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π-π Scattering with Unitarity and Crossing

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A model of pion-pion scattering is presented in which each partial-wave amplitude in the s channel satisfies unitarity including inelastic states. This is accomplished by the use of the N/D technique. The full amplitude, i.e., the explicit, continued, partial-wave sum does not satisfy unitarity in the t and u channels, but it does have the correct branch points in these channels. These characteristics are guaranteed by construction and do not depend on arbitrary parameters occurring in the N functions. The parameters are fixed by imposing the crossing relations and by demanding the existence of a Pomeranchuk trajectory in the $T=0$ channel. The existence of the ρ is not assumed. Since crossing is not satisfied exactly, the parameters are determined as those that yield the best fit to the crossing relations. Once the parameters have been fixed, the behavior of each partial wave in each isospin channel is determined. Phase shifts are presented for the S, P, and D waves. The P wave is repulsive at low energies but becomes quite attractive at higher energies due to inelastic effects. A resonance occurs at $s=33.7\ \mu^2$ with a width about three times that experimentally observed; this width is extremely sensitive to the parameters, the position is not. An S-wave ghost occurs at $s=-57.1\ \mu^2$ (whose residue is zero) as well as the Pomeranchuk trajectory at $s=0$. The S waves are strongly repulsive at low energies. In fact, they are so repulsive that rather broad peaks are produced in their cross sections near 400 MeV when the phase shifts pass through $-\pi/2$. The peak in the $T=0$ channel may very well correspond to the Abashian, Booth, and Crowe (ABC) anomaly. There are no resonances in the D waves. In particular, the existence of an f_0 is incompatible with any choice of the parameters unless it is accompanied by another strong D-wave resonance at low energy, and even this possibility violates crossing badly. This may be due to a poor choice for our trial function.

I. INTRODUCTION

In the past few years, several attempts[1–3] have been made to determine the pion-pion scattering amplitude. It is clear that this is an important amplitude to determine since a knowledge of it is necessary for a determination of all other amplitudes. Among other characteristics, the π-π amplitude is unique in the sense that the crossing relations are relations between the same amplitudes, no other amplitudes being involved. In addition to the constraint of crossing, one has of course the usual restrictions of unitarity, threshold behavior, etc. Not surprisingly, the solution of this complete problem is very difficult and various approximation schemes have been developed to make the problem tractable. The goal of any given calculation using such a scheme has been primarily to reproduce the pronounced resonance in the $T=1$, $l=1$ channel, the ρ meson. These calculations have shown, if nothing else, that the basic difficulty in a study of the π-π amplitude is the imposition of the crossing relations in the following sense. One conventionally decomposes the amplitude into partial waves and then analyzes these partial waves by some method, say the N/D technique, by means of which one can proceed a reasonable distance, but the crossing relations apply to the full amplitude, i.e., the partial-wave *sum*.

To avoid the total complexity of the crossing relations, the concept of a "self-consistent" calculation[4] was introduced, i.e., the presence of a ρ meson in the crossed channels will itself provide an exchange force which may produce a resonance in the direct channel. Such calculations have encountered divergence difficulties (which can be eliminated by the use of Regge poles in the crossed channel[5]), but they have been useful. They do suffer, however, from the essential difficulty of principle in that they assume the existence of the ρ meson itself. This, of course, is independent of the violence done to the proper crossing relations which is related to the divergence difficulties alluded to above. Also inelastic states have been shown to play an important role in producing resonances which occur at higher energies,[6,7] and although such states have not been included in previous calculations, these states are certainly important in a discussion of the ρ meson. In any event, it is clear that a calculation which can produce the ρ meson by imposing the constraints of crossing, unitarity, etc., and without assuming its

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existence, is certainly of interest. This, in fact, is the object of this paper.

Finally, our main motivation here is not to perform a "better" calculation than those done by others since at this stage of development, the word "better," like the rest of the theory, is probably not well defined. On the other hand, we feel that it is not sensible to believe a model of the strong interactions simply because there happens to be some approximation scheme which when applied to the model yields results more or less in agreement with experiment. The natural question to ask is the following: Is this agreement a property of the model or the approximations? This question is of the utmost importance for those who want to "bootstrap" the phenomena associated with strong-interaction physics as we know it. We have tried to develop a scheme which is *different* from those already reported in the literature, and yet which is at the same time sensible. If the reader will bear this in mind, many obscure points of logic will become clear. This paper is probably unique in that we make very little use of Regge poles (none from a strictly logical point of view) simply because it is simpler to avoid them in our analysis. It should also be kept in mind that our results, which come from a numerical variational procedure, are strongly limited by our choice of a trial function and lack of time on a computer.

In this analysis, we will construct a model which includes nearly all of the restrictions inherent in the $\pi-\pi$ problem and show that, in fact, the ρ meson is a necessary consequence of these constraints. Basically, our procedure will be to construct scattering amplitudes which intrinsically satisfy certain of the usual constraints, e.g., unitarity, threshold behavior, etc., while the remaining constraint and crossing will be satisfied by choosing specific values for parameters which will be introduced into the amplitude. Once these parameters are determined, the complete amplitude is known. Section II will provide a construction of those amplitudes which satisfy all of the constraints except crossing. In Sec. III, we discuss the parameter determination by the use of crossing in detail. Finally, in Sec. IV, we present the numerical results of our analysis including the phase shifts for the S, P, and D waves in all three isospin channels.

II. CONSTRUCTION OF THE SCATTERING AMPLITUDES

A. Elastic States

As stated in the Introduction, our goal is to construct a parameterized scattering amplitude which intrinsically satisfies some of the constraints imposed on any scattering amplitude and such that the remaining requirements can be approximately satisfied by choosing the parameters judiciously. This has been the object of several studies[1–3] from various points of view, but we believe the analysis given here is a more properly self-consistent one than those previously presented and, as such, is more comprehensive. Although we will include the effects of some inelastic states, we will begin by imposing only elastic unitarity.

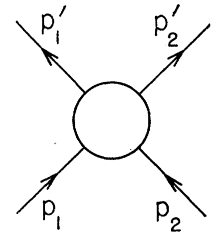


FIG. 1. Elastic pion-pion scattering.

Our procedure will be to choose a specific form for the partial-wave amplitudes in some channel such that each amplitude satisfies unitarity and has the correct threshold behavior in the relevant variable. We will further insist that the full amplitude, i.e., partial-wave sum, have, at least, discontinuities in the remaining variables over the correct range, although unitarity and the correct threshold behavior may not be satisfied in these variables. Even though the chosen partial-wave amplitudes will contain parameters, it is to be emphasized that the above requirements must be satisfied independent of a particular choice of the parameters. Other constraints on the amplitudes, such as crossing, will be approximately satisfied by the choice of the parameters.

We begin by defining the usual variables, see Fig. 1, as

$$\begin{aligned} s&=-(P_1+P_2)^2,\\ t&=-(P_1-P_1')^2,\\ u&=-(P_1-P_2')^2, \end{aligned} \tag{1}$$

where $P_1^2=P_2^2=P_1'^2=P_2'^2=-1$, and we introduce the scattering angle in the s channel as

$$Z=(\mathbf{P}_1\cdot\mathbf{P}_1')/P^2, \tag{2}$$

where $\mathbf{P}_i$ and $\mathbf{P}_i'$ denote the initial and final pion momenta, and $P^2=\frac{1}{4}(s-4)$. All quantities refer to the barycentric system in the s channel. We now expand the amplitude, using superscripts to denote the isotopic spin values, in a partial-wave series in the s channel as

$$M^{(0,2)}(s,t)=\tfrac{1}{2}\sum_l(2l+1)M_l^{(0,2)}(s)[P_l(s)+P_l(-z)], \tag{3}$$

$$M^{(1)}(s,t)=\tfrac{1}{2}\sum_l(2l+1)M_l^{(1)}(s)[P_l(z)-P_l(-z)]. \tag{4}$$

The requirements that $M_l(s)$ satisfy unitarity and also have the correct threshold behavior are usually accomplished by first explicitly displaying the threshold behavior and then writing the remaining amplitude in terms of the N/D procedure.[1] Therefore one writes for all s,

$$M_l(s)=P^{2l}F_l(s)=P^{2l}(N_l(s)/D_l(s)). \tag{5}$$

For the purposes of this analysis, however, Eq. (5)

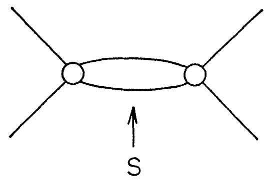


FIG. 2. Born approximation: two-pion state in the crossed channel.

is not a favorable choice. Especially, within the framework of approximations we wish to employ, it will not lead to the correct cuts in the t plane. A more suitable choice is provided by the modification of the simple, Regge-pole formula due to Khuri,[8] in which the full amplitude exhibits the correct t cuts explicitly. Rather than Eq. (5) we write,

$$M_l(s)=e^{-\xi l}F_l(s)\,, \tag{6}$$

$$\cosh\xi=1+(2M/(s-4))\,, \tag{7}$$

where M is the lowest mass exchanged in the t and u channels. If we regard the diagram in Fig. 2 as the driving term of the s-channel, partial-wave, dispersion relations, then $M=4$. By use of Eq. (7), we may write

$$e^{-\xi}=(s^{1/2}-2)/(s^{1/2}+2)\,, \tag{8}$$

which leads to the same threshold behavior as that given in Eq. (5). Thus, rewriting Eqs. (3) and (4), we have

$$M^{(0,2)}(s,t)=\tfrac{1}{2}\sum_l(2l+1)e^{-\xi l}F_l^{(0,2)}(s)\times[P_l(z)+P_l(-z)]\,, \tag{9}$$

$$M^{(1)}(s,t)=\tfrac{1}{2}\sum_l(2l+1)e^{-\xi l}F_l^{(1)}(s)\times[P_l(z)-P_l(-z)]\,, \tag{10}$$

where

$$F_l^{(T)}(s)=N_l^{(T)}(s)/D_l^{(T)}(s)\,, \tag{11}$$

and

$$D_l^{(T)}(s)=1-\frac{1}{\pi}\int_4^\infty ds'\frac{\rho(s')e^{-\xi' l}N_l^{(T)}(s')}{s'-s}\,, \tag{12}$$

with

$$\rho(s')=((s'-4)/s')^{1/2}. \tag{13}$$

Extraneous constant factors in the phase-space function $\rho(s)$ are absorbed into the multiplicative parameters of the N_l; these factors will be the same for all channels. Once a choice of the form of $N_l^{(T)}(s)$ is made, the amplitude is determined except for the values of the parameters occurring in $N_l^{(T)}(s)$. This will be discussed in Sec. III.

