first order in H' is

$$-i\tilde{f}(m_{N'}+\omega_2-m_N-\omega_1)(N\theta_{\mathbf{k}_2}|U_+|N\theta_{\mathbf{k}_1}). \quad \text{(E21)}$$

Evaluating (E21) to second order in g^2, we obtain according to Eqs. (A16) and (A17) given in Appendix A,

$$-i\tilde{f}(m_{N'}+\omega_2-m_N-\omega_1)\Bigg(\delta_{\mathbf{k}_2,\mathbf{k}_1}+\frac{g^2}{(4\omega_1\omega_2)^{1/2}\Omega} \times[(\omega_2-\omega_1+2i\alpha)(m_N+\omega_1-m_V+i\alpha)]^{-1}\Bigg). \quad \text{(E22)}$$

The first and second terms in (E22) are, respectively, the amplitudes for diagrams (ii) and (iii), Fig. 3. To order g^2, the contribution to the probability of finding the θ particle in the energy interval $m_\theta \leqq \omega \leqq \epsilon$ due to the interference between these amplitudes is given by

$$-\frac{g^2}{4\pi^2}|\tilde{f}(m_{N'}-m_N)|^2\int_{m_\theta}^{\epsilon}\frac{d\omega k}{(m_N+\omega-m_V)^2}. \quad \text{(E23)}$$

In Eq. (E23), we have summed over initial states of the θ particle in the same energy interval. Now if we add Eqs. (E20) and (E23) we see that the combined transition probability is finite in the limit $\mu\rightarrow 0$.

Remark

Note that Eq. (E22) differs from the usual Feynman amplitude in the factor 2 multiplying α. This factor can be neglected in the nondiagonal elements of the U matrix, but is essential here, since we are evaluating the interference term with the disconnected process, diagram (ii) Fig. 3, at $\mathbf{k}_1=\mathbf{k}_2$.

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Crossed Graphs in the Feinberg-Pais Theory of Weak Interactions*

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A possible damping mechanism is suggested to prevent the occurrence of essential singularities, such as that found on the light cone by Bardakci, Bolsterli, and Suura, when finite order expansions of the irreducible Bethe-Salpeter amplitude are iterated in configuration space without prior regularization. An infinite number of irreducible Feynman graphs are considered and approximated by a "peratization" method; a simple example is found in which the light cone damping, obtained by Feinberg and Pais by summing over the regularized ladder graphs, is reproduced by this crossed graph method.

RECENTLY, Feinberg and Pais[1] have developed a theory of higher-order corrections to weak interactions mediated by charged W mesons of spin one. Their discussion of the leptonic processes, based on an approximate solution to a regularized ladder approximation BS equation, has been verified by Pwu and Wu.[2] Recently, however, Bardakci, Bolsterli, and Suura[3] have remarked that the sum of the unregularized ladder graphs has, in configuration space, an essential singularity on the light cone which cannot be regularized away. Thus the procedures of regularization and summation apparently do not commute, and in the sense of BBS, this interaction is not renormalizable.

The purpose of this paper is to suggest a mechanism whereby the crossed graphs without the aid of regularization may provide sufficient damping to prevent the occurrence of an essential singularity. This conjecture is made here within the context of the weak interactions, but the mechanism might be expected to be relevant to the renormalization of other vector meson theories.

A standard way of writing the BS amplitude (omitting self-energy, vertex, and closed fermion loop complications) is in terms of the iteration of an irreducible kernel or amplitude

$$T=T^i+T^i\times T, \quad (1)$$

where, as illustrated in Fig. 1, the irreducible amplitude is defined to be the sum of all the irreducible Feynman graphs. The use of a finite-order expansion ($\sim g^{2n}$) of the irreducible amplitude leads, in the approximation of neglecting 4-momenta but not momentum transfer,[4] to BS equations whose solutions apparently contain essential singularities, with the severity of the singularity increasing with order n. For example, for $n=2$, one obtains for the "forbidden" crossed graph amplitude

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[1] G. Feinberg and A. Pais, Phys. Rev. **131**, 2724 (1963).
[2] Y. Pwu and T. T. Wu, Phys. Rev. **133**, B1299 (1964).
[3] K. Bardakci, M. Bolsterli, and H. Suura, Phys. Rev. **133**, B1273 (1964).

[4] An additional simplifying approximation, equivalent to iterating only the "most singular part" of the irreducible amplitude expansion, has been made here. For $n=2$ this corresponds to iterating not the simplest crossed graph but, rather, its value between spinors.

a light cone singularity of form $\exp(1/x^2)$; in comparison, the behavior found by BBS for the ladder graphs has the form $\exp[1/(x^2)^{1/2}]$.

