One expects that $\langle \mathcal{E} \rangle$ and $\langle \mathbf{Q} \rangle$ obey the conservation equation

$$(\partial\langle \mathcal{E}\rangle/\partial T)+\nabla\cdot\langle \mathbf{Q}\rangle=0. \quad (17)$$

This equation follows easily from taking the energy moment of Eq. (2) and noting that the dispersion relations for $\mathbf{v}(\mathbf{q})$ and $\mathcal{E}(\mathbf{q})$ are not $\mathbf{x}$-dependent. The zero on the right-hand side is a consequence of local energy conservation by the relaxation time approximation to $\partial N/\partial t|_{\text{coll}}$. Using (12) in (17) yields

$$\int \Delta n_0(\mathbf{x},t)\,\mathcal{E}(\mathbf{k}\cdot\mathbf{v}-\omega_0) \times\Bigg[\frac{1/\tau_c+i(\mathbf{k}\cdot\mathbf{v}-\omega_0)\exp-[1/\tau_c+i(\mathbf{k}\cdot\mathbf{v}-\omega_0)]t}{1/\tau_c+i(\mathbf{k}\cdot\mathbf{v}-\omega_0)} -i\mathbf{k}\cdot\mathbf{v}\int_0^t dt'\cdots\Bigg]d^3q=0. \quad (18)$$

The consistent solution of (18) and the β Eq. (14) constrains the relationship of ω_0, $\mathbf{k}$, and $\mathbf{v}$; this constraint is the dispersion relation. Exact solutions of (14) and (18) are difficult even in the isotropic case, $\mathbf{v}=(s/q)\mathbf{q}$. Up to this point our equations have been general; we will now begin to reduce the generality. In Sec. VI and the Appendices we show that no significant error is incurred due to the approximations we make.

One is aided in going further by the introduction of rate considerations and approximations on the relaxation times.

The rate considerations are the relations between τ_N, τ_0, and ω_0; they are referred to as the timing relations. The timing relations restrict the solution of (2) to the region where one expects second-sound propagation.[10] These are

I. $\omega\tau_N \ll 1$, The temperature perturbation has a characteristic period long compared to the times required for temperature relaxation; the phonon gas can follow the perturbation.

II. $\omega\tau_0 \gg 1$, A large number of temperature oscillations occur in the time required for the momentum to relax.

If the relaxation times depend on q these inequalities are true in the sense of some part of average over q. As a consequence of I and II $\tau_0 \gg \tau_N$; temperature equilibrium (assumed to be produced primarily by fast elastic N processes) is achieved much more rapidly than momentum relaxation. In addition to the timing relations, the results in the remainder of this section are subject to the assumptions: (a) the relaxation times τ_N, τ_0, and τ_c are independent of $\mathbf{q}$, (b) β does not depend on t (for $t\gg\tau_c$ this is quite reasonable), and (c) the phonon velocities are independent of $\mathbf{q}$; i.e., $\mathbf{v}=s\mathbf{q}/q$. (Appendix B shows that no essential features of the problem are altered by these assumptions).

For $t\gg\tau_c$ we have for the energy conservation equation

$$\int \mathcal{E}\frac{\partial N_0}{\partial T} i(\mathbf{k}\cdot\mathbf{v}-\omega)\left[\frac{1-i\mathbf{k}\cdot\mathbf{v}\beta(\tau_c/\tau_N)}{1+i\omega_0\tau_c(\mathbf{k}\cdot\mathbf{v}/\omega_0-1)}\right]d^3q\equiv0. \quad (19)$$

And for the β equation,

$$\int \mathcal{E}\mathbf{v}\frac{\partial N_0}{\partial T}\Bigg[1-\frac{1}{1+i\omega_0\tau_c(\mathbf{k}\cdot\mathbf{v}/\omega_0-1)} -i\mathbf{k}\cdot\mathbf{v}\beta\bigg\{1+\frac{\tau_c}{\tau_N}\frac{1}{1+i\omega_0\tau_c(\mathbf{k}\cdot\mathbf{v}/\omega_0-1)}\bigg\}\Bigg]d^3q=0, \quad (20)$$

where we have used $\hbar\mathbf{q}=\mathcal{E}\mathbf{v}/s^2$. The dispersion relation follows from simultaneous solution of (19) and (20); the algebra is reproduced in reasonable detail below.

[10] A good qualitative discussion of the physical requirements for second sound in solids is given in the thesis by Prohofsky, Ref. 2.