It is obvious from this construction that $M_l(s)$ has the proper threshold behavior and satisfies unitarity in the s channel. We now show that $M^{(T)}(s,t)$ has the correct positions of the t and u cuts. It is well known that the N/D procedure starts with a Born-approximation diagram (N) and iterates this to satisfy unitarity (D). Thus the diagram in Fig. 2 yields in an N/D calculation the sum in Fig. 3. We may explicitly display such diagrams by expanding the D function as,

$$\frac{1}{D_l(s)}=\frac{1}{1-d_l(s)}=1+d_l(s)+\cdots\,, \tag{14}$$

where we have suppressed the isotopic-spin labels. Of course, if the series is truncated, it will not be a good approximation of the D function itself, especially if the value of s is near that of a resonance, but term by term the series is useful in determining the analytic properties. Thus

$$M_l(s)=h^l(s)N_l(s)[1+d_l(s)+\cdots]\,, \tag{15}$$

where we have introduced $h(s)$ as

$$h(s)=e^{-\xi}=(s^{1/2}-2)/(s^{1/2}+2)\,. \tag{16}$$

The first term in Eq. (15) when summed over l is simply the Born approximation, i.e., the first diagram in Fig. 3, and the second term represents the second diagram in Fig. 3. However, Eq. (15) implies that the positions of the left-hand cuts arising from the diagrams in Fig. 3 are the same, which is not true. Thus if we wish to keep the first two terms in the expansion of $1/D_l(s)$, we must modify $N_l(s)$ if we are to obtain the correct left-hand cuts. We therefore truncate the expansion of $1/D_l(s)$ but replace $N_l(s)$ by

$$N_l(s)\to N_l(s)[1-d_l(s)+\gamma_l(s)]\,, \tag{17}$$

where $\gamma_l(s)$ is a correction to $d_l(s)$ and is proportional to an integral over $N_l(s)$. Equation (15) now becomes,

$$\begin{aligned}M_l(s)&=h^l(s)N_l(s)[1-d_l(s)+\gamma_l(s)][1+d_l(s)]\\&\simeq h^l(s)N_l(s)[1+\gamma_l(s)]\,,\end{aligned} \tag{18}$$

where we have neglected the quadratic terms. In order that we preserve the fundamental spirit of the N/D method we must insist that $N(s)$, as modified in Eq. (17), is still analytic on the right. Since $d_l(s)$ is cut from $s=4$ to ∞, $\gamma_l(s)$ must also be so cut and thus let us choose for $\gamma_l(s)$

$$\gamma_l(s)=\frac{1}{\pi}\int_4^\infty\frac{ds'}{s'-s}\rho(s')h^{-l}(s)H^l(s,s')K(s,s')\,, \tag{19}$$

for reasons that will become clear shortly. With this choice for $\gamma_l(s)$, the requirement that $N_l(s)$ have no right-hand discontinuity leads to the relation

$$h^l(s)N(s)=h^{-l}(s)H^l(s,s)K(s,s)\,, \tag{20}$$

which may be satisfied by requiring that

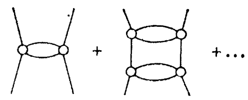
$$\begin{aligned}H(s,s)&=h^2(s)\,,\\K(s,s)&=N(s)\,.\end{aligned} \tag{21}$$

It is easy to see that $M_l(s)$ satisfies unitarity, by use of Eqs. (19) and (21) to first order, i.e.,

$$\mathrm{Im}[M_l(s)/h^l(s)]=\rho(s)h^l(s)N_l^2(s)\,, \tag{22}$$

[8] N. N. Khuri, Phys. Rev. 130, 429 (1963).

FIG. 3. Iteration of two-pion exchange by means of one-channel unitarity.



and this is the principal reason for the choice in Eq. (19).

The total amplitude is given now as

$$M(s,t)=\sum_l (2l+1)h^l(s)N_l(s)[1+\gamma_l(s)]P_l(z). \quad (23)$$

If we assume that we may neglect the l dependence in $N_l(s)$, say as compared to $h^l(s)$, we may sum the series explicitly. In particular,

$$\begin{aligned}M(s,t)&=N(s)\sum_l (2l+1)h^l(s)P_l(z)\\&\quad+(1/\pi)\int_4^\infty \frac{ds'}{s'-s}\rho(s')N(s)K(s,s')\\&\qquad\times\sum_l (2l+1)H^l(s,s')P_l(z)\\&=N(s)\frac{1-h^2(s)}{[(1-h(s))^2-2h(s)(z-1)]^{3/2}}\\&\quad+\frac{1}{\pi}\int_4^\infty \frac{ds'}{s'-s}\rho(s')N(s)K(s,s')\\&\qquad\times\frac{1-H^2(s,s')}{[(1-H(s,s'))^2-2H(s,s')(z-1)]^{3/2}},\end{aligned}$$

which follows by use of the generating function of the Legendre polynomials. By use of Eqs. (16) and (2), this may be reduced to

$$\begin{aligned}M(s,t)&=\frac{s^{1/2}(s^{1/2}+2)N(s)}{8(4-t)^{3/2}}\\&\quad+\frac{1}{\pi}\int_4^\infty \frac{ds'}{s'-s}\rho(s')N(s)K(s,s')\frac{P^3(s)(1-H^2(s,s'))}{H^{3/2}(s,s')}\\&\qquad\times\frac{1}{[(P^2(s)/H(s,s'))(1-H(s,s'))^2-t]^{3/2}}. \quad (24)\end{aligned}$$

The first term in this expression clearly represents the singularities in t arising from the first diagram in Fig. 3. However, the second term is quite unsatisfactory in representing the singularities in t for the second diagram in Fig. 3. If one insists on a product, cut-plane representation (i.e., a Mandelstam representation). In particular, the cuts in t depend on s as well as s'. Since $H(s,s')$ is arbitrary except for the condition in Eq. (21), we will insist that $H(s,s')$ satisfy the additional restriction

$$(P^2(s)/H(s,s'))(1-H(s,s'))^2=f(s'), \quad (25)$$

where $f(s')$ is some function of s' alone. We may determine $f(s')$ immediately by setting s equal to s' and using Eq. (21). This yields

$$f(s')=16s'/(s'-4), \quad (26)$$

which is, of course, the equation of the boundary of the Mandelstam spectral function. The function $H(s,s')$ itself may now be determined from Eq. (25), and one obtains

$$H(s,s')=1+\frac{f(s')}{2P^2(s)}-\left[\frac{f(s')}{P^2(s)}+\frac{f^2(s')}{4P^4(s)}\right]^{1/2}. \quad (27)$$

The opposite sign for the square root does not satisfy Eq. (21).

However, Eq. (24) is still unsatisfactory because the integrand still contains a dependence on s other than that occurring in the denominator $(s'-s)$. Again we insist that

$$N(s)K(s,s')\frac{P^3(s)(1-H^2(s,s'))}{H^{3/2}(s,s')}=F(s'), \quad (28)$$

and determine $F(s')$ by setting s equal to s' and using Eq. (21). This gives

$$F(s')=N^2(s')(P^3(s')/h^3(s'))(1-h^4(s')),$$

and

$$K(s,s')=\left[\frac{P(s')}{P(s)}\right]^3\left[\frac{H(s,s')}{h^2(s')}\right]^{3/2}\frac{1-h^4(s')}{1-H^2(s,s')}\frac{N^2(s')}{N(s)}. \quad (29)$$

We may therefore write Eq. (24) as

$$\begin{aligned}M(s,t)&=\frac{s^{1/2}(s^{1/2}+2)N(s)}{8(4-t)^{3/2}}\\&\quad+\frac{1}{\pi}\int_4^\infty \frac{ds'\rho(s')P^3(s')}{(s'-s)[(16s'/(s'-4))-t]^{3/2}}\\&\qquad\times\left[\frac{1-h^4(s')}{h^3(s')}\right]N^2(s'), \quad (30)\end{aligned}$$

which correctly represents the singularities in s and t arising from the diagrams in Fig. 3. The proper cuts in u follow immediately from the symmetrization of $M(s,t)$ given in Eqs. (3) and (4), specifically if $z\to -z$, then $t\to u$.

Although we would not expect $M(s,t)$ as given in Eq. (30) to yield reliable results in any calculation, especially those involving resonances, it does show that the form given in Eqs. (9) and (10) satisfies our requirements for a scattering amplitude. Further, we note that the l dependence of $N_l(s)$ played no essential role in the development of the proper t and u singularities. By neglecting this l dependence we have only allowed ourselves the possibility of summing the series explicitly. In everything that follows, we will assume it is a good approximation to neglect the l dependence of $N_l(s)$ especially when compared to that of $e^{-\xi l}$. Therefore, our model, in the approximation of elastic unitarity, is defined by Eqs. (9) to (13) and by the parameters which occur in the function $N(s)$.

B. Inelastic States

We would now like to extend the analysis of the previous section to include inelastic states in the unitarity relation. The motivation for this follows primarily from the observation that important experimental effects are present considerably above the lowest inelastic threshold ($s=16$) and also that the effects of inelastic channels can be large below the inelastic thresholds.[7] In order to include, in a general way, the inelastic effects, we will consider two "two-particle" inelastic channels which represent in some way the states (π–ω) and (ρ–ρ).

In the $T=0$ and $T=2$ channels the (π–ω) state cannot contribute, and thus we represent the inelastic states for even isospin by the (ρ–ρ) state alone. Since we have even isospin, Bose statistics requires that the spin-zero and spin-two states couple to even orbital angular momentum (L) in the (ρ–ρ) system, while the spin-one state couples to odd L. The complications here due to the spin may be considerably reduced by neglecting, for the moment, all values of L except $L=0$. This is a reasonable physical assumption since the primary inelastic contribution to π–π scattering below, say, 1500 MeV is probably dominated by the low-energy region of this production reaction; this region is dominated, in turn, by the S-wave, (ρ–ρ) phase space. Thus under this assumption, in each isospin state, the spin-zero (ρ–ρ) state contributes to S-wave π–π scattering while the spin-two state contributes to D-wave π–π scattering; the spin-one state does not contribute at all. Actually, we shall construct all the partial waves not just those corresponding to $L=0$. These remarks concerning the importance of $L=0$ motivate the inelastic model we shall use to hold for all l.

If we continue to impose the condition that $L=0$ in the (ρ–ρ) state, then this state does not contribute to the $T=1$ channel since $L=0$ requires even parity while Bose statistics requires odd parity in the (π–π) system. Even if the (ρ–ρ) state could contribute to the $T=1$ channel, e.g., by removing the restriction that $L=0$, we would neglect its effect because its threshold ($112\ \mu^2$) is so much higher than that of the (π–ω) state ($42\ \mu^2$). While it is necessary to assume some model for the weighting of the spin zero and two in the inelastic (ρ–ρ) amplitude, spin presents no problem for the inelastic (π–ω) amplitude. Since the orbital angular momentum in the (π–ω) channel is uniquely related to l, we can readily include all l in the construction.

There are, of course, many other two-particle states which contribute to the inelastic effects, e.g., the (ω–ω) state. However, since all we wish to accomplish is an approximate representation of the inelastic states, we will ignore these additional states. As a matter of fact the (ω–ω) state is well approximated by the (ρ–ρ) state; their thresholds and the analysis of their spin dependences are similar, and they will contribute to the $T=0$ (π–π) state in approximately the same way. (They do not couple to the $T=2$ state.) Intermediate ($K\bar{K}$) states, etc., are neglected altogether.

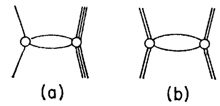


FIG. 4. Two-pion exchange states in elastic $\pi\omega$ and $\rho\rho$ scattering.

If we do confine our attention, and we will, to the (π–ω) and (ρ–ρ) states then it is necessary to show that including these inelastic states does not affect the remarks made in Sec. IIA concerning threshold behavior and crossed-channel cuts. In order to show that the previous analysis still applies, let us ignore all questions of spin, both intrinsic and isotopic. We now have three channels (π–π), (π–ω), and (ρ–ρ) which we will label 1, 2, and 3, respectively. For channel 1 we relabel the relevant quantities as:

$$\begin{aligned} \rho_1&=[(s-4)/s]^{1/2},\\ h_1&=(s^{1/2}-2)/(s^{1/2}+2)=e^{-\xi_1},\\ \cosh\xi_1&=1+(4/2P_1^2),\\ P_1^2&=(s-4)/4. \end{aligned} \tag{31}$$

From the preceding section we know that if the partial-wave amplitudes exhibit the factor h_1^l, then the correct threshold behavior is present.

For channel 2 we consider the diagram in Fig. 4(a) and define

$$\begin{aligned} \rho_2&=\frac{\{[s-(\omega+1)^2][s-(\omega-1)^2]\}^{1/2}}{s},\\ h_2&=e^{-\xi_2}=\frac{[s^{1/2}-1]^2-\omega^2}{[s^{1/2}+1]^2-\omega^2},\\ \cosh\xi_2&=1+(4/2P_2^2),\\ P_2^2&=\frac{[s-(\omega+1)^2][s-(\omega-1)^2]}{4s}, \end{aligned} \tag{32}$$

where we note that $h_2\rightarrow P_2^2$ as $s\rightarrow(\omega+1)^2$. Similarly for channel 3 we consider the diagram in Fig. 4(b) and write

$$\begin{aligned} \rho_3&=((s-4\rho^2)/s)^{1/2},\\ h_3&=e^{-\xi_3}=\frac{[s-4\rho^2+4]^{1/2}-2}{[s-4\rho^2+4]^{1/2}+2},\\ \cosh\xi_3&=1+(4/2P_3^2),\\ P_3^2&=(s-4\rho^2)/4, \end{aligned} \tag{33}$$

and, of course, $h_3\rightarrow P_3^2$ as $s\rightarrow 4\rho^2$. Just as in the case of channel 1, if the partial-wave amplitudes in channels 2 and 3 exhibit the factors h_2^l and h_3^l, the correct threshold behavior will be present.

