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Surface Contour of Rotating Liquid Helium II*

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The curvature of the free surface of a thin layer of liquid helium II resting on the horizontal bottom of a rotating cylindrical container (radius = 1.26 cm) was measured by a sensitive optical method. For a liquid depth of $h=5.0\times10^{-3}$ cm and $T=1.11$°K, the steady-state curvature was found to be indistinguishable from that of an ordinary viscous liquid for rotational speeds down to $\omega=0.29$ rad/sec. Transient effects were observed which differ qualitatively from those of a normal viscous liquid. These results are discussed in relation to various theories of rotating helium and are compared with other measurements.

INTRODUCTION

When we rotate a container of liquid helium II, does the superfluid component rotate? To answer this question, Osborne[1] observed the contour of the free surface of helium II in a rotating container and concluded that, at the rotational speeds used, the superfluid came to a steady rotational state indistinguishable from that of an ordinary viscous liquid. Although rotation of the superfluid was not anticipated, perhaps, from the hydrodynamic theories of Tisza[2] and Landau,[3] the experimental result was interpreted by London[4] in the following way: Adopting a suggestion of Onsager[5] that the circulation was quantized in concentric cylindrical regions separated by vortex sheets, London minimized the free energy and concluded that in equilibrium the superfluid should rotate practically as a solid body at high velocities, but should not rotate at all below a certain critical velocity. This critical angular velocity is $\omega\approx\hbar/2\,ma$, where m is the mass of the helium atom and a is the radius of the cylindrical container, so that for $a=1$ cm, the critical angular velocity has the very small value of about 10^{-4} rad/sec. Landau and Lifshitz,[6] on the other hand, considered the energy of the vortex sheets separating cylindrical regions of quantized circulation and concluded that the superfluid velocity field would approximate that of a rotating solid body only at much higher angular velocities. Hall and Vinen[7] have given a revised version of London's calculation, assuming a regular array of vortex lines as suggested by Onsager[5] and Feynman.[8] In this vortex-line model the energy associated with the velocity singularities is assumed small, and the final result as given by Hall is not greatly different from the London theory. In contrast to the above models, all of which predict a finite critical velocity, Lin[9] has suggested that the superfluid might have nonzero viscosity and would thus reach a steady state of uniform rotation in which boundary slip would cause the superfluid to rotate more slowly than the normal fluid at low velocities.

Experimentally, no definite and reproducible critical velocities have been observed in steady rotation. On the one hand, Reppy and Lane[10] have shown that a container of helium II can be rotated at a rather high speed

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[1] D. V. Osborne, Proc. Roy. Soc. (London) A63, 909 (1950).

[2] L. Tisza, Nature **141**, 913 (1938); Compt. Rend. **207**, 1035, 1186 (1938); J. Phys. Radium (8) **1**, 164, 350 (1940); Phys. Rev. **72**, 838 (1947).

[3] L. D. Landau, Zh. Eksperim. i Teor. Fiz. **5**, 71 (1941).

[4] F. London, *Superfluids* (John Wiley & Sons, Inc., New York, 1954). Vol. II.

[5] L. Onsager, Nuovo Cimento **6**, Suppl. 2, 249 (1949). See also footnote in Ref. 4, p. 151.

[6] Landau and Lifshitz, Dokl. Akad. Nauk SSSR **100**, 669 (1955).

[7] H. E. Hall and W. F. Vinen, Proc. Roy. Soc. (London) A238, 215 (1956).

[8] R. P. Feynman, *Progress in Low Temperature Physics* (Interscience Publishers, Inc., New York, 1955), Vol. I, p. 17.

[9] C. C. Lin, Phys. Rev. Letters **2**, 345 (1959).

[10] J. D. Reppy and C. T. Lane, *Proceedings of the Seventh International Conference on Low-Temperature Physics* (The University of Toronto Press, Toronto, 1961), p. 443.