Clearly, what is of interest here is the behavior of the entire irreducible amplitude. This is too ambitious a project; however, it is possible to write a modification of (1) which will incorporate an infinite sum of a particular class of irreducible graphs. The summation may be extended to include the iterations of these graphs, and further, the class of irreducible graphs can easily be enlarged. Only the simplest of these possibilities is considered here. Specifically, (1) is approximated by

$$T=T_2{}^i+T_2{}^i\times T+\Sigma_n{}'T_n{}^i, \tag{2}$$

where $T_2{}^i$ denotes the two lowest order (g^2 and g^4) terms of T^i, and $\Sigma_n{}'T_n{}^i$ denotes a partial sum over the irreducible graphs generated by the insertion of lower order (reducible and irreducible) graphs between a pair of crossed lines, as illustrated in Fig. 2. For the moment, to simplify the discussion, we neglect the isotopic factors which separate the contributions of different graphs into one "allowed" and two "forbidden" amplitudes.[5] (In a neutral vector-meson theory, which results from the neglect of all isotopic factors, one, of course, needs no such mechanism to eliminate mass shell singularities, since the latter are removed by very special gauge-type cancellations resulting from the coherent addition of all ladder and crossed graphs.)

Consider now the "gradient" portion of the last term of (2). This is obtained by the replacement of $\Delta_F{}^{\mu\nu}(a-b)$ by $\mu^{-2}\partial_\mu{}^a\partial_\nu{}^b\Delta_F(a-b)$ in both (of the explicit) crossed boson lines; here, μ represents the W-meson mass, and the configuration space coordinates bearing subscripts 1 are to refer to lepton A, while those with subscript 2 refer to lepton B. Neglecting isotopic $\gamma_{A,B}{}^5$ and unimportant proportionality factors, this is given by

$$-\left(\frac{g}{\mu}\right)^4\partial_\mu{}^{x_1}\partial_\lambda{}^{y_2}\Delta_F(x_1-y_2)\partial_\nu{}^{x_2}\partial_\sigma{}^{y_1}\Delta_F(x_2-y_1)\gamma_\mu{}^A\gamma_\nu{}^B$$
$$\times\int S_F{}^A(x_1-u_1)S_F{}^B(x_2-u_2)T(u;v)$$
$$\times S_F{}^A(v_1-y_1)S_F{}^B(v_2-y_2)\gamma_\sigma{}^A\gamma_\lambda{}^B, \tag{3}$$

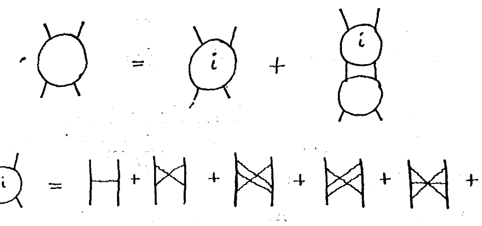

FIG. 1. The BS equation defined in terms of the iteration of the irreducible amplitude.

where $T(x;y)=T(x_1y_1,x_2y_2)$ denotes the configuration-space BS amplitude. In momentum space (3) becomes

$$-\left(\frac{g}{\mu}\right)^4\int dk_1\tilde{\Delta}_F(k_1)\int dk_2\tilde{\Delta}_F(k_2)[\boldsymbol{k}_1(\boldsymbol{k}_1+\boldsymbol{q}_1)^{-1}]^A$$
$$\times[\boldsymbol{k}_2(\boldsymbol{k}_2+\boldsymbol{q}_2)^{-1}]^B\tilde{T}(q_1+k_1,\, p_1-k_2,\, q_2+k_2,\, p_2-k_1)$$
$$\times[(\boldsymbol{k}_2-\boldsymbol{p}_1)^{-1}\boldsymbol{k}_2]^A[(\boldsymbol{k}_1-\boldsymbol{p}_2)^{-1}\boldsymbol{k}_1]^B, \tag{4}$$

where $\tilde{T}(q_1p_1,q_2p_2)$ denotes the Fourier transform of $T(x_1y_1,x_2y_2)$. We now make a peratization approximation to (4), defined by dropping all external momenta, compared to virtual momenta in the spinor factors *only*,

$$-(g/\mu)^4\int dk_1\tilde{\Delta}_F(k_1)\int dk_2\tilde{\Delta}_F(k_2)$$
$$\times\tilde{T}(q_1+k_1,\, p_1-k_2,\, q_2+k_2,\, p_2-k_1). \tag{5}$$