If one proceeds in analogy with the previous section, we now write the many-channel, partial-wave amplitudes as[7,9]

$$\mathbf{M}_l^{(T)}=\mathbf{h}^{(T)l/2}\mathbf{F}_l^{(T)}\mathbf{h}^{(T)l/2}, \tag{34}$$

where

$$\mathbf{h}^{(0)}=\mathbf{h}^{(2)}=\begin{pmatrix} h_1 & 0 \\ 0 & h_3 \end{pmatrix},$$

and

$$\mathbf{h}^{(1)}=\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}. \tag{35}$$

The amplitudes $\mathbf{F}_l^{(T)}$ may be written in terms of a many-channel, N/D analysis so that

$$\mathbf{F}_l^{(T)}\mathbf{D}_l^{(T)}=\mathbf{N}_l^{(T)}, \tag{36}$$

where, suppressing the isotopic spin label,

$$\begin{aligned}
D_{11l}^{(s)}&=1-\frac{1}{\pi}\int_4^\infty \frac{ds'}{s'-s}h_1'^l\rho_1'N_{11l}',\\
D_{1il}^{(s)}&=-\frac{1}{\pi}\int_4^\infty \frac{ds'}{s'-s}h_1'^l\rho_1'N_{1il}',\\
D_{i1l}^{(s)}&=-\frac{1}{\pi}\int_{s_i}^\infty \frac{ds'}{s'-s}h_i'^l\rho_i'N_{i1l}',\\
D_{iil}^{(s)}&=1-\frac{1}{\pi}\int_{s_i}^\infty \frac{ds'}{s'-s}h_i'^l\rho_i'N_{iil}',
\end{aligned} \tag{37}$$

in which the primed functions under the integrals are functions of s' and where $i=2(3)$ for $T=1(0,2)$. Also $s_2=(\omega+1)^2$ and $s_3=4\rho^2$, and we have suppressed the spin-state quantum numbers in the case of $i=3$ in the above equations.

To show that the correct cuts in the crossed channels are still retained, we proceed exactly as before. Let us consider the $T=1$ channel and expand the determinant, $D=1-d$, which now replaces the simple denominator function, as well as replacing $\mathbf{N}$ by $\mathbf{N}(1-d+\gamma)$. As before, we assume the matrix elements of $\mathbf{N}_l$ to be l independent. To second order, i.e., keeping the diagrams in Fig. 5, we obtain for the π–π channel,

$$\begin{aligned}
M_{11l}=h_1^lN_{11}&\left[1+\frac{1}{\pi}\int_4^\infty \frac{ds'}{s'-s}h_1'^l\rho_1'N_{11}'\right]\\
&+h_1^lN_{12}\frac{1}{\pi}\int_{(\omega+1)^2}^\infty \frac{ds'}{s'-s}h_2'^l\rho_2'N_{21}'.
\end{aligned} \tag{38}$$

The series

$$M_{11}(s,t)=\sum_l (2l+1)M_{11l}(s)P_l(z)$$

[9] R. Blankenbecler, Phys. Rev. **122**, 983 (1961).

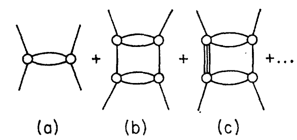


FIG. 5. Iteration of two-pion exchange by means of two-channel unitarity.

may be again summed exactly and one obtains

$$\begin{aligned}
M_{11}(s)=N_{11}&\frac{1-h_1^2}{(1+h_1^2-2zh_1)^{3/2}}\\
&+\frac{N_{11}}{\pi}\int_4^\infty \frac{ds'}{s'-s}\rho_1'K_{11}(s,s')\\
&\times\frac{1-H_{11}^2(s,s')}{(1+H_{11}(s,s')^2-2zH_{11}(s,s'))^{3/2}}\\
&+\frac{N_{12}}{\pi}\int_{(\omega+1)^2}^\infty \frac{ds'}{s'-s}\rho_2'K_{21}(s,s')\\
&\times\frac{1-H_{12}^2(s,s')}{(1+H_{12}^2(s,s')-2zH_{12}(s,s'))^{3/2}}.
\end{aligned} \tag{39}$$

The three terms in Eq. (39) correspond to the diagrams in Figs. 5(a), (b), (c), respectively, and the definition of the K_{ij} and H_{ij} is obvious from Eqs. (20) and (21). Proceeding in precisely the same way as Sec. IIA, we find the appropriate denominators corresponding to Fig. 5 as

$$[4-t]^{3/2}, \tag{40a}$$

$$[(16s'/(s'-4))-t]^{3/2}, \tag{40b}$$

$$\left[4\frac{(2s'+1-\omega^2)^2}{(s'+1-\omega^2)^2-4s'}-t\right]^{3/2}. \tag{40c}$$

Equations (40b) and (40c), of course, give the Mandelstam spectral curves for diagrams (b) and (c), respectively, in Fig. 5. One may also determine the remaining functions, but we will not do so here. The same arguments apply equally well to the $T=0$ and $T=2$ channels.

The complications introduced by including inelastic contributions can be considerably reduced by making two approximations. First, the fact that the (π–ω) state consists of two nonidentical particles played no crucial role in the above and will not do so in what follows. It is therefore convenient to replace the (π–ω) state by an identical particle state for which:

$$\begin{aligned}
P_2^2&=(s-(\omega+1)^2)/4,\\
\rho_2&=(s-(\omega+1)^2)/s,\\
h_2&=\frac{[s-(\omega+1)^2+4]^{1/2}-2}{[s-(\omega+1)^2+4]^{1/2}+2}.
\end{aligned} \tag{41}$$

Second, we have no *a priori* way of relating the

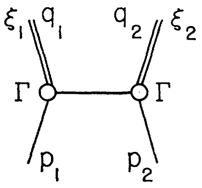


FIG. 6. One-pion exchange diagram in the inelastic process $\pi\pi \to \rho\rho$.

contributions of the $S=0$ and $S=2$ states in the reaction $\pi+\pi\to\rho+\rho$, even when we only allow $(\rho$–$\rho)$ s waves, and this forces us to consider several amplitudes. It is much more convenient to consider a specific diagram, compute the relative spin weightings, and apply that weighting to all l. In particular, let us consider the diagram in Fig. 6. The symmetrized amplitude is

$$M_{13}=\frac{\Gamma^2}{(P_1-q_1)^2+1}\xi_{1\mu}P_{1\mu}\xi_{2\nu}P_{2\nu}+\frac{\Gamma^2}{(P_1-q_2)^2+1}\xi_{1\mu}P_{2\mu}\xi_{2\nu}P_{1\nu}, \quad (42)$$

where ξ_1 and ξ_2 are the spin 4-vectors for the odd-parity, spin-one particles, satisfying $\xi_{i\mu}q_{i\mu}=0$. In Fig. 7 we have defined the angles θ and ϕ relative to P and q, the center-of-mass momenta in the initial and final states, respectively. We have, setting $z=\cos\theta$,

$$\begin{aligned}(P_1-q_1)^2+1&=(s/2)-\rho^2-2Pqz,\\(P_2-q_1)^2+1&=(s/2)-\rho^2+2Pqz,\end{aligned} \quad (43)$$

where, of course,

$$P^2=(s-4)/4 \quad\text{and}\quad q^2=(s-4\rho^2)/4.$$

By use of the polarization conditions on ξ_1 and ξ_2, we obtain, suppressing the index (13),

$$M=\frac{\Gamma^2}{2Pq}\left[\frac{\xi_1\cdot(\mathbf{P}-\mathbf{q})\xi_2\cdot(\mathbf{P}-\mathbf{q})}{\lambda-z}+\frac{\xi_1\cdot(\mathbf{P}+\mathbf{q})\xi_2\cdot(\mathbf{P}+\mathbf{q})}{\lambda+z}\right], \quad (44)$$

where

$$\lambda=(s-2\rho^2)/4Pq.$$

Since we wish to know the relative weighting between the $S=0$ and $S=2$ states, we must express Eq. (44) in terms of total spin states. In order to do this, it is convenient to introduce the quantities

$$\begin{aligned}\xi_{1+}&=-(1/\sqrt{2})(\xi_{1x}+i\xi_{1y}),\\\xi_{1-}&=(1/\sqrt{2})(\xi_{1x}-i\xi_{1y}),\\\xi_{10}&=\xi_{1z},\end{aligned} \quad (45)$$

and similarly for ξ_2, $\mathbf{P}$ and $\mathbf{q}$, so that

$$\boldsymbol{\xi}\cdot\mathbf{A}=-A_+\xi_--A_-\xi_++A_o\xi_0, \quad (46)$$

where $\mathbf{A}$ is any vector. The total spin states, $\chi_s^{M_s}$ may then be expressed as, e.g.,

$$\begin{aligned}\chi_o^o&=(1/\sqrt{3})[\xi_{1+}\xi_{2-}+\xi_{1-}\xi_{2+}-\xi_{1o}\xi_{2o}],\\\chi_2^o&=(1/6^{1/2})[\xi_{1+}\xi_{2-}+\xi_{1-}\xi_{2+}+2\xi_{1o}\xi_{2o}].\end{aligned} \quad (47)$$

After some algebra one obtains,

$$\begin{aligned}\xi_1\cdot(\mathbf{P}-\mathbf{q})\xi_2\cdot(\mathbf{P}-\mathbf{q})=&(1/\sqrt{3})[2PqP_1(z)-P^2-q^2]\chi_o^o\\&+q_+^2\chi_2^{-2}+\sqrt{2}q_+[P-qP_1(z)]\chi_2^{-1}\\&+(2/3)^{1/2}[P^2-2PqP_1(z)+q^2P_2(z)]\chi_2^o\\&+\sqrt{2}q_-[P-qP_1(z)]\chi_2^1+q_-^2\chi_2^2,\end{aligned} \quad (48)$$

and similarly for $\xi_1\cdot(\mathbf{P}+\mathbf{q})\xi_2\cdot(\mathbf{P}+\mathbf{q})$. At the inelastic threshold, Eq. (48) becomes

$$\xi_1\cdot(\mathbf{P}-\mathbf{q})\xi_2\cdot(\mathbf{P}-\mathbf{q})\to-(1/\sqrt{3})P^2\chi_o^o+(2/3)^{1/2}P^2\chi_2^o, \quad (49)$$

and in fact we will use the amplitudes near threshold to determine the weighting. The partial-wave amplitudes at threshold are determined by projecting with a state of total angular momentum l and spin 0 or 2, with an orbital angular momentum, L, in the $(\rho$–$\rho)$ channel equal to l or $l-2$, respectively; these states are

$$\begin{aligned}\psi_{l,l,o}{}^o&=Y_l{}^o\chi_o{}^o,\\\psi_{l,l-2,o}{}^o&=\sum_m C(l-2,m,2,-m;l-2,2,l,o)\times Y_{l-2}{}^m\chi_2{}^{-m},\end{aligned} \quad (50)$$

and the projections are as follows:

$$\begin{aligned}(\psi_{l,l,o}{}^o,M)&=-\frac{1}{\sqrt{3}}\Gamma^2\frac{P}{q}\int d\Omega\left(\frac{2l+1}{4\pi}\right)^{1/2}\frac{P_l(z)}{\lambda-z}\\&=-(4\pi/3)^{1/2}\Gamma^2(P/q)(2l+1)^{1/2}Q_l(\lambda),\\(\psi_{l,l-2,2}{}^o,M)&=\left(\frac{2}{3}\right)^{1/2}\Gamma^2\frac{P}{q}\times\int d\Omega C(l-2,o,2,o;l-2,2,l,o)\times\left(\frac{2l-3}{4\pi}\right)^{1/2}\frac{P_{l-2}(z)}{\lambda-z}\\&=\left(\frac{8\pi}{3}\right)^{1/2}\Gamma^2\frac{P}{q}\left(\frac{l(l-1)}{2l-1}\right)^{1/2}Q_{l-2}(\lambda).\end{aligned} \quad (51)$$

We use a superscript to denote the spin-0 and spin-2 components of the partial-wave amplitudes of our model and we write (suppressing the index 13):

$$M_l=M_l{}^{(o)}\chi_o+M_l{}^{(2)}\chi_2, \quad (52)$$

in which

$$M_l{}^{(o)}=h_1{}^{l/2}Ah_3{}^{l/2},$$

and

$$M_l{}^{(2)}=h_1{}^{l/2}\left(\frac{l(l-1)}{(2l+1)(2l-1)}\right)^{1/2}\frac{B}{h_3}h_3{}^{l/2}.$$

We now assume $L=0$ to dominate and determine the ratio of A to B by setting

$$\frac{M_o^{(o)}}{M_2^{(2)}}=\frac{[(1/(2l+1)^{1/2})(\psi_{l,l,o}{}^oM)]_{l=o}}{[(1/(2l+1)^{1/2})(\psi_{l,l-2,2}{}^oM)]_{l=2}}, \tag{53}$$

and evaluate the result at threshold. By this procedure we obtain:

$$A=-\frac{\rho-1}{\rho+1}\frac{B}{\sqrt{2}}\approx-\frac{B}{2}. \tag{54}$$

Thus the inelastic amplitude M_{13l} is determined by a single function as

$$M_{13l}(s)=h_1^{l/2}F_{13l}\times\left\{\chi_o-\frac{1}{h_3}\left(\frac{l(l-1)}{(l+\frac{1}{2})(l-\frac{1}{2})}\right)^{1/2}\chi_2\right\}h_3^{l/2}. \tag{55}$$

C. Choice of the $N_{ij}(s)$

Our task of constructing a scattering amplitude which satisfies unitarity in the s channel, with the correct "crossed cuts" and threshold behavior has been completed, and there remains only the explicit choice of the $N_{ij}(s)$. The $N_{ij}(s)$ will contain a certain set of parameters which will allow us to satisfy the crossing relations.