(0.3 rad/sec for a 1.25 cm radius container) without the superfluid *always* coming into rotation. This result is difficult to explain on the basis of Lin's theory, and it can only be explained in terms of the models of London or Feynman as a metastability. On the other hand, the fact that the superfluid usually comes into rotation, even at much lower speeds, does not agree with the prediction of Landau and Lifshitz.

Thus none of these theories appears to be completely adequate, and we must seek what empirical evidence there is in the results of other types of flow experiments exhibiting critical velocities at which the flow of the superfluid apparently becomes rotational. Atkins[11] has shown that for flow in channels larger than 10^{-3} cm many flow experiments can be correlated with a critical superfluid velocity given by the relation

$$(Vd)_C \approx 0.01 \text{ cm/sec}, \qquad (1)$$

where d is the characteristic lateral dimension of the channel. A phenomenological theory by the author correlates various kinds of critical velocities and their temperature dependence but does not differ in order of magnitude from Eq. (1). This equation can perhaps be derived from vortex mechanics, but here it is used simply as an empirical relation which is approximately valid for many experiments. In Osborne's experiment the value of the product of the lowest peripheral velocity and the container raduis was 24.5 cm^2/sec. In Andronikashvili and Kaverkin's[12] experiment the value of this parameter was 5.7 cm^2/sec. Donnelly *et al.*[13] reached a value of 8.4 cm^2/sec, and Donnelly,[14] using a small capillary to reduce the characteristic distance, achieved a value of 0.216 cm^2/sec which, although much smaller, still exceeded the value of the above critical velocity criterion. Recently, Turkington, Brown, and Osborne[15] reached about 0.3 cm^2/sec. In all of these experiments it was found that the superfluid was entrained into rotation as would be predicted by Eq. (1).

The present experiment was designed to measure the surface contour under conditions in which this critical velocity parameter would not be exceeded. To achieve this, a sensitive optical method was used to measure the curvature of a thin layer of helium II on a horizontal rotating substrate.

STEADY ROTATION OF LIQUID HELIUM II

Using the hydrodynamic equations essentially as given by Landau[3] and also by Zilsel,[16] we derive the pressure and temperature fields and the surface contour of liquid helium II in a cylinder which is closed at its bottom and rotates about its (vertical) axis with an angular velocity ω. The resulting surface contour is well known, but the derivation is given since it shows explicitly how the second-order velocity terms allow two different steady-state solutions without entropy production. The following equations of motion are assumed:

$$\begin{aligned}\rho_s[\partial \mathbf{V}_s/\partial t+\nabla(\mathbf{V}_s^2/2)] &= -(\rho_s/\rho)\nabla p-\rho_s\nabla\Omega+\rho_s S\nabla T \\ &\quad +(\rho_s\rho_n/2\rho)\nabla|\mathbf{V}_n-\mathbf{V}_s|^2, \end{aligned}\qquad (2)$$

$$\begin{aligned}\rho_n[\partial \mathbf{V}_n/\partial t+\nabla(\mathbf{V}_n^2/2)-\mathbf{V}_n\times(\nabla\times\mathbf{V}_n)] &= -(\rho_n/\rho)\nabla p-\rho_n\nabla\Omega-\rho_s S\nabla T \\ &\quad -(\rho_s\rho_n/2\rho)\nabla|\mathbf{V}_n-\mathbf{V}_s|^2 \\ &\quad -\eta\nabla\times\nabla\times\mathbf{V}_n+(4/3)\eta\nabla\nabla\cdot\mathbf{V}_n. \end{aligned}\qquad (3)$$

The gravitational potential Ω will be taken as $\Omega=gz$ where z, the vertical coordinate, is measured from a horizontal plane. We consider the following special case.

(a) Steady state; no explicit dependence on time.

(b) $\mathbf{V}_s=0$; (this can be considered as the consequence of a simply-connected container and the condition $\nabla\times\mathbf{V}_s=0$, inherent in the assumed equations of motion).

(c) $\mathbf{V}_n=\mathbf{\Phi}_1\omega r$, where $\mathbf{\Phi}_1$ is the unit vector in the azimuthal direction.