This is exactly what would occur if (4) were sandwiched between the appropriate zero-mass lepton spinors; that is, each irreducible graph generated by the iteration of (2) will have the form (5) between spinors, although it is not necessarily true that the sum of all such terms is equivalent to the solution of (2) between spinors. This replacement of (4) by (5) is, in configuration space, equivalent to the replacement of (3) by

$$-(g/\mu)^4\Delta_F(x_1-y_2)\Delta_F(x_2-y_1)T(x;y). \tag{6}$$

Hence the peratized part of this infinite sum of irreducible graphs is simply proportional to the scattering amplitude, and (2) can be rewritten in the form

$$T\approx[1+(g/\mu)^4\Delta_F(x_1-y_2)\Delta_F(x_2-y_1)]^{-1}$$
$$\times\{T_2{}^i+T_2{}^i\times T\}; \tag{7}$$

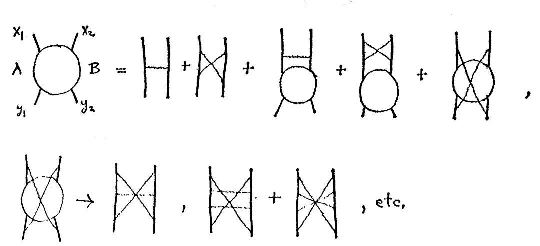

FIG. 2. A representation of an approximate BS equation, whose iteration generates an infinite number of irreducible graphs.

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[5] Corresponding to the interaction Lagrangian

$$\mathcal{L}'=i(g/\sqrt{2})\bar{\psi}_e\gamma_\mu(1+\gamma_5)\boldsymbol{\sigma}\cdot\mathbf{W}_\mu\psi_e+(e\leftrightarrow\mu),$$

where

$$\psi_e=\begin{pmatrix}e\\ \nu_e\end{pmatrix},\ \psi_\mu=\begin{pmatrix}\mu\\ \nu_\mu\end{pmatrix},\ W_\mu{}^{(1)}=(1/\sqrt{2})[W_\mu+W_\mu{}^+],$$
$$W_\mu{}^{(2)}=(i/\sqrt{2})[W_\mu-W_\mu{}^+],\ W_\mu{}^{(3)}=0,$$

the scattering amplitude may be split into three parts; $T=p_aT_a+p_fT_f+p_cT_c$. Here, T_a denotes the "allowed" amplitude (e.g., for the process $e^-+\nu_\mu\rightarrow\nu_e+\mu^-$), T_f denotes the "forbidden" amplitude (e.g., $e^-+\nu_\mu\rightarrow e^-+\nu_\mu$) which is represented in lowest order by a two-rung ladder graph, and T_c denotes the "forbidden" amplitude (e.g., $e^-+\mu^-\rightarrow e^-+\mu^-$) which in lowest order is given by the simplest crossed graph. The operators $p_{a,f,c}$ are given by

$$p_a=\tfrac{1}{2}\sum_{j=1}^{2}\sigma_j{}^A\sigma_j{}^B,\ p_f=\tfrac{1}{2}[1-\sigma_3{}^A\sigma_3{}^B],\ p_c=\tfrac{1}{2}[1+\sigma_3{}^A\sigma_3{}^B],$$

and are linearly related to the three projection operators $p_{1,2,3}$;

$$p_1=\tfrac{1}{2}[p_f+p_a],\ p_2=\tfrac{1}{2}[p_f-p_a],\ p_3=p_c.$$

for simplicity, the corrections to (6) and other higher order graphs have been omitted from the right side of (7).

Introducing the proper isotopic and γ_5 factors, one finds in place of (7) three related equations for the amplitudes T_a, T_f, T_c;

$$T_a=[1+16(g/\mu)^4\Delta_F(x_1-y_2)\Delta_F(x_2-y_1)]^{-1}\times\{T_B{}^i+T_B{}^i\times T_F\}. \quad (8)$$

$$T_f=T_B{}^i\times T_a, \quad (9)$$

$$T_c=[1+16(g/\mu)^4\Delta_F(x_1-y_2)\Delta_F(x_2-y_1)]^{-1}\times\{T_C{}^i+T_C{}^i\times T_c\}, \quad (10)$$

where $T_B{}^i$ and $T_C{}^i$ denote the one-boson exchange and simplest crossed graph, respectively. Equations (8) and (9) differ from the coupled-ladder graph equations by the inclusion of the extra denominator term; a similar remark holds for the decoupled crossed-graph equation (10).