The choice of the $N_{ij}(s)$ is, of course, rather arbitrary with simplicity and ease of manipulation being the primary considerations. However, referring to the previous sections, especially Eq. (30), we see that one choice will lead to a correct analytic behavior of the Born term, viz.,

$$N(s)=a/s^{1/2}(s^{1/2}+2). \tag{56}$$

This function is defined in the plane cut from $s=0$ to $s=-\infty$, and has a pole on its second sheet at $s=4$.

We must choose the N_{ij} for all isospins and let us begin with $T=1$. If we refer to Eq. (56) then we would write

$$N_{11}^{(1)}(s)=16a_1^{(1)}/s^{1/2}(s^{1/2}+2), \tag{57}$$

and the simplest choice for $N_{12}^{(1)}(s)$ would be, therefore,

$$N_{12}^{(1)}(s)=16b^{(1)}/s^{1/2}(s^{1/2}+2).$$

However, in order to allow more freedom in satisfying the constraints discussed in Sec. III, we define $N_{12}^{(1)}(s)$ with two additional parameters as

$$N_{12}^{(1)}(s)=((s-s_o^{(1)})/(s+s_1^{(1)}))\times(16b^{(1)}/s^{1/2}(s^{1/2}+2)). \tag{58}$$

Equations (57) and (58) allow us to determine the

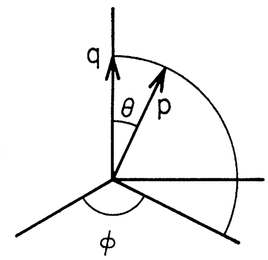


FIG. 7. Polar and azimuthal angles used in the computation of M_{13}.

$D_{ij}(s)$, and

$$D_{11l}^{(1)}(s)=1-\frac{16a_1^{(1)}}{\pi}\int_4^\infty ds'\frac{1}{s'-s}\left(\frac{s'-4}{s'}\right)^{1/2}\times\left(\frac{s'^{1/2}-2}{s'^{1/2}+2}\right)^l\frac{1}{s'^{1/2}(s'^{1/2}+2)},$$

$$D_{12l}^{(1)}(s)=-\frac{16b^{(1)}}{\pi}\int_4^\infty ds'\frac{1}{s'-s}\left(\frac{s'-4}{s'}\right)^{1/2}\times\left(\frac{s'-2}{s'+2}\right)^l\frac{s'-s_o^{(1)}}{s'+s_1^{(1)}}\frac{1}{s'^{1/2}(s'^{1/2}+2)}. \tag{59}$$

It is now convenient to introduce the quantity y as

$$4y=s^{1/2}-2, \tag{60}$$

so that Eq. (57) becomes

$$N_{11}^{(1)}(s)=a_1^{(1)}/(y+1)(y+\tfrac{1}{2}).$$

Equation (59) now takes the form

$$D_{11l}^{(1)}(s)=1-a_1^{(1)}\frac{2}{\pi}\int_0^\infty dy'\left(\frac{y'}{y'+1}\right)^{l+\frac{1}{2}}\times\frac{1}{(y'-y)(y'+y+1)(y'+\frac{1}{2})}$$
$$=1-a_1^{(1)}\alpha_l(s), \tag{61}$$

and

$$D_{12l}^{(1)}(s)=-b^{(1)}\left[\frac{s-s_o^{(1)}}{s+s_1^{(1)}}\alpha_l(s)+\frac{s_o^{(1)}+s_1^{(1)}}{s+s_1^{(1)}}\alpha_l(-s_1^{(1)})\right]$$
$$=-b^{(1)}\beta_l^{(1)}(s). \tag{62}$$

In order to define N_{21} and N_{22} we introduce the quantity x, in analogy to y, as

$$4x=[s-(\omega+1)^2+4]^{1/2}-2, \tag{63}$$

which vanishes at the $(\pi-\omega)$ threshold, and make the following choices:

$$N_{21}{}^{(1)}(s)=\frac{b^{(1)}}{a_2{}^{(1)}}\frac{s-s_o{}^{(1)}}{s+s_1{}^{(1)}}N_{22}(s),$$

$$N_{22}{}^{(1)}(s)=\frac{a_2{}^{(1)}}{(x+\frac{1}{2})(x+1)}\times\frac{[x(x+1)+((\omega+1)/4)^2]^{1/2}}{x+\frac{1}{2}}. \tag{64}$$

The perhaps surprising choice for $N_{22}{}^{(1)}(s)$ is easily understood in terms of the resulting $D_{2il}{}^{(1)}(s)$; in particular, we find with Eq. (64)

$$\begin{aligned}D_{21l}{}^{(1)}(s)&=-b^{(1)}\beta_l{}^{(1)}[s-(\omega+1)^2+4],\\ D_{22l}{}^{(1)}(s)&=1-a_2{}^{(1)}\alpha_l[s-(\omega+1)^2+4].\end{aligned} \tag{65}$$

The $T=0$ and 2 channels will be handled in somewhat the same way. We write, for $T=0$,

$$\begin{aligned}N_{11}{}^{(o)}(s)&=\frac{a_1{}^{(o)}}{b^{(o)}}\frac{s-s_o{}^{(o)}}{s+s_1{}^{(o)}}N_{13}{}^{(o)}(s),\\ N_{13}{}^{(o)}(s)&=\frac{b^{(o)}}{(y+1)(y+\frac{1}{2})},\\ N_{31}{}^{(o)}(s)&=\frac{b^{(o)}}{(w+1)(w+\frac{1}{2})}\frac{[w(w+1)+\rho^2/4]^{1/2}}{w+\frac{1}{2}},\\ N_{33}{}^{(o)}(s)&=\frac{a_3{}^{(o)}}{b^{(o)}}\frac{s-s_o{}^{(o)}}{s+s_1{}^{(o)}}N_{31}{}^{(o)}(s),\end{aligned} \tag{66}$$

where we have used the quantity

$$4w=[s-4\rho^2+4]^{1/2}-2, \tag{67}$$

which is the appropriate variable for the $(\rho\text{-}\rho)$ channel. These lead to the $D_{ijl}(s)$ as

$$\begin{aligned}D_{11l}{}^{(o)}(s)&=1-a_1{}^{(o)}\beta_l{}^{(o)}(s),\\ D_{13l}{}^{(o)}(s)&=-b^{(o)}\alpha_l(s),\\ D_{31l}{}^{(o)}(s)&=-b^{(o)}\left[\alpha_l(s-4\rho^2+4)+\frac{l(l-1)}{(l+\frac{1}{2})(l-\frac{1}{2})}\alpha_{l-2}(s-4\rho^2+4)\right],\\ D_{33l}{}^{(o)}(s)&=1-a_3{}^{(o)}\left[\beta_l{}^{(o)}(s-4\rho^2+4)+\frac{l(l-1)}{(l+\frac{1}{2})(l-\frac{1}{2})}\beta_{l-2}{}^{(o)}(s-4\rho^2+4)\right],\end{aligned} \tag{68}$$

where the l dependence in D_{31} and D_{33} is determined by Eq. (55). The $N_{ij}{}^{(2)}(s)$ and $D_{ij}{}^{(2)}(s)$ are identical to those in Eqs. (66) and (68) with (o) replaced by (2). It is to be understood that a highly simplifying assumption has been made with regard to the spin dependence in channel 3. The matrix elements N_{33l} and D_{33l} are themselves 9×9 matrices in the spin variables; we take these to be proportional to the unit matrix. Likewise the (13) and (31) elements have been taken to be proportional to the unit row and column, respectively, where the proportionality factors involve that weighting of the spin components which we have already indicated.

The $N_{ij}(s)$ and $D_{ij}(s)$ are now completely specified and are determined by the single function $\alpha_l(s)$, defined in Eq. (61). The integral defining $\alpha_l(s)$ may be performed and yields hypergeometric functions; in particular, we find

$$\begin{aligned}\alpha_l(s)=\frac{4}{\pi(l+\frac{3}{2})}\frac{1}{s}\{&(s^{1/2}+2)\,{}_2F_1(1,1;l+\tfrac{5}{2};(s^{1/2}-2)/4)\\ &-(s^{1/2}-2)\,{}_2F_1(1,1;l+\tfrac{5}{2};(2-s^{1/2})/4)\\ &-4\,{}_2F_1(1,1;l+\tfrac{5}{2};\tfrac{1}{2})\},\end{aligned} \tag{69}$$

which is defined by analytic continuation outside the interval $0<s<4$. We note that the $D_{ij}(s)$ have discontinuities for $s\geqslant4$ or $s\geqslant(\omega+1)^2$, $s\geqslant4\rho^2$, depending on the argument of the α_l since the hypergeometric function has a branch point at one.

III. CONSTRAINTS AND DETERMINATION OF THE PARAMETERS

The analysis in Sec. II specifies the π–π scattering amplitude completely except for the explicit numerical value of the parameters. The method of determination of these numerical values is the subject of this section.

The primary constraint on the amplitudes will be that of crossing. This constraint may be expressed by requiring that the amplitudes satisfy the following relations[1]:

$$\begin{aligned}M^{(o)}(s,t,u)&=\tfrac{1}{3}M^{(o)}(t,s,u)+M^{(1)}(t,s,u)+(5/3)M^{(2)}(t,s,u),\\ M^{(1)}(s,t,u)&=\tfrac{1}{3}M^{(o)}(t,s,u)+\tfrac{1}{2}M^{(1)}(t,s,u)-\tfrac{5}{6}M^{(2)}(t,s,u),\end{aligned} \tag{70}$$

and

$$M^{(2)}(s,t,u)=\tfrac{1}{3}M^{(o)}(t,s,u)-\tfrac{1}{2}M^{(1)}(t,s,u)+\tfrac{1}{6}M^{(2)}(t,s,u).$$

If we define $\Delta_i(s,t)$ as

$$\Delta_i(s,t)=M^{(i)}(s,t,u)-M^{(i)}(t,s,u), \tag{71}$$

then Eq. (70) implies that

$$\Delta_o(s,t)=-2\Delta_1(s,t)=-2\Delta_2(s,t), \tag{72}$$

which must be satisfied for all s and t.

In order to determine the values of the parameters we must be able to obtain numerical expressions for the $M^{(i)}(s,t)$ and $M^{(i)}(t,s)$. However, since we can only compute $M^{(i)}(s,t)$ from the partial-wave sum, we must restrict our attention to those values of s and t for which the sum converges. In particular, our formulation does not allow us to obtain numerical expressions for $M^{(i)}(s,t)$ if $t\geqslant4$ or $u\geqslant4$, and since we must use the same values of s and t, we are restricted to the range: $t<4$,

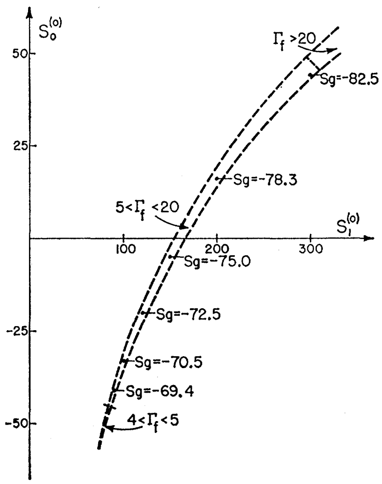


FIG. 8. $T=0$ parameters leading to the occurrence of an f_0 resonance with indicated width. Sensitivity of the position of the ghost is also shown.