Since the q and ϕ integrals cannot give zero, we get for (19)

$$\int_0^\pi \left\{\left(1+\frac{\beta}{\tau_N}\right)-\left[1+\frac{\beta}{\tau_N}(1-i\gamma)\right] \times\frac{1}{1+i\gamma(x\cos\theta-1)}\right\}\sin\theta d\theta=0, \quad (21)$$

and from (20)

$$\int_0^\pi \left\{\left[1+\frac{\beta}{\tau_N}(1-i\gamma)\right]\left[1-\frac{1-i\gamma}{1+i\gamma(x\cos\theta-1)}\right] -\omega_0\gamma\beta x^2\cos^2\theta\right\}\sin\theta d\theta=0, \quad (22)$$

where $\gamma=\omega_0\tau_c$, $x=ks/\omega_0$, and the z axis for the $\mathbf{q}$ system has been taken in the $\mathbf{k}$ direction. Both (21) and (22) can be solved for β with the results: from (21)

$$\beta=\tau_N\{[I_0-2]/[2-(1-i\gamma)I_0]\}, \quad (23)$$

and from (22)

$$\beta=\tau_N\frac{2-(1-i\gamma)I_0}{\frac{2}{3}(\tau_N/\tau_c)\gamma^2x^2-(1-i\gamma)[2-(1-i\gamma)I_0]}, \quad (24)$$

where

$$I_0=\int_0^\pi \frac{\sin\theta d\theta}{1+i\gamma(x\cos\theta-1)}. \quad (25)$$

Equations (23) and (24) are a redundant condition on β; this condition constrains the relationship of ω_0, $\mathbf{k}$, and $\mathbf{v}$. From (25)

$$I_0=\frac{i}{x\gamma}\log_e\left[\frac{1-i\gamma(x+1)}{1+i\gamma(x-1)}\right]. \quad (26)$$

From the timing relations $\gamma=\omega_0\tau_c\ll 1$, one can generate power series in γ for the two expressions for β. These are from (23).

$$\beta=\frac{3}{x^2}\tau_N\frac{i-\gamma(1+x^2/3)-i\gamma^2(1+x^2)+O(\gamma^3)+\cdots}{\gamma(1+2i\gamma)-3\gamma^2(1+x^2/5)+O(\gamma^3)+\cdots}, \quad (27)$$

and from (24),

$$\beta=\frac{\gamma[1+2i\gamma-3\gamma^2(1+x^2/5)+O(\gamma^3)+\cdots]}{\omega_0\{1-(\tau_c/\tau_N)[1+i\gamma-\gamma^2(1+3x^2/5)+O(\gamma^3)+\cdots]}. \quad (28)$$

Equating (27) and (28) and solving for $1/x^2=(\omega_0/ks)^2$ yields

$$(\omega_0/ks)^2=\tfrac{1}{3}\{1+-i[\tfrac{4}{5}\omega_0\tau_N+(1/\omega_0\tau_0)] +O(\gamma^2)+\cdots\}. \quad (29)$$

Our dispersion relation for second sound in solids, Eq. (29), is essentially that of Prohofsky and Chester; however, the damping term $\mathrm{Im}(\omega_0/ks)^2$ contains both the momentum relaxation time τ_0 and the temperature relaxation time $\sim\tau_N$. If the damping of the second-sound wave is to be small, the period of the temperature perturbation, $\tau=2\pi/\omega$, must be in the "window" between τ_N and τ_0; i.e., $\tau_N\ll\tau\ll\tau_0$. We believe it essential to call attention to the two distinctly different sources of damping; the entire feasibility of any experiment rests on the existence of a "window." Of course, the specific mathematical approximations, as well as the implied statistical nature of the Boltzmann description, assumed $\omega_0\tau_c\ll 1$, and we are implying that (29) can apply when $\omega_0\tau_N$ is not small. The content of the assertion is not that the specific functional form for the damping term is necessarily accurate then, but that strong damping will occur and (29) is a first approximation to its frequency dependence. Equation (29) suggests that an experimental investigation of the damping of second sound waves should yield information about the two relaxation processes operative in a phonon gas.

V. MACROSCOPIC EQUATIONS

The dispersion relation for first sound in a particle gas can be derived by appropriately combining the macroscopic equations for number density and velocity.[11] Chester has used a macroscopic equation for $\langle\mathcal{E}\rangle$ and an *ad hoc* equation for $\langle\mathbf{Q}\rangle$ to derive a dispersion relation for second sound. In this section the results of Sec. III are used to examine the macroscopic equations for $\langle\mathcal{E}\rangle$ and $\langle\mathbf{Q}\rangle$ which follow from the equation of motion of the phonon gas, Eq. (2).