Adding Eqs. (2) and (3) and using the above assumptions we have

$$\nabla p+\rho\nabla\Omega=-\rho_n[\nabla(\mathbf{V}_n^2/2)-\mathbf{V}_n\times\nabla\times\mathbf{V}_n]. \qquad (4)$$

Along the surface of the liquid $\nabla p=0$ and the slope of the liquid surface is

$$\frac{dh}{dz}=\frac{\rho_n}{\rho}\frac{\omega^2 r}{g}. \qquad (5)$$

After integrating, we obtain a parabolic surface,

$$h=\frac{\rho_n}{\rho}\frac{\omega^2 r^2}{2g}+h_0, \qquad (6)$$

whose maximum curvature is less than that of a normal viscous liquid by the factor ρ_n/ρ.

From Eqs. (2) and (4) we have

$$\begin{aligned}\rho_s S\nabla T=-(\rho_s/\rho)[\rho_n\nabla(\mathbf{V}_n^2/2) \\ -\rho_n\mathbf{V}_n\times\nabla\times\mathbf{V}_n]-(\rho_s\rho_n/2\rho)\nabla|\mathbf{V}_n-\mathbf{V}_s|^2,\end{aligned}$$

which in this special case yields

$$\nabla T=0. \qquad (7)$$

This result of uniform temperature is dependent on the inclusion of the terms in the square of the relative velocity and demonstrates that the solution with $\mathbf{V}_s=0$ is a possible one in thermodynamic equilibrium.

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[11] K. R. Atkins, *Liquid Helium* (Cambridge University Press, Cambridge, England, 1959), Chap. 6.

[12] E. L. Andronikashvili and I. P. Kaverkin, Zh. Eksperim. i Teor. Fiz. **28**, 126 (1955) [English transl.: Soviet Phys.—JETP **1**, 174 (1955)].

[13] R. J. Donnelly, G. V. Chester, R. H. Walmsley, and C. T. Lane, Phys. Rev. **102**, 3 (1956).

[14] R. J. Donnelly, Phys. Rev. **109**, 1461 (1958).

[15] R. R. Turkington, J. B. Brown, and D. V. Osborne, Can. J. Phys. **41**, 820 (1963).

[16] P. R. Zilsel, Phys. Rev. **79**, 309 (1950); **92**, 1106 (1953).

It is also apparent that these terms explain how the superfluid, although motionless, is held up in a parabolic shape, a point which has troubled the intuition of many.

Another solution in which there is no entropy production can be obtained if we assume that $\langle\nabla\times\mathbf{V}_s\rangle_{av}\neq 0$ and if we add to the left side of Eq. (2) and acceleration term $-\rho_s\mathbf{V}_s\times\langle\nabla\times\mathbf{V}_s\rangle_{av}$. The average vorticity designates the circulation around a small but macroscopic region divided by the area of the region and does not necessarily imply that the flow must be microscopically rotational. Thus, hollow vortex discontinuities could make the region multiply connected and give large-scale rotation of the superfluid without violating the microscopic condition $\nabla\times\mathbf{V}_s=0$ implicit in Eq. (2). When we have $\langle\mathbf{V}_s\rangle_{av}=\mathbf{V}_n=\mathbf{\Phi}_1\,\omega r$ ($\langle\mathbf{V}_s\rangle_{av}$ again being an average over a small, but macroscopic, region), the steady state has the same surface as a viscous liquid and Eq. (7) is again valid;

$$h=(\omega^2r^2/2g)+h_0. \qquad (8)$$

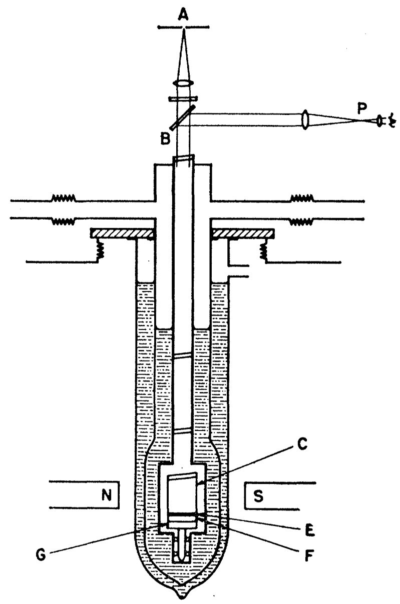

FIG. 1. Schematic diagram of the apparatus used to measure the curvature of the liquid surface.