$s<4$, $s+t>0$. Further, each partial wave becomes complex for $s<0$. Although the summation over l must yield a full amplitude which is analytic down to $s=-t$, the technique for carrying out this summation and continuation with $s<0$ is not treated in this paper. Consequently, we are limited to the region: $0<t<4$, $0<s<4$.

In principle, we would proceed as follows: We choose a set of points, say $s=\frac{1}{2}, 1, \frac{3}{2}, \cdots, \frac{7}{2}$ and similarly for t, compute $\Delta_i(s,t)$ for various choices of the parameters and finally obtain a set of parameters which satisfies the requirements in Eq. (72) most accurately. Within our framework we would have no *a priori* reason to believe that this set would be unique and, in fact, it would probably change as we change the mesh of points in the (s–t) plane. Such a program is simply not feasible at this point because we have so many parameters that a search, even by machine techniques, is impractical. Therefore, it will be necessary first for us to reduce the number of parameters in a more straightforward manner.

Let us confine our attention to the $T=0$ channel first. In this channel we have five parameters, $a_1^{(o)}$, $a_2^{(o)}$, $b^{(o)}$, $s_o^{(o)}$ and $s_1^{(o)}$, and we wish to reduce this to a smaller number of independent parameters. In order to do this we will require that this channel exhibit a Pomeranchuk trajectory,[10,11] i.e., we insist that the parameters be such that the determinant of $\mathbf{D}_l^{(o)}(s)$, $D_l^{(o)}(s)$, vanish for $l=1$, $s=0$. Such a requirement will lead to the vanishing of $D_o^{(o)}(s)$ for some $s<0$, i.e., for an S-wave ghost, and we will also require that the residue of this ghost state vanish. To be more explicit we have

$$D_l^{(o)}(0)=D_{11l}^{(o)}(0)D_{33l}^{(o)}(0)-D_{13l}^{(o)}(0)D_{31l}^{(o)}(0)=0 \quad \text{for} \quad l=1. \tag{73}$$

[10] See e.g., S. Pomeranchuk, Zh. Eksperim. i Teor. Fiz. 34, 725 (1958) [English transl.: Soviet Phys.—JETP 7, 499 (1958)].

[11] G. F. Chew and S. C. Frautschi, Phys. Rev. Letters 8, 41 (1962).

From Eq. (68), this yields

$$[1-a_1^{(o)}\beta_1^{(o)}(0)][1-a_3^{(o)}\beta_1^{(o)}(-4\rho^2+4)]-[b^{(o)}]^2\alpha_1(0)\alpha_1(-4\rho^2+4)=0. \tag{74}$$

The existence of an S-wave ghost implies

$$D_{11o}^{(o)}(s_g)D_{33o}^{(o)}(s_g)-D_{13o}^{(o)}(s_g)D_{31o}^{(o)}(s_g)=0,$$

or

$$[1-a_1^{(o)}\beta_o^{(o)}(s_g)][1-a_3^{(o)}\beta_o^{(o)}(s_g-4\rho^2+4)]-(b^{(o)})^2\alpha_o(s_g)\alpha_o(s_g-4\rho^2+4)=0, \tag{75}$$

where s_g is the position of the ghost and is negative. Since the vanishing of the S-wave determinant will produce a ghost in all of the reactions $\pi+\pi\leftrightarrow\pi+\pi$, $\pi+\pi\leftrightarrow\rho+\rho$ and $\rho+\rho\leftrightarrow\rho+\rho$, we must insist that all of the numerator functions vanish at $s=s_g$. As a matter of fact, it is sufficient to require that

$$\begin{aligned} N_{11o}^{(o)}D_{33o}^{(o)}-N_{13o}^{(o)}D_{31o}^{(o)}&=0, \quad s=s_g \\ N_{33o}^{(o)}D_{11o}^{(o)}-N_{31o}^{(o)}D_{13o}^{(o)}&=0, \quad s=s_g. \end{aligned} \tag{76}$$

To see that this implies the vanishing of the off-diagonal elements, consider F_{13}. Multiply the first equation in Eq. (76) by $D_{13o}^{(o)}(s_g)D_{11o}^{(o)}(s_g)$ to obtain, dropping the angular momentum and isospin labels,

$$N_{11}(s_g)D_{13}(s_g)D_{11}(s_g)D_{33}(s_g)-D_{11}(s_g)N_{13}(s_g)D_{13}(s_g)D_{31}(s_g)=0.$$

But by Eq. (75) this yields

$$D_{11}(s_g)N_{13}(s_g)-N_{11}(s_g)D_{13}(s_g)=0,$$

which is the numerator function of $F_{13}(s_g)$.

The constraints contained in Eqs. (75) and (76) may be reduced to

$$\begin{aligned} &a_1^{(o)}\left[\beta_o^{(o)}(s_g)-\frac{s_g-s_o^{(o)}}{s_g+s_1^{(o)}}\alpha_o(s_g)\right]=1, \\ &a_3^{(o)}\left[\beta_o^{(o)}(s_g-4\rho^2+4)-\frac{s_g-s_o^{(o)}}{s_g+s_1^{(o)}}\times\alpha_o(s_g-4\rho^2+4)\right]=1, \\ &(b^{(o)})^2=a_1^{(o)}a_3^{(o)}\left(\frac{s_g-s_o^{(o)}}{s_g+s_1^{(o)}}\right)^2, \end{aligned} \tag{77}$$

or expressing these in a more tractable form

$$\begin{aligned} a_1^{(o)}&=\frac{s_g+s_1^{(o)}}{s_o^{(o)}+s_1^{(o)}}\frac{1}{\alpha_o(-s_1^{(o)})}, \\ a_3^{(o)}&=\frac{s_g+s_1^{(o)}}{s_o^{(o)}+s_1^{(o)}}\frac{1}{\alpha_o(-s_1-4\rho^2+4)}, \\ (b^{(o)})^2&=\left[\frac{s_g-s_o^{(o)}}{s_o^{(o)}+s_1^{(o)}}\right]^2\frac{1}{\alpha_o(-s_1^{(o)})\alpha_o(-s_1^{(o)}-4\rho^2+4)}. \end{aligned} \tag{78}$$

The reduction of the parameters now proceeds as follows: We choose values of $s_o^{(o)}$ and $s_1^{(o)}$ which, by Eq. (77), determine $a_1^{(o)}$, $a_3^{(o)}$, and $(b^{(o)})^2$ as a function of s_g. The quantity s_g is then determined by substituting these functional expressions into Eq. (74). This equation is an algebraic equation, quadratic in s_g, and can be solved immediately. In particular, we have

$$As_g^2+2Bs_g+C=0, \tag{79}$$

where

$$A=\frac{1}{[s_o^{(o)}+s_1^{(o)}]^2}\frac{\beta_1^{(o)}(0)\beta_1^{(o)}(-4\rho^2+4)-\alpha_1(0)\alpha_1(-4\rho^2+4)}{\alpha_o(-s_1^{(o)})\alpha_o(-s_1^{(o)}-4\rho^2+4)},$$

$$B=\frac{1}{[s_1^{(o)}+s_1^{(o)}]^2}\frac{s_1^{(o)}\beta_1^{(o)}(0)\beta_1^{(o)}(-4\rho^2+4)+s_o^{(o)}\alpha_1(0)\alpha_1(-4\rho^2+4)}{\alpha_o(-s_1^{(o)})\alpha_o(-s_1^{(o)}-4\rho^2+4)}-\frac{1}{2[s_o^{(o)}+s_1^{(o)}]}\times\left[\frac{\beta_1^{(o)}(0)}{\alpha_o(-s_1^{(o)})}+\frac{\beta_1^{(o)}(-4\rho^2+1)}{\alpha_o(-s_1^{(o)}-4\rho^2+4)}\right],$$

$$C=1-\frac{s_1^{(o)}}{s_1^{(o)}+s_1^{(o)}}\left[\frac{\beta_1^{(o)}(0)}{\alpha_o(-s_1^{(o)})}+\frac{\beta_1^{(o)}(-4\rho^2+4)}{\alpha_o(-s_1^{(o)}-4\rho^2+1)}\right]+\frac{1}{[s_o^{(o)}+s_1^{(o)}]^2}\times\frac{s_1^{(o)2}\beta_1^{(o)}(0)\beta_1^{(o)}(-4\rho^2+4)-s_o^{(o)2}\alpha_1(0)\alpha_1(-4\rho^2+4)}{\alpha_o(-s_1^{(o)})\alpha_o(-s_1^{(o)}-4\rho^2+4)}.$$

The nature of the roots of Eq. (77) restricts somewhat the possible choices of $s_o^{(o)}$ and $s_1^{(o)}$; specifically we allow only those solutions which contain at least one real, negative root. As we will see this is not an important restriction of the possible range of the $s_o^{(o)}$ and $s_1^{(o)}$. In the case that there are two negative roots we choose the more negative one on the basis of continuity.

Finally, we remark that it is not possible to choose $(1/a_1^{(o)})N_{11}^{(o)}(s)=(1/a_3^{(o)})N_{13}^{(o)}(s)$ as we pointed out in Sec. II. This, of course, could be accomplished by choosing $s_o^{(o)}=-s_1^{(o)}$ [see Eq. (66)], but we see from Eq. (78) that this implies $a_1^{(o)}=a_3^{(o)}=(b^{(o)})^2=\infty$. Actually this means that the equations in Eq. (77) or (78) are reduced to identities for all values of the parameters and thus the parameters cannot be determined.

Thus the $T=0$ channel is characterized by two input parameters, $s_o^{(o)}$ and $s_1^{(o)}$, and these are to be determined by crossing.

Let us now turn to the $T=1$ channel. In this case there are no restrictions that can be imposed in analogy to the $T=0$ channel. However, because there are no restrictions to satisfy, we can reduce the number of parameters rather arbitrarily keeping simplicity in mind. In particular, we may choose $s_o^{(1)}=-s_1^{(1)}$ in contrast to the $T=0$ channel, and this possibility will be discussed in detail in Sec. IV. Also it is possible to take the point of view that *rescattering* of the $(\pi$–$\omega)$ state plays no crucial role in determining $M_{11}(s,t)$ although the existence itself of an inelastic state is significant, i.e., the $(\pi$–$\omega)$ state can contribute to $(\pi$–$\omega)$ elastic scattering only by means of the $(\pi$–$\pi)$ state through unitarity, but the $(\pi$–$\omega)$ state can contribute directly to the elastic $(\pi$–$\pi)$ amplitude since the inelastic amplitude is nonzero. We will, in fact take this point of view and set $a_2^{(1)}=0$ in everything that follows. The remaining parameters, as well as those for the $T=0$ channel, are then determined by the first crossing relation in Eq. (72).

It should be emphasized that there is no choice of the parameters which will satisfy crossing exactly. After all, we have certainly not been clever enough to guess the correct solution to the problem. The values of the parameters will simply be those that lead to amplitudes that satisfy crossing most accurately. These will then determine, within the model presented here, the behavior of the various partial-wave amplitudes in each isospin channel.

IV. NUMERICAL RESULTS

In this section we shall discuss the consequences of the parameter variation defined in the preceding section. For the moment, we will restrict our attention to simply one of the crossing relations, viz., $\Delta_o=-2\Delta_1$. It is to be admitted that parameters determined in this way may lead to parameters in the $T=2$ channel which yield undesirable consequences, e.g., ghosts or bound states in the $T=2$ channel. (As we shall see, this does not occur.) However, with the wealth of parameters available in this model, a simultaneous variation of all the parameters is not practical even with the use of machines. Further, there are many sets of parameters which can be used, each set satisfying the crossing relations to some extent. In any case, what we wish to show is that the main features of pion-pion scattering are a necessary consequence of the symmetries imposed,

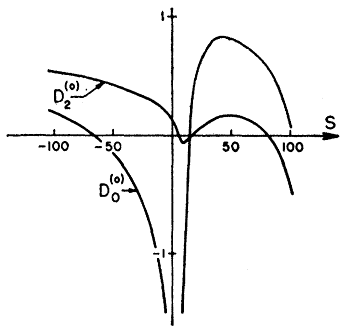


FIG. 9. $T=0$ determinants for $l=0$ and 2 for the case $s_1^{(o)}=74$, $s_o^{(o)}=-56.6$. Above threshold the real parts are plotted.

and that the model presented here is a reasonable representation of this fact.