The macroscopic equation for $\langle\mathcal{E}\rangle$ is derived from (2) by multiplying by $\mathcal{E}$ and integrating over $\mathbf{q}$. As remarked above (Sec. III) one obtains

$$c(\partial T/\partial t)+\nabla\cdot\langle\mathbf{Q}\rangle=-\langle(\partial\mathcal{E}/\partial t)|_{\text{collision}}\rangle =-(T-T_0)/\tau_c, \quad (30)$$

where c is the differential specific heat per unit volume.

The right-hand side of Eq. (30) is identically zero for the exact collision operator. Our development employing the relaxation time approximation to $\partial N/\partial t|_{\text{collision}}$, included a parameter β, to guarantee the appropriate conservation of momentum. We have apparently included no parameter in the relaxation time approximation to adjust $\partial N/\partial t|_{\text{collision}}$ to conserve energy. But by setting $\langle\partial\mathcal{E}/\partial t|_{\text{collision}}\rangle$ equal to zero in Eq. (17) we have in fact required that the solutions for the class of perturbations $\delta Te^{i(\mathbf{k}\cdot\mathbf{x}-\omega_0 t)}$ be energy conserving for a collision term of the form (4). Thus (30) is

$$c(\partial T/\partial t)+\nabla\cdot\langle\mathbf{Q}\rangle=0. \quad (31)$$

The macroscopic equation for $\langle\mathbf{Q}\rangle$ is from (2);

$$\frac{\partial}{\partial t}\langle\mathbf{Q}\rangle+\frac{\langle\mathbf{Q}\rangle}{\tau_0}+\int\mathbf{Q}L_0Nd^3q=0. \quad (32)$$

[11] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon Press, Inc., New York, 1959).

For the third term in (32), using Eq. (12) for N, one finds for $t \gg \tau_c$

$$\int \mathbf{Q} L_0 N d^3q = \frac{K_{\text{eff}}}{\tau_0} |\nabla T| , \tag{33}$$

where $K_{\text{eff}} = K_{\text{dc}}(1 - \frac{4}{5} i\omega\tau_N + O(\gamma^2) + \cdots)$ and

$$K_{\text{dc}} = \left| \tau_0 \int \mathbf{Q} \mathbf{v} (\partial N_0 / \partial T) d^3q \right| .$$

Hence, the equation for energy current is

$$\frac{\partial\langle\mathbf{Q}\rangle}{\partial t} + \frac{\langle\mathbf{Q}\rangle}{\tau_0} + (1 - \tfrac{4}{5} i\omega\tau_N + O(\gamma^2) + \cdots) \frac{K_{\text{dc}}}{\tau_0} |\nabla T| = 0 . \tag{34}$$

Equations (31) and (34) are the two equations to be combined in analogy to the procedure for treating first sound in a particle gas. The resulting T equation is

$$\partial^2 T/\partial t^2 + (1/\tau_0)\partial T/\partial t - [1 - \tfrac{4}{5} i\omega\tau_N + O(\gamma^2) + \cdots] \times K_{\text{dc}}/c\tau_0 \nabla^2 T = 0 . \tag{35}$$

Substitution of $T = T_0 + \delta T e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}$ into (35) leads to the dispersion relation equation (29). Although Chester's discussion of second sound is along the lines we have just indicated, we emphasize that we differ quite fundamentally. Our result differs in that he has used an equation for $\langle\mathbf{Q}\rangle$ where the third term of (34) is simply $K_{\text{dc}}/\tau_0 |\nabla T|$. The term $\langle\mathbf{Q}\rangle/\tau_0$ in (34) is of the same order as $\omega\tau_N(K_{\text{dc}}/\tau_0)\nabla T$ so that to get a result consistent to order γ^2 we need the first two terms of K_{eff}. Hence, Chester's dispersion relation does not explicitly exhibit the need for a window as does Eq. (29).

VI. DISCUSSION

The phenomena of second sound in solids depends critically on the existence of two characteristic relaxation mechanisms for the phonon distribution function. The collision term proposed by Callaway immediately acknowledges this fact and is the reason for choosing the Callaway form of the collision term as the starting point of the analysis. In this paper an equation of motion for the phonon distribution function has been solved for a phonon gas in the presence of a periodic thermal disturbance. The solution is restricted to the region of second-sound propagation by the introduction of the timing relations. The dispersion relation for second sound then follows from combining the statement of energy conservation with $\langle\mathcal{E}\rangle$ and $\langle\mathbf{Q}\rangle$ computed using the solution to the equation of motion. The dispersion relation explicitly exhibits the need for a "window" in the relaxation time spectrum. A macroscopic equation for $\langle\mathbf{Q}\rangle$ is derived from (2) with the aid of the microscopic solution. We obtain a correct macroscopic equation for $\langle\mathbf{Q}\rangle$ similar to that introduced by Chester on plausibility grounds. We are able to treat the second sound phenomena with this macroscopic equation and obtain the appropriate dispersion relation.