Whether or not these extra denominator factors lead, in general, to well-damped Fourier transforms is not immediately apparent; that this may be the case is suggested by the explicit damping which does occur in simple approximations to these equations. Working with (10), for example, it is possible to define

$$T_c(x;y)=C(x_1-x_2)\times[1+16(g/\mu)^4\Delta_F(x_1-y_2)\Delta_F(x_2-y_1)]^{-1}M_c(x;y),$$

where

$$C(x)=4(g/\mu)^4[\gamma_\mu(1+\gamma_5)]_A[\gamma_\nu(1+\gamma_5)]_B\times\partial_\mu\Delta_F(x)\partial_\nu\Delta_F(x),$$

and[4]

$$M_c(x;y)=\delta(x_1-y_1)\delta(x_2-y_2)+\int S_F{}^A(x_1-u_1)S_F{}^B(x_2-u_2)C(u_1-u_2)\times\left[1+16\left(\frac{g}{\mu}\right)^4\Delta_F(u_1-y_2)\Delta_F(u_2-y_1)\right]^{-1}M_c(u;y). \quad (11)$$

If $M_c(x;y)=M_c(x_1-y_1,\,x_2-y_2,\,x_1-x_2)\equiv M_c(\xi_1,\xi_2,x)$, and the partial Fourier transform $\tilde{M}_c(p_1,p_2,x)$ is defined with respect to ξ_1, ξ_2, one obtains an equation in which the full damping enters explicitly if the $p_{1,2}$ dependence of $\tilde{M}_c$ is ignored; that is, if $\tilde{M}_c(p_1,p_2,x)$ is replaced by $M_c(x)$, one finds the equation

$$\partial_A\partial_B M_c(x)=-C(x)[1+16(g/\mu)^4\Delta^2{}_F(x)]^{-1}M_c(x) \quad (12)$$

whose solution does not contain an essential singularity.[6] In contrast, the iteration of just the simplest crossed graph, in the approximation of treating T_c as a function of momentum transfer only,[4] produces (12) without the damping denominator. It should be emphasized that this is only an indication of damping; this example can certainly be criticized by noting, e.g., that the approximation of neglecting the $p_{1,2}$ dependence is not compatible with the general form of (11).

Perhaps the simplest and most relevant example follows from the "Born approximation" to (8),

$$T_a(x;y)\approx[1+16(g/\mu)^4\Delta_F{}^2(x_1-x_2)]^{-1}\times B(x_1-x_2)\delta(x_1-y_1)\delta(x_2-y_2), \quad (13)$$

where $B(x)=ig^2[\gamma_\mu(1+\gamma_5)]_A[\gamma_\nu(1+\gamma_5)]_B\Delta_F{}^{\mu\nu}(x)$. Hence, treating $\tilde{T}_a$ as a function of momentum transfer only, one has

$$T_a(x)\approx B(x)[1+16(g/\mu)^4\Delta_F{}^2(x)]^{-1}, \quad (14)$$

which is just the (unregularized) ladder graph result of Ref. 1. Here, however, the light cone damping has been explicitly generated by a class of irreducible crossed graphs. If the damping is such that (13) represents a decent approximation to (8), i.e., if subsequent corrections are not divergent and are of higher order, one may give new credence to the famous factor[1] of $(-\delta_{\mu\nu}/4\mu^2)$ in the zero momentum transfer limit of the Fourier transform of (14).

Details of the elementary functional techniques useful in the formulation of these and related approximations will be given elsewhere. It is a pleasure to thank Professor M. Ruderman for his patient ear, and to acknowledge several informative discussions with Dr. K. Bardakci and Professor H. Suura.

[6] The equation

$$\partial_A\partial_B F(x)=\{(\gamma_A\cdot x)(\gamma_B\cdot x)I(x^2)+(\gamma_A\cdot\gamma_B)J(x^2)\}\times(1+\gamma_5)_A(1+\gamma_5)_B F(x)$$

will have solutions of form

$$F=\{F_0(x^2)+(\gamma_A\cdot\gamma_B)(\gamma_A\cdot x)(\gamma_B\cdot x)F_1(x^2)\}(1+\gamma_5)_A(1+\gamma_5)_B$$

which contain an essential singularity if near the light cone $I\sim(x^2)^{-\alpha}$, $\alpha>2$. This does not necessarily imply that the corresponding mass shell scattering amplitude will contain that singularity, since there is always the possibility of using divergence-free combinations, or of finding gauge-type cancellations (depending upon the form of J).