A. $T=0$ Channel

As shown in Sec. III, the $T=0$ channel is determined by two parameters: $s_o^{(o)}$ and $s_1^{(o)}$. For any choice of these parameters we have a Pomeranchuk trajectory and an S-wave ghost whose residue vanishes. Before we impose crossing symmetry, let us examine the behavior of the relevant partial waves as we vary $s_o^{(o)}$ and $s_1^{(o)}$. We remark first that the position of the ghost, s_g, is not a sensitive function of $s_o^{(o)}$ and $s_1^{(o)}$, and in fact occurs in the neighborhood of the conjectured Chew and Frautschi[11] ghost. Second, it is possible to find values of the parameters for which $D_2^{(o)}(s)$, the D-wave determinant, has a vanishing real part in the neighborhood of the observed f_o resonance,[12] although it does not seem possible for our model to yield a width for the f_o which is as small as the observed value. If we require that $\mathrm{Re}\, D_2^{(o)}(s)=0$ for $76\leqslant s\leqslant 86$, then we obtain the values displayed in Fig. 8. This figure also demonstrates the insensitivity of s_g to $s_o^{(o)}$ and $s_1^{(o)}$. The width of the resonance Γ_f is determined by $\mathrm{Im}\, D_2^{(o)}$ and ranges over the values indicated in the figure. We determine Γ_f as

$$\Gamma_f=\mathrm{Im}\, D_2^{(o)}(s_f)/s_f\lambda_f\,, \tag{80}$$

where s_f is the position of the resonance and λ_f is the slope of $\mathrm{Re}\, D_2^{(o)}(s)$ at $s=s_f$. Equation (80) gives

$$\begin{aligned}\Gamma_f=-\frac{16}{\lambda_f s_f^{3/2}}\left(\frac{s_f^{1/2}-2}{s_f^{1/2}+2}\right)^{5/2}\Bigg\{ & a_1^{(o)}\frac{s_f-s_o^{(o)}}{s_f+s_1^{(o)}}\\ &\times[1-a_2^{(o)}[\beta_2^{(o)}(s_f-4\rho^2+4)\\ &+(8/15)\beta_o^{(o)}(s_f-4\rho^2+4)]\\ &+b^{(o)2}[\alpha_2(s_f-4\rho^2+4)\\ &+(8/15)\alpha_2(s_f-4\rho^2+4)]]\Bigg\}\,. \end{aligned}\tag{81}$$

We emphasize, however, that in *every* case considered in the parameter variation the D-wave resonance near the f_o was accompanied by a D-wave resonance at a lower energy. This is illustrated in Fig. 9 for the case $s_o^{(o)}=-56.6$ and $s_1^{(o)}=74$. This leads to an S-wave ghost at $s_g=-66$, and D-wave resonances at $s=6$ and $s_f=81$. This behavior is not supported by the experimental evidence. In the parameter search to obtain the best fit to crossing—to be discussed shortly—we incorporate the fact that $\mathrm{Re}\, D_2^{(o)}$ shall not vanish for $s\leqslant 60$. It happens that this restriction improves the fit to crossing.

B. $T=1$ Channel

In this channel there are four parameters: $a_1^{(1)}$, $b^{(1)}$, $s_o^{(1)}$ and $s_1^{(1)}$; there is no ghost constraint here analogous to the ghost constraint in $T=0$, so that all of these are independent. Because of this independence, it is possible to find an enormous number of sets of parameters which reproduce the P-wave resonance,[13] i.e., the ρ with the correct width. If we insist that the crossing relations determine a set which yields the correct properties of the ρ, then we must decide how to begin the parameter search. This can be established by determining those sets which do yield a ρ with the correct properties and then to use these sets as a starting point. This will, in fact, be our procedure. To determine the initial set of parameters then, we insist that $\mathrm{Re}\, D_1^{(1)}(s)=0$ at $s=s_\rho$, the position of the ρ. This yields a linear relation between $a_1^{(1)}$ and $b^{(1)2}$

$$b^{(1)2}=\frac{1-a_1^{(1)}[\mathrm{Re}\alpha_1(s_\rho)]}{[\mathrm{Re}\beta_1^{(1)}(s_\rho)]\beta_1^{(1)}[s_\rho-(\omega+1)^2+4]}\,. \tag{82}$$

The width of the resonance so obtained is

$$\Gamma_\rho=\mathrm{Im}\, D_1^{(1)}(s_\rho)/\lambda_\rho s_\rho^{1/2}\,, \tag{83}$$

in analogy to Eq. (80) for the f^o, or

$$\begin{aligned}\Gamma_\rho=&-\frac{16}{\lambda_\rho s_\rho^{3/2}}\left(\frac{s_\rho^{1/2}-2}{s_\rho^{1/2}+2}\right)^{3/2}\\ &\times\left[a_1^{(1)}+b^{(1)2}\frac{s_\rho-s_o^{(1)}}{s_\rho+s_1^{(1)}}\beta_1^{(1)}(s_\rho-(\omega+1)^2+4)\right].\end{aligned}\tag{84}$$

If we insist that $0.6\leqslant\Gamma_\rho\leqslant 0.8$, we obtain sets of parameters such as those displayed in Fig. 10. A given curve in the figure, belonging to a given value of $s_1^{(1)}$, represents a very narrow region of values of $s_o^{(1)}$ and $a_1^{(1)}$ for which there is a ρ with the correct mass and width. Once $a_1^{(1)}$ is chosen and $s_o^{(1)}$ and $s_1^{(1)}$ determined so that we have the correct mass and width, $b^{(1)2}$ is obtained by Eq. (82). Figure 10 represents only a very small sample of the results of the search.

The case of $b^{(1)}=0$, i.e., no inelastic effects in the

[12] W. Selove, V. Hagopian, H. Brody, A. Baker, and E. Leboy, Phys. Rev. Letters **9**, 272 (1962); see also I. J. R. Aitchison, Phys. Rev. **131**, 1797 (1963) and references given there, as well as V. Hagopian and W. Selove, Phys. Rev. Letters **10**, 533 (1963).

[13] A. Erwin, R. March, W. D. Walker, and E. West, Phys. Rev. Letters **6**, 628 (1961).

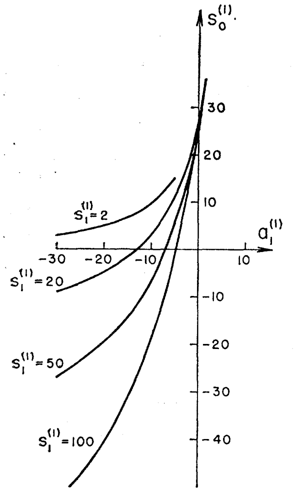


FIG. 10. $T=1$ parameters leading to the occurrence of a ρ resonance with width between 0.6 and 0.8.

elastic amplitude, is not among these results. If we set $b^{(1)}=0$ and demand that $\operatorname{Re} D_1^{(1)}(s_\rho)=0$, we find that the slope of this function is positive and cannot represent a true resonance. In fact, there is a ghost near $s=-90$ and the zero we have demanded is the *second* zero of $\operatorname{Re} D_1^{(1)}(s)$. This result is independent of $s_o^{(1)}$ and $s_1^{(1)}$. In all the cases considered in the search, the zero at $s=s_\rho$ was required to be the first one.

As we showed in Sec. III, it was necessary to introduce the parameters $s_o^{(o)}$ and $s_1^{(o)}$ if we were to satisfy the Pomeranchuk trajectory condition. However, there is no such constraint in the $T=1$ channel, and there would seem to be no need for the introduction of the parameters $s_o^{(1)}$ and $s_1^{(1)}$. In particular, we could have simply written

$$N_{12}^{(1)}/b^{(1)}=N_{11}^{(1)}/a_1^{(1)}. \tag{85}$$

If, in fact, we set $s_o^{(1)}=-s_1^{(1)}$, we can eliminate these parameters from all formulas and obtain a ρ with the correct position and width. The parameters which accomplish this are $a_1^{(1)}=-349$ and $b^{(1)2}=2919$. However, this case does not provide nearly as good a fit to crossing symmetry as do the sets for which $s_o^{(1)}\neq -s_1^{(1)}$. One could argue here that this makes little difference because it is clear that one can improve the crossing-symmetry fit by introducing more parameters. This of course is true, but it is important to remember that these parameters determine *all* of the partial waves, not just the P wave.

C. Cross Symmetry

At this point we now impose the crossing relation

$$\Delta_o(s,t)=-2\Delta_1(s,t).$$

As we have stated, the parameters are to be determined as those which best satisfy the crossing relation, but we must first establish a quantitative criterion as to what constitutes the best fit. Such a criterion is not unique, and we have considered many, but the following seems convenient: We consider the following mesh of twenty-one points in the s–t plane: $s=\frac{1}{2}, 1, \frac{3}{2}, \cdots, \frac{7}{2}$; $t=\frac{1}{2}, 1, \frac{3}{2}, \cdots, \frac{7}{2}$ with $t<s$. For each of these points we evaluate $\Delta_o(s,t)$ and $-2\Delta_1(s,t)$, and represent this calculation by a point in the $(\Delta_o, -2\Delta_1)$ plane. If the crossing relation were satisfied exactly, then all these points would lie on the 45° line in Fig. 11. In fact, these points will be scattered about this line as indicated in Fig. 11. We now define the angle $\epsilon(s,t)$ for each point as shown and whose sine is given by

$$\sin\epsilon(s,t)=\frac{|\Delta_o(s,t)+2\Delta_1(s,t)|}{\sqrt{2}[\Delta_o^2(s,t)+4\Delta_1^2(s,t)]^{1/2}}. \tag{86}$$

The average sine for the $T=0$ and $T=1$ case is then

$$\psi_{o1}=\frac{1}{N}\sum_{\{s,t\}}\sin\epsilon(s,t), \tag{87}$$

where the sum is taken over the twenty-one points in the mesh. The best fit to the crossing relation is defined to be that which minimizes ψ_{o1}.

The first variation of parameters was made with only those $T=1$ parameters for which there is a ρ with the correct mass and width. The minimum obtained in this way is

$$\psi_{o1}=0.358,$$

and the parameters so determined are:

$$\begin{array}{ll} s_o^{(o)}=\ 24 & s_o^{(1)}=\ 12 \\ s_1^{(o)}=\ 10 & s_1^{(1)}=\ 10 \\ a_1^{(o)}=\ -4.3 & a_1^{(1)}=-5.8 \\ a_2^{(o)}=-15.0 & b^{(1)2}=144 \\ b^{(o)2}=\ 192. & \end{array}$$

The points in Fig. 11 in fact represent this situation. The position of the ghost in this case is

$$s_g=-57.1.$$

The determinants of the lower partial waves are plotted in Fig. 12. This shows the S-wave ghost, $D_o^{(o)}(-57.1)=0$, the P-wave resonance (by construction), $D_1^{(1)}(29)=0$, and the fact that there is no D-wave resonance.

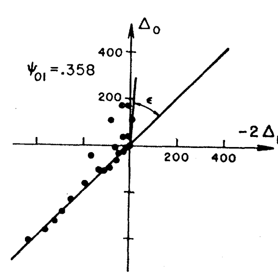


FIG. 11. Map of Δ_o versus $-2\Delta_1$. The angle (s,t) is defined in the figure.

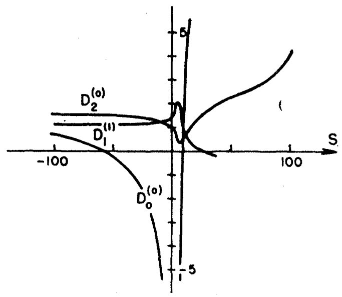


FIG. 12. Determinants for $T=0$, $l=0$ and 2, for $s_1^{(o)}=10$ and $s_o^{(o)}=24$, and $T=1$, $l=1$, for the optimum case in which the exact ρ occurs. Above threshold, the real parts are plotted.