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APPENDIX A: ENERGY CONSERVATION

In setting the total collision term in the relaxation time approximation equal to zero [see Eq. (17) or (30)] we have required that

$$\int \mathcal{E} \frac{\partial N_0}{\partial T} \frac{\mathbf{k}\cdot\mathbf{v}(1+\beta/\tau_N) - \omega_0}{1 + i\omega_0\tau_c(\mathbf{k}\cdot\mathbf{v}/\omega_0 - 1)} d^3q = 0 . \tag{A1}$$

We really want the "N process" and "U process" collision term to separately lead to energy conservation. If the normal process and umklapp process terms are to separately conserve energy, we must have

$$\int \mathcal{E} \frac{\partial N_0}{\partial T} \frac{\tau_c}{\tau_N} \frac{\mathbf{k}\cdot\mathbf{v}(1+\beta/\tau_N) - \omega_0}{1 + i\omega_0\tau_c(\mathbf{k}\cdot\mathbf{v}/\omega_0 - 1)} d^3q \equiv 0 \tag{A2}$$

and

$$\int \mathcal{E} \frac{\partial N_0}{\partial T} \frac{\tau_c}{\tau_0} \frac{\mathbf{k}\cdot\mathbf{v}(1+\beta/\tau_N) - \omega_0}{1 + i\omega_0\tau_c(\mathbf{k}\cdot\mathbf{v}/\omega_0 - 1)} d^3q \equiv 0 , \tag{A3}$$

respectively. For τ's independent of $\mathbf{q}$, (A1)≡(A2)≡(A3); the two conditions reduce to one. For τ's which depend on $\mathbf{q}$, the two conditions can be satisfied by the introduction of another parameter in the expression for N_λ. Choose N_λ of the form

$$N_\lambda = N_0 + \Delta n(\mathbf{x}, t)(\alpha - i\beta\mathbf{k}\cdot\mathbf{v}) . \tag{A4}$$

In Appendix B, the consequences of choosing (A4) for N_λ, requiring

$$\langle \partial N/\partial t |_{\text{normal collisions}} \rangle = \langle \partial N/\partial t |_{\text{umklapp collisions}} \rangle = 0 ,$$

and permitting the τ's to depend on $\mathbf{q}$ are examined. Since $\tau_c/\tau_N \simeq 1$ and $\tau_c/\tau_0 \ll 1$ replacing (A1) by (A2) and (A3) should not lead to drastic changes.

APPENDIX B: GENERALIZATIONS OF THE DISTRIBUTION FUNCTION

As remarked above the problem solved in the major part of this paper is not the most general one within the spirit of the relaxation time approximation. In particular, (a) the relaxation times were assumed to be independent of $\mathbf{q}$ and (b) the entire collision term was taken to be energy conserving instead of the normal and umklapp terms separately. In this appendix we briefly remark on the consequences of removing (a) and (b).

The starting point of the analysis is to replace the expression (6) for N_λ with (A4). Then, proceeding as in Sec. III, we find that for $t \gg \tau_c$

$$N(t)=N_0+\Delta n_0(\mathbf{x},t) \times\left[\frac{1+\Delta(\tau_c/\tau_N)-i\beta\mathbf{k}\cdot\mathbf{v}(\tau_c/\tau_N)}{1+i\omega_0\tau_c(\mathbf{k}\cdot\mathbf{v}/\omega_0-1)}\right] \tag{B1}$$

where $\alpha=1+\Delta$. $N(t)$ contains β and Δ which are determined by the conditions for momentum and energy conservation;

$$\left\langle\frac{\partial\mathbf{P}}{\partial t}\bigg|_{\text{normal collision}}\right\rangle=0,\quad \left\langle\frac{\partial\mathcal{E}}{\partial t}\bigg|_{\text{normal collision}}\right\rangle=0,$$

and

$$\left\langle\frac{\partial\mathcal{E}}{\partial t}\bigg|_{\text{umklapp collision}}\right\rangle=0.$$

The two energy conditions can be combined to give Δ.