EXPERIMENTAL METHOD

Figure 1 shows the equipment schematically. The thin layer of liquid helium to be studied was condensed on a lightly ground flat glass plate *F*, which formed the bottom of the sealed chamber C. This chamber, which had brass walls and a glass top, was filled with helium gas under pressure at room temperature to give the desired liquid depth. The glass top and bottom were retained with brass flanges and sealed with indium wire O rings. Chamber C rested on a rotating platform G and was wrapped with a thin layer of soft iron, which, coupling with rotating magnets outside the Dewars, served to rotate the chamber. The top surface of the rotating platform could be moved relative to its bottom by adjusting screws, thereby allowing the axis of the container C to be moved horizontally and tilted until it coincided with the axis of rotation of the platform.

Parallel light from a collimator (after passing through a heat-absorbing glass filter) was projected vertically into the Dewar, and the beam reflected by the surface of the liquid layer E was directed by beam splitter B into an objective lens (focal length 26.2 cm) which formed an image at P of the pinhole light source A. Since the reflection from a liquid helium surface is about 0.5% of that from glass, the glass substrate was lightly ground until most of the light was scattered, just enough of the original optical surface being left to give an image in the focal plane equal in brightness to that from the liquid surface. This substrate reflection provided a reference image at the principal focus corresponding to zero curvature and allowed adjustment of the substrate and liquid film until they were parallel and their normals aligned with the axis of rotation.

The position of the image was determined by its coincidence with the illuminated crosshair of a filar eyepiece whose position was read on an optical-bench scale. The curvature of the surface was calculated by using the measured displacement of the focal point from the principal focus of the objective lens.

The main difficulty in determining the focus was the eye's power of accommodation. The standard parallax method of eliminating this difficulty was not useful in rotation because the focal spot usually described a small circular orbit in the focal plane corresponding to a residual angle (of about 10^{-4} rad) between the axis of rotation and the optical axis of the liquid surface. A useful technique was to increase the illumination of the crosshair until it was brighter than the focal spot, thus tending to fix the focus of the eye at the crosshair. It was also found that by placing a mask consisting of three rather thick radial spokes over the aperture of the collimator, the off-focus setting gave three small sector-shaped light areas, which at focus merged into a single spot. These techniques, plus the procudure of approaching the focus from opposite directions on alternate readings, made the determination of the focus reasonably objective.

The effect of the viewing light on the behavior of the liquid helium was investigated in a number of ways. The total power in the light beam incident on the cell was between 4×10^{-5} W and 4×10^{-7} W, depending on the size and intensity of the source used, and it is estimated that perhaps one third of this was absorbed in the cell. Since the light was uniform over the substrate, the temperature difference between the center and the edge was calculated to be at most, 2×10^{-6} deg. From other measurements on the flow of heat in thin layers of helium II (to be published) it was known that this temperature difference would not lead to flow velocities exceeding the critical velocity. Actually, the effect of the light was never observed to change the results in rotation, even when the beam intensity was increased from the weakest one used to one 1000 times as strong. Tests were also made by opening a shutter on the light beam only after the steady rotational state had been reached. No difference in the results were found even though the liquid could not have changed its contour in the time of observation.

An attempt was also made to measure the surface contour by means of Fizeau fringes formed by the interference of the reflections from the liquid surface and the lightly ground substrate. With proper grinding these reflections were of equal amplitude and resulted in continuous saturated fringes by which a depth change of about 5×10^{-6} cm could be detected. Unfortunately these fringes could only be obtained in the nonrotating state since surface ripples introduced by rotation were large enough (about 1 μ in height at low rotational speeds) to blur the fringes and render them useless.