We now fix the $T=0$ parameters and vary $a_1^{(1)}$ and $b^{(1)2}$ in a second variation. We let $a_1^{(1)}$ and $b^{(1)2}$ vary in small increments until a minimum of ψ_{o1} is found. In principle, we then fix these new parameters and vary the $T=0$ parameters, etc., until a stable set of parameters is found, *if any*. If we perform the second variation mentioned above we find that the new minimum is very nearby, and it is not necessary to continue the parameter variation. It is to be emphasized that *this result is by no means built into the searching technique—the minimum could equally well have been unstable and "run away."*

At the minimum we obtain $\psi_{o1}=0.356$ with parameters

$$a_1^{(1)}=-6.0\,,$$
$$b^{(1)2}=138\,.$$

The map of Δ_o versus $-2\Delta_1$ for this case is not appreciably different from that shown in Fig. 11. In Fig. 13, the determinants $D_1^{(o)}$ and $D_1^{(1)}$ are plotted; the first of these shows the (by construction) Pomeranchuk trajectory at $s=0$ and returns through zero near $s=10$; the second shows the P-wave resonance to be slightly removed from the observed position. The parameters of the resonance are:

$$s_\rho=33.7\ =(5.8)^2=(812\text{ MeV})^2$$
$$\Gamma_\rho=\ 2.52=353\text{ MeV}.$$

These results are remarkably sensitive to the variation of parameters. In particular, a 3% change in $a_1^{(1)}$ and a 4% change in $b^{(1)2}$ have shifted the position of the resonance by 17% and have increased the width by more than 350%.

D. $T=2$ Channel

Only the $T=2$ channel remains undetermined. To determine the remaining parameters $s_o^{(2)}$, $s_1^{(2)}$, $a_1^{(2)}$,

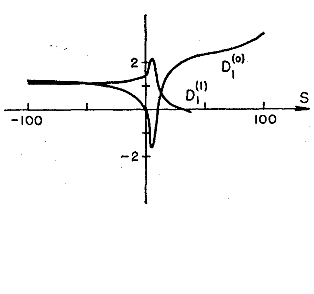


FIG. 13. $l=1$ determinants for $T=0$ and 1 in the case of minimum ψ_{o1}.

$a_2^{(2)}$, and $b^{(2)}$, we must optimize the fit to the other two crossing relations

$$\Delta_o(s,t)=-2\Delta_2(s,t)\,, \tag{88}$$
$$\Delta_1(s,t)=\Delta_2(s,t)\,. \tag{89}$$

In analogy to the preceding section we introduce the following ψ's:

$$\psi_{o2}=\frac{1}{N}\sum_{\{s,t\}}\frac{|\Delta_o+2\Delta_2|}{\sqrt{2}[\Delta_o^2+4\Delta_2^2]^{1/2}}\,, \tag{90}$$

$$\psi_{12}=\frac{1}{N}\sum_{\{s,t\}}\frac{|\Delta_1-\Delta_2|}{\sqrt{2}[\Delta_1^2+\Delta_2^2]^{1/2}}\,. \tag{91}$$

The choice of parameters $s_1^{(2)}=s_1^{(o)}=10$ and $s_o^{(2)}=s_o^{(o)}=24$ suggests itself and proves to be superior to the case $s_o^{(2)}=-s_1^{(2)}$. However, we again have the problem of where to begin our parameter variation. In order to obtain a starting point for the variation we will use the following argument. If we assume that $T=1$

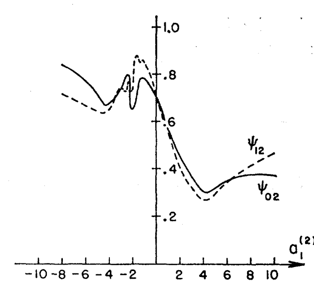


FIG. 14. Minimization of ψ_{o2} and ψ_{12} as a function $a_1^{(2)}$.

scattering dominates in the crossed channel then we have, from the crossing relations

$$M^{(o)}(s,t)\approx M^{(1)}(t,s) \tag{92}$$

and

$$M^{(2)}(s,t)\approx-\tfrac{1}{2}M^{(1)}(t,s)\approx-\tfrac{1}{2}M^{(o)}(s,t)\,. \tag{93}$$

This suggestion that we begin the variation with:

$$a_1^{(2)}=-\tfrac{1}{2}a_1^{(o)}\,,$$
$$a_2^{(2)}=-\tfrac{1}{2}a_2^{(o)}\,,$$
$$b^{(2)2}=\tfrac{1}{4}b^{(o)2}\,.$$

If in fact we do this we find that ψ_{12} reaches a minimum as a function of $b^{(2)2}$ when $b^{(2)2}=0$. We argue from this that the $T=2$ channel prefers to be a one-channel problem, and thus as far as the elastic π–π amplitude is concerned, only one parameter, $a_1^{(2)}$, remains to be varied. We have no reason to suppose that ψ_{o2} and ψ_{12} will have their minima for the same value of $a_1^{(2)}$; however, their minima are very close together as one can see from Fig. 14. This is a very satisfactory result, and it probably implies that we have obtained an essentially optimum fit to all of the crossing relations. A good compromise is reached if we take $a_1^{(2)}=4.2$.

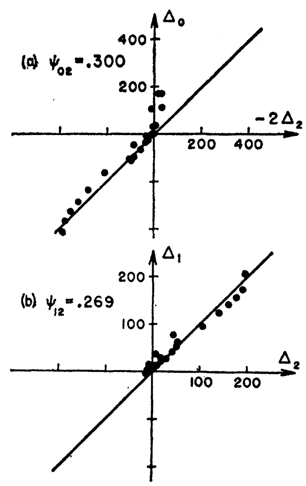


FIG. 15. (a) Map of Δ_o versus $-2\Delta_2$. (b) Map of Δ_1 versus Δ_2.

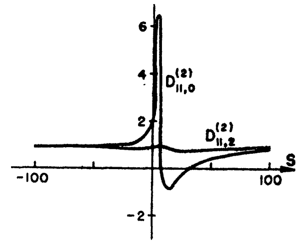


FIG. 16. $T=2$ denominator functions for $l=0$ and 2. Above threshold, real parts are plotted.

The maps of Δ_o versus $-2\Delta_2$ and Δ_1 versus Δ_2 are given in Fig. 15.

In the $T=2$ channel, for this determination of the parameters, the relevant partial-wave denominator function for elastic π–π scattering is not $D_l^{(2)}$ but simply $D_{11l}^{(2)}$ since only $a_1^{(2)}$ is involved. This function is plotted in Fig. 16 for $l=0$ and $l=2$. The $l=0$ case seems to show a resonance near $s=8$; however, the imaginary part of $D_{11o}^{(2)}$ is negative there so that the scattering is repulsive for low energies. This repulsive behavior of the low-energy $T=2$ scattering is more obvious from a plot of the phase shifts. This is given in the next section.

E. Phase Shifts

Here we will present the results of the previous analysis in terms of phase shifts for the lower partial waves in each of the three isospin channels. The phase of the determinant $D_l^{(T)}$ is $-\delta_l^{(T)}$, where $\delta_l^{(T)}$ is the eigenphase shift. Therefore we have

$$\tan\delta_l^{(T)} = -(\operatorname{Im} D_l^{(T)}/\operatorname{Re} D_l^{(T)}). \tag{94}$$

For $T=0$ and 2 we use:

$$\operatorname{Re} D_l^{(T)} = [1-a_1^{(T)}\beta_l^{(T)}(s)]\left\{1-a_2^{(T)}\left[\beta_l^{(T)}(s-4\rho^2+4)+\frac{l(l-1)}{(l+\frac{1}{2})(l-\frac{1}{2})}\beta_{l-2}^{(T)}(s-4\rho^2+4)\right]\right\}$$
$$-b^{(T)2}\alpha_l(s)\left[\alpha_l(s-4\rho^2+4)+\frac{l(l-1)}{(l+\frac{1}{2})(l-\frac{1}{2})}\alpha_{l-2}(s-4\rho^2+4)\right], \tag{95}$$

$$\operatorname{Im} D_l^{(T)} = -\frac{16}{s}\left(\frac{s^{1/2}-2}{s^{1/2}+2}\right)^{l+\frac{1}{2}}\left\{a_1^{(T)}\frac{s-s_o^{(T)}}{s+s_1^{(T)}}\left[1-a_2^{(T)}\left[\beta_l^{(T)}(s-4\rho^2+4)+\frac{l(l-1)}{(l+\frac{1}{2})(l-\frac{1}{2})}\beta_{l-2}^{(T)}(s-4\rho^2+4)\right]\right]\right.$$
$$\left.+b^{(T)2}\left[\alpha_l(s-4\rho^2+4)+\frac{l(l-1)}{(l+\frac{1}{2})(l-\frac{1}{2})}\alpha_{l-2}(s-4\rho^2+4)\right]\right\}; \tag{96}$$

for $T=1$ we use:

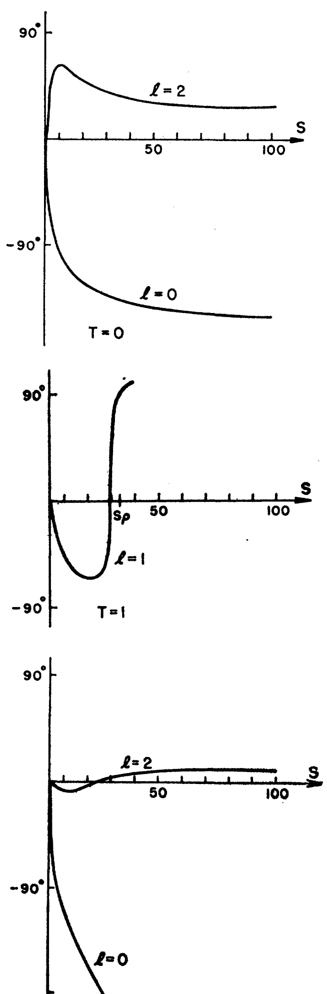
$$\operatorname{Re} D_l^{(1)} = 1-a_1^{(1)}\alpha_l(s)-b^{(1)2}\beta_l^{(1)}(s)\left[\beta_l^{(1)}(s-(\omega+1)^2+4)+\frac{l(l-1)}{(l+\frac{1}{2})(l-\frac{1}{2})}\beta_{l-2}^{(1)}(s-(\omega+1)^2+4)\right], \tag{97}$$

$$\operatorname{Im} D_l^{(1)} = -\frac{16}{s}\left(\frac{s^{1/2}-2}{s^{1/2}+2}\right)^{l+\frac{1}{2}}\left\{a_1^{(1)}+b^{(1)2}\frac{s-s_o^{(1)}}{s+s_1^{(1)}}\left[\beta_l^{(1)}(s-(\omega+1)^2+4)+\frac{l(l-1)}{(l+\frac{1}{2})(l-\frac{1}{2})}\beta_{l-2}^{(1)}(s-(\omega+1)^2+4)\right]\right\}. \tag{98}$$

For the parameters determined above we have computed $\delta_o^{(o)}$, $\delta_2^{(o)}$, $\delta_1^{(1)}$, $\delta_o^{(2)}$, and $\delta_2^{(2)}$; these are plotted as functions of s in Fig. 17. As expected, the $T=1$, $l=1$ phase shift increases through $\pi/2$ at $s=33.7$. The remarkable feature about the phase shifts to be noted in Fig. 17 is that in every single case, except $T=0$, $l=2$, the scattering is repulsive at threshold. That this is true of the P wave is an unusual feature of the model presented here, but crossing symmetry apparently requires channel one to be repulsive. The attraction necessary to produce a resonance is provided by the inelastic contributions.

In addition to the P-wave resonance discussed above, we note from Figs. 17(a) and (c) that peaks will occur at low energy in the S-wave cross sections for both the $T=0$ and $T=2$ channels. In the $T=0$ case, a peak will occur at $s=9\mu^2$ (420 MeV) because the phase shift passes rapidly through $-\pi/2$. On the basis of the

Levinson theorem one might argue that the phase shift should be π at threshold because of the existence of a "bound state," i.e., the S-wave ghost. However, since the residue of the ghost is zero, it will not contribute to the contour integral in the usual derivation, and thus the phase shift is taken to be zero at threshold. Of course, the correct value of the phase shift at threshold should be determined by defining $\delta_o^{(o)}(\infty)=0$ and then tracing the variation of $\delta_o^{(o)}(s)$ down to threshold. This will fix unambiguously the phase shift at either 0 or π. In any case, it may very well be that this peak at 420 MeV should be identified with the ABC[14] anomaly at 310 MeV. However, the peak obtained here certainly cannot be considered a resonance in any sense of the word since the peak is so broad and asymmetrical as seen in Fig. 18(a), where the $\sin^2\delta_o^{(o)}$ is plotted. On the other hand, it is not impossible that



(a)

(b)

(c)

FIG. 17. Phase shifts as functions of s.