$$\Delta=\frac{\langle H/\tau_N\rangle_1\langle M/\tau_0\rangle_1-\langle H/\tau_0\rangle_1\langle M/\tau_N\rangle_1}{2\langle 1/\tau_N\rangle_1\langle M/\tau_0\rangle_1-\{\langle J/\tau_N\rangle_1\langle M/\tau_0\rangle_1-\langle J/\tau_0\rangle_1\langle M/\tau_N\rangle_1\}} \tag{B2}$$

where $H=I_0-2$, $M=1/\tau_N[2-(1-i\gamma)I_0]$, $J=(\tau_c/\tau_N)I_0$, and

$$\langle f\rangle_1=\int\mathcal{E}\frac{\partial N_0}{\partial T}f(|q|)q^2dq\bigg/\int\mathcal{E}\frac{\partial N_0}{\partial T}q^2dq. \tag{B3}$$

Δ is zero for relaxation times which do not depend on $|q|$; for q dependent relaxation times (B-2) can be shown to be small compared to $\langle\omega\tau_c\rangle_1$. We can also obtain redundant conditions on β similar to Eqs. (23) and (24). From the normal process energy conservation condition we have

$$\beta=\frac{\langle(I_0-2)/\tau_N\rangle_1+\Delta\langle(I_0\tau_c/\tau_N-2)/\tau_N\rangle_1}{\langle(1/\tau_N^2)[2-(1-i)I_0]\rangle_1}, \tag{B4}$$

and from the condition for momentum conservation by the N process we have

$$\beta=\frac{\langle[1+\Delta(\tau_c/\tau_N)](1/\tau_c\tau_N)[2-(1-i\gamma)I_0]\rangle_1}{\frac{2}{3}\omega_0^2x^2\langle 1/\tau_N\rangle_1-\langle[(1-i\gamma)/\tau_N](1/\tau_c\tau_N)[2-(1-i\gamma)I_0]\rangle_1}. \tag{B5}$$

The similarity of these equations to (23) and (24) is evident. As in Sec. IV the dispersion relation follows from equating (B3) and (B4); we find

$$\left(\frac{\omega_0}{ks}\right)^2=\frac{1}{3}\frac{\left\langle\left(\frac{\tau_c}{\tau_N}\right)^2\right\rangle_1\left\langle 1+\Delta\frac{\tau_c}{\tau_N}\right\rangle_1+2i\left[\left\langle\gamma\left(\frac{\tau_c}{\tau_N}\right)^2\right\rangle_1\left\langle 1+\Delta\frac{\tau_c}{\tau_N}\right\rangle_1+\left\langle\left(\frac{\tau_c}{\tau_N}\right)^2\right\rangle_1\left\langle\gamma\left(1+\Delta\frac{\tau_c}{\tau_N}\right)\right\rangle_1\right]+O(\gamma^2)+\cdots}{(1+\Delta)\left\{\left\langle\frac{\tau_c}{\tau_N}\right\rangle_1\left\langle\left(\frac{\tau_c}{\tau_N}\right)^2\right\rangle_1+i\left(\left\langle\frac{\tau_c}{\tau_N}\right\rangle_1\left\langle\frac{\tau_c}{\tau_N}\frac{1}{\omega_0\tau_0}\right\rangle_1+2\left\langle\left(\frac{\tau_c}{\tau_N}\right)^2\right\rangle_1\left\langle\frac{\tau_c}{\tau_N}\gamma\right\rangle_1+\frac{14}{5}\left\langle\frac{\tau_c}{\tau_N}\right\rangle_1\left\langle\left(\frac{\tau_c}{\tau_N}\right)^2\gamma\right\rangle_1\right)\right\}+O(\gamma^2)+\cdots}. \tag{B6}$$

Recall $\tau_c/\tau_N=1-(\tau_N/\tau_0)+(\tau_N/\tau_0)^2+\cdots$ and $\tau_N/\tau_0\ll 1$. Keeping terms up to order Δ^2 and γ^2 in (B6) yields a dispersion relation which is identically (29) with the damping term modified to read

$$i\{\tfrac{4}{5}\langle\omega_0\tau_c\rangle_1+\langle 1/\omega_0\tau_0\rangle_1\}. \tag{B7}$$

A window continues to exist if the period of the thermal wave satisfies the relation $\langle\tau_N\rangle_1\ll\tau\ll\langle 1/\tau_0\rangle_1^{-1}$. In the paragraph following the timing relations we remarked that if the relaxation times depended on q then the timing relations must be true in the sense of some sort of average; the kind of average required follows directly from the details of the analysis sketched above.