SURFACE CURVATURE IN STEADY ROTATION

It follows from Eq. (6) that, if the superfluid velocity is zero, the maximum surface curvature (at $r=0$) is

$$\gamma=(\rho_n/\rho)(\omega^2/g). \qquad (9)$$

When both fluids rotate together at a common angular velocity ω, Eq. (8) gives the curvature

$$\gamma=\omega^2/g, \qquad (10)$$

which is the same as that of an ordinary liquid.

Figure 2 shows the measured curvature in steady rotation as a function of angular velocity. The result is that the surface curvature of the helium II in this thin layer (average depth 5.0×10^{-3} cm) is very nearly given by Eq. (10) and is certainly not that of Eq. (9); in short, heluim II behaved in steady rotation like a viscous liquid.

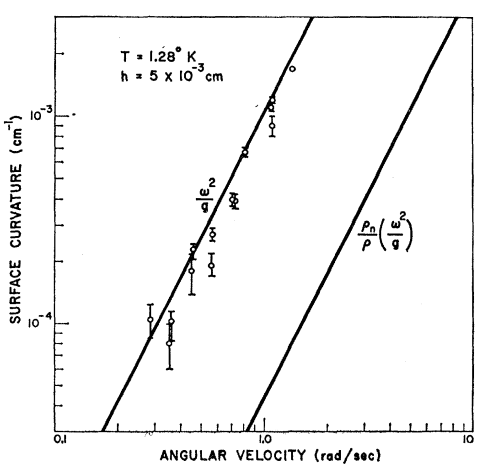

FIG. 2. The maximum measured curvature of liquid helium II in steady rotation. The line ω^2/g is the curvature expected of a normal viscous liquid; the line $(\rho_n/\rho)(\omega^2/g)$ is the curvature expected if the superfluid remains at rest.

The lowest angular velocity at which measurements were practical was 0.29 rad/sec. Actually, the best fit to the measured points is somewhat less than predicted by Eq. (10); but since this divergence varies little with the angular velocity, it cannot be explained on the basis of a theory such as Landau's, in which the curvature is velocity dependent, and it is probably a systematic error in the curvature determination not connected with the superfluid properties of helium II.

TRANSIENT EFFECTS

For comparison with helium II, we first consider the angular acceleration of a thin layer of ordinary viscous liquid. The liquid rests on the horizontal bottom of a circular cylindrical container, and both the liquid and the container are initially at rest. The container is then rapidly accelerated to a constant angular velocity about its vertical axis, and motion is transmitted to the liquid, whose surface eventually reaches the parabolic shape of steady-state rotation. The time required to approach the steady state is determined, in a thin layer, by two nearly independent processes. First, the velocity field in the liquid is set up by the diffusion of vorticity from the solid bottom upward in the liquid; and second, the centrifugal force acting on the rotating liquid causes radial flow until the required equilibrium contour in the gravitational field is reached.

For a very thin liquid layer, the first process will be completed before the second fairly begins. In this first process the velocity field is determined by the diffusion equation

$$\partial V/\partial t=\nu(\partial^2V/\partial z^2),$$

where $\nu=\eta/\rho$ is the kinematic viscosity and z is the vertical coordinate. For impulsive acceleration from rest, Carslaw and Jaeger[17] give a series solution. When we have $\nu t/h^2\geq1$ (where h is the liquid depth), the series converges rapidly, and we can write the characteristic time to set up the velocity field as

$$\tau_1=(4/\pi^2)(\rho h^2/\eta).$$

We define τ_1 as the time to reach one-half the final centrifugal pressure.