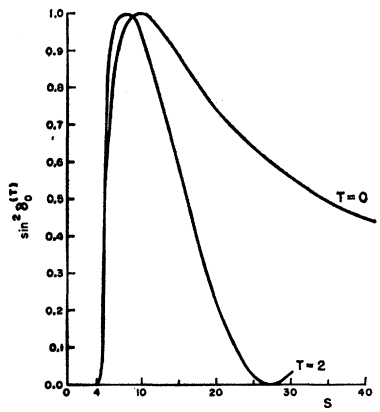


FIG. 18. $\sin^2\delta_o^{(T)}$ versus s for $T=0$, 2.

such behavior is consistent with the data, and that this type of situation is what is being observed.

In close similarity to the $T=0$ channel, the $T=2$ exhibits virtually the same behavior, the peak occurring at $s=8\mu^2$ (390 MeV) as seen in Fig. 18(b). To the best of our knowledge such a situation has not been observed experimentally, but the crossing relations within our model appear to demand this behavior.

This repulsive nature of the $T=0$ S wave is in considerable contrast to previous studies.[14,15] In terms of the Lagrangian parameter λ of Ref. 1, this means that $\lambda>0$, in opposition to the results of Desai[15] who found $\lambda\sim-0.2$. Further, the scattering length is negative here whereas others obtain a positive value. Of course, positive scattering lengths are to be expected in these other analyses since it was assumed that the $T=0$ S-wave scattering must be attractive to explain the ABC data. As we have seen, it may not be necessary to understand these experimental results in terms of an attractive S-wave scattering.

One might ask whether the rapidly decreasing S-wave phase shifts in fact violate Wigner's[15] remarks concerning the rate at which phase shifts can decrease. Wigner's arguments are based on causality and a finite range potential of range R, and he has found that

$$d\delta/dk>-R.$$

(The correct quantum mechanical relation differs but little from this.) All we wish to show here is that the minimum ranges necessary for the results given in Figs. 17(a) and (c) are not too large. In particular, if we calculate $R^{(T)}$ when $\delta_o^{(T)}$ passes through $-\pi/2$, we find

$$R^{(o)}=0.79(1/\mu),$$
$$R^{(2)}=1.25(1/\mu).$$

These are reasonable ranges.

Finally, let us obtain the scattering lengths them-

[14] A. Abashian, N. E. Booth, and K. M. Crowe, Phys. Rev. Letters **5**, 238 (1960); **7**, 35 (1961).

[15] T. N. Truong, Phys. Rev. Letters **6**, 308 (1961); B. Desai, *ibid.* **6**, 497 (1961).

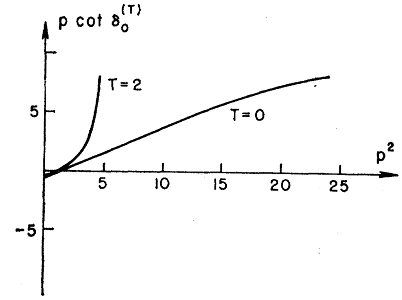


FIG. 19. Effective-range plots for the S-wave phase shifts.

selves. For this purpose we would write

$$P^{2l+1}\cot\delta_l^{(T)}=-\mathrm{Re}D_l^{(T)}[P^{2l+1}/\mathrm{Im}\,D_l^{(T)}],\quad(99)$$

and for the term in brackets simply make the replacement

$$\frac{16}{s}\left(\frac{s^{1/2}-2}{s^{1/2}+2}\right)^{l+\frac{1}{2}}\rightarrow\frac{4^{l+\frac{1}{2}}}{s(s^{1/2}+2)^{2l+1}}.\quad(100)$$

We have plotted $p\cot\delta_o^{(T)}$ against P^2 in Fig. 19, and if we use the relation

$$p\cot\delta_o^{(T)}=+(1/a_o^{(T)}),\quad(101)$$

we find

$$a_o^{(o)}=-1.72,\quad a_o^{(2)}=-1.85\quad(102)$$

in units of $(1/\mu)$. As we have indicated above, the $T=0$ result differs considerably from other studies. The result in the $T=2$ channel is also inconsistent with that obtained by Kirz *et al.*,[16] who find $|a_o^{(2)}|\lesssim 0.15$. From Eq. (102) one finds that

$$a_o^{(2)}-a_o^{(o)}=-0.13.\quad(103)$$

This is in agreement with the experimental results of Botusov *et al.*,[17] which yield, using an analysis of Anselm and Gribov,[18]

$$a_o^{(2)}-a_o^{(o)}=-0.35\pm0.30.$$

On the other hand, Khuri and Treiman[19] have found

$$a_o^{(2)}-a_o^{(o)}=+0.7.$$

V. DISCUSSION

We have presented in this paper a model of π–π scattering which we believe includes all of the essential properties of the true scattering problem. In particular, these are unitarity-including inelastic states in the s channel with branch points located at the correct positions in the t and u channels, the correct threshold behavior in the s channel, and crossing symmetry. Basically, we have found that these constraints, when a specific behavior in the $T=0$ channel is assumed, i.e., the Pomeranchuk trajectory, *force* a resonance in the $T=1$ channel at approximately the position of the ρ meson with a rather large width. It is important to emphasize that we have not assumed the existence of the ρ meson in any essential way, but that it is a *necessary consequence* of the constraints imposed. By "essential" we mean it is the existence of the inelastic states themselves that is important and not the fact that one of the inelastic states is the (π–ω) state; the fact that we knew of such a resonance experimentally and used this information to begin our parameter search is not significant. If we had not known that the ρ meson existed, we would have predicted such a resonance (practically, we admit that the parameter search may have been somewhat difficult in this case).

If we consider the success of this model in reproducing the ρ meson, it does not seem unreasonable to have some confidence in the results for other partial waves, especially the $T=0$ channel since our model may very well enable one to understand the ABC anomaly. Thus let us consider the $T=0$, $l=2$ case. As we have remarked, our model will not reproduce the f_o. However, considering the experimental data with respect to the f_o, it may be worth noting that the determinant function, $D_2^{(o)}(s)$, dipped near zero in the general neighborhood of the f_o in many cases although it never did pass through zero without producing a lower energy D-wave resonance. In terms of phase shifts, this means that the phase shift approached $\pi/2$, but then receded. In any case, it is not clear that one can believe the results of the model presented here at such energies.

Because of the many aspects of the model it is certainly reasonable to ask whether any particular aspect is more important than the others. This is a difficult question to answer, but a few remarks can be made. One can construct an amplitude in which all aspects of the full model are retained except for inelastic unitarity. This would certainly appear to be a reasonable approximation and in fact is basically the form of nearly all previous calculations. We have made such calculations when $s_o^{(1)}=-s_1^{(1)}$ so that there is only one parameter in the $T=1$ channel, and we find that the P-wave resonance cannot have a mass greater than $s=6$. This result does not depend on crossing symmetry but simply on the form of the amplitude given in Eqs. (9)–(13), and (57). A resonance can be produced at the correct position by using $s_o^{(1)}$ and $s_1^{(1)}$ but this is not very satisfactory. It would thus appear that elastic unitarity is not adequate to reproduce the ρ meson with our choice of a trial function. By including inelastic unitarity, and ignoring crossing, we can produce the ρ. This is in contrast to the results of Balázs[3] who found that inelastic effects were not particularly significant.

[16] J. Kirz, J. Schwartz, and R. D. Tripp, Phys. Rev. **126**, 763 (1962).
[17] Y. A. Botusov, S. A. Bunyatov, V. M. Sidorov, and V. A. Yarba, Proc. Ann. Intern. Conf. High Energy Phys. Rochester **10**, 79 (1960). In this paper see also the table of scattering lengths.
[18] A. A. Ansel'm and V. N. Gribov, Zh. Eksperim. i Teor. Fiz. **37**, 501 (1959) [English transl.: Soviet Phys.—JETP **10**, 354 (1960)].
[19] N. Khuri and S. B. Treiman, Phys. Rev. **119**, 1115 (1960).

However, he assumed that the maximum effect would be produced by a totally black disc, and it is not clear that this is true. If we add only inelastic states to the model, however, we find that we can produce a P-wave resonance virtually anywhere. The *unique* value for the position and width of the ρ meson is obtained when one imposes crossing symmetry as well. Thus it would appear that all aspects are equally important; elastic unitarity and the correct crossed cuts yield a dasically repulsive interaction, the attraction necessary to produce a resonance is provided by the inelastic states (a phenomenon observed in other calculations[7]), and the actual value of the resonance is determined by the crossing relations.

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Asymptotic Behavior of Partial-Wave Amplitudes

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For infinite energies, we determine the asymptotic behavior of partial-wave amplitudes when the full scattering amplitude satisfies Mandelstam representation and has itself a Regge asymptotic behavior. Particular attention is paid to the behavior of the partial-wave-amplitude discontinuities on their cuts. They are shown to behave as $|t|^{\alpha(0)-1}$, where t is the energy squared and $\alpha(0)$ is the leading Regge-pole position at zero energy. This result removes an old-standing difficulty in the Chew-Mandelstam calculation of amplitudes and provides a precise justification of the nearest singularity technique. As an application, we show that no subtraction is necessary in partial-wave-amplitude dispersion relations at physical values of the angular momentum, even for the case of S waves.

I. INTRODUCTION

In their original program, Chew and Mandelstam stressed that a particle or a resonance in a crossed channel contributes to the forces acting between two particles.[1] More precisely, the partial-wave amplitudes for pion-pion scattering have both a left- and a right-hand cut as functions of the energy, and the resonances in the crossed channels determine the discontinuity across the left-hand cut or, equivalently, the forces. Unfortunately, it appeared that the discontinuity obtained from that mechanism increased at a rate in conflict with unitarity when the energy became infinite and negative, as soon as the spin of the resonance or of the bound state in the crossed channel was larger than or equal to one. Such is the case for the ρ meson (and now also for the f^0 meson). The problem of determining the exact high-energy behavior of amplitudes became a necessary preliminary to the dispersion theory of elementary particles.

It was indeed felt that a simple solution of the problem had to exist since, in several cases, the simple trick of introducing a cutoff for the left-hand cut discontinuity leads to sensible results. This idea has been expressed as the nearest singularity hypothesis, by which one meant that a physical process was mostly determined by the effects of the singularities nearest to the physical region and was not affected by any misbehavior of the amplitudes at infinity.[2]

[1] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).
[2] See, for instance, G. F. Chew, *S-Matrix Theory of Strong Interactions* (W. A. Benjamin, Inc., New York, 1961).

The clue to a solution of the problem was provided by the observation, due to Regge,[3] that the asymptotic behavior of the nonrelativistic-scattering amplitudes, as functions of the angle, are determined by the singularities of the partial-wave amplitudes as functions of a continuous angular momentum.[3] Actually, these singularities are only poles. Chew and Frautschi[4] and Mandelstam[5] pointed out that the high-energy difficulties of the S-matrix theory of strong interactions could be eliminated if one takes as an ansätz that the asymptotic behavior of the total amplitude in relativistic theory is analogous to the one found in nonrelativistic theory.

Although it was clear that the asymptotic difficulties were removed by that hypothesis, one had yet to exhibit a practical way of resuming the Chew-Mandelstam program, now enlarged to be a program for self-consistently computing the leading Regge-pole trajectories. Chew and Jones are currently investigating such an approach in which they work both with the full amplitude and with the partial-wave amplitudes.[6,7] However, it is not clear whether only using the partial-wave amplitudes, which has the advantage of leading to one-dimensional well-known equations, could lead

[3] T. Regge, Nuovo Cimento **18**, 947 (1960).
[4] G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **7**, 394 (1961).
[5] S. Mandelstam (unpublished).
[6] G. F. Chew and C. E. Jones, Lawrence Radiation Laboratory Report UCRL-10992, August 1963 (unpublished).
[7] G. F. Chew, Conferences at the Department of Applied Mathematics and Theoretical Physics, University of Cambridge, England, 1963 (unpublished); see also Ref. 9.