Assuming that the velocity field discussed above reaches its final value very rapidly, a measure of the time for the liquid surface to deform to its final parabolic shape is found, from a result given by Emslie *et al.*,[18] to be

$$\tau_2=\tfrac{3}{8}(\eta r^2/\rho g h^3).$$

Here, g is the acceleration of gravity, h is the average depth, and r is the radius of the container. If we con-

[17] H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids* (Clarendon Press, Oxford, 1947), 1st ed., p. 83.
[18] A. G. Emslie, F. T. Bonner, and L. G. Peck, J. Appl. Phys. **29**, 858 (1958).

sider both of these processes, a reasonably upper bound for the time for the liquid surface to approach its final parabolic contour is

$$\tau=\tau_1+\tau_2.$$

For liquid helium in a layer 5×10^{-3} cm deep, the time calculated for the curvature to reach 0.7 of its final value is about one second, if we use the total fluid density and the normal fluid viscosity. Although the time resolution and the precision of these curvature measurements were not very good, it was found that the time to accelerate the liquid was of the order of 10 sec, and that at the lowest speeds the steady-state curvature was reached only after about 40 sec. Thus, we can conclude that for speeds up to 1 rad/sec, liquid helium accelerates considerably slower than an ordinary viscous liquid. In stopping, the characteristic time to reach one-half of the steady-state curvature was 2 sec or less, a time not far from that expected for a viscous liquid with kinematic viscosity η/ρ. This slow acceleration and rapid deceleration have previously been reported by Hall.[19]

DISCUSSION

For a value of peripheral velocity times liquid depth greater than $Vh=1.8\times10^{-3}$ cm^2/sec, the superfluid rotates with the container. Previous measurements of other types can be interpreted as predicting a critical velocity condition $Vd=10^{-2}$ cm^2/sec, below which the superfluid should remain stationary, d being the characteristic lateral dimension. Are these two results incompatible?

Certainly, on the basis of a simple vortex theory we might expect the pertinent characteristic dimension to be the radius of the container rather than the liquid depth (that is, essentially the *longest* distance perpendicular to the superfluid flow). In the present case this would lead to a critical velocity perhaps 50 times smaller than the lowest velocity attained, and is consistent with the experimental result.

However, such a simple vortex model does not predict the measured critical velocities in other experiments. In flow through slits, it has been shown that the pertinent characteristic distance is the *shortest* distance perpendicular to the flow, not the *longest*. In oscillating boundary experiments, the pertinent distance is the viscous penetration depth, not the radius of the containing vessel. Even in steady rotation the superfluid can apparently remain at rest in a container rotating at about 3000 times the velocity corresponding to one vortex line.[10] Thus, neither a simple vortex model nor Eq. (1) is adequate to explain all these results.

It seems probable that the metastable nonrotating state sometimes observed by Reppy and Lane was not observed in the present experiment because of the roughness of the boundaries in the present case as compared with the microscopically smooth blown-glass container used in their experiment. In this regard one is reminded that the metastability reported by Brewer, Edwards, and Mendelssohn[20,21] in heat conduction was observed in a smooth glass capillary. It is also possible that the free surface is of primary importance since surface ripples, which were always present to some extent, would presumably be unfavorable to the occurence of metastability.

We may conclude that a stable rotating state of the superfluid with the present boundary conditions exists down to $\omega=0.29$ rad/sec. A question, which is still unanswered, is to what minimum velocity this will remain true. Is it necessary to go to the very low rotational speeds predicted by a simple vortex-line theory, or does a surface energy associated with velocity discontinuities, as suggested by Landau and Lifshitz[6] and by Mott,[22] lead to a *stable* nonrotating state at higher speeds? Such a surface energy could perhaps exclude vorticity from the bulk of the superfluid in a manner analogous to the Meissner effect in which the exclusion of the magnetic field from a superconductor can be attributed to a surface energy. The present technique could, with more attention to vibration isolation, be used to investigate this question to rotational speeds down to about 0.1 rad/sec.

ACKNOWLEDGMENTS

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[19] H. E. Hall, Phil. Trans. Roy. Soc. London **A250**, 359 (1957).

[20] D. F. Brewer, D. O. Edwards, and K. Mendelssohn, Phil. Mag. **1**, 1130 (1956).

[21] D. F. Brewer and D. O. Edwards, Proc. Roy. Soc. (London) **A251**, 247 (1959).

[22] N. F. Mott, Phil Mag. **40**, 61 (1949).