Correlations in $\bar{p}p \to 4\pi$*

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Under the assumption that the $p\bar{p}$ annihilation takes place from an S state, the unique final state of $p\bar{p} \to 2\rho^0 \to 2\pi^+ + 2\pi^-$ has been constructed within the formalism of Jacob and Wick. The mean value of $\cos\theta_{12}$ between pion pairs has been computed and the results for annihilation at rest are $\langle\cos\theta_{12}{}^{++}\rangle = -0.29$ and $\langle\cos\theta_{12}{}^{+-}\rangle = -0.31$. The di-pion effective mass distributions have also been computed.

I. INTRODUCTION

In the search for ρ mesons in $p+\bar{p} \to n\pi$ it is not possible to determine which pions belong to a ρ since the lifetime of the ρ is so short. Thus, for example, in making di-pion effective mass measurements, one must include all possible pions pairs. In order to determine the extent of ρ production, it is necessary to know the effect of measuring all possible pion pairs on the effective mass distributions.

In this paper we shall calculate the two-body effective mass distributions in the final state of

$$p+\bar{p} \to 2\rho^0 \to 2\pi^+ + 2\pi^-. \quad (1)$$

We shall find that if we assume the annihilation proceeds from an S state, then the final 4π state is uniquely determined, and we can derive an expression for the expectation value of any two-body scalar operator which commutes with the momenta of the individual pions. In Sec. V this expression is used to compute the mean value of $\cos\theta_{12}$ between $\pi^+\pi^-$ and $\pi^-\pi^-$ pairs. In Sec. VI we compute the two-particle effective mass distributions and find that although there are two ρ mesons, the need to measure all possible pion pairs tends to mask the resonance.

II. THE INITIAL $p\bar{p}$ STATE

With regard to the angular momentum, we have the following picture: The antiproton enters the sample of matter and loses energy just as any charged particle does. As it slows down, probability of capture into an atomic orbital increases and it is captured in some atomic n, l orbital. Now just as in the classical case of negative pion capture,[1] one assumes that the proton will capture the antiproton (that is, the $p-\bar{p}$ system will annihilate) if the wave functions overlap. One must decide if the capture rate from a high l state is much smaller than the rate of radiative decay to an S state. If this is so, then one may conclude that usually the annihilation takes place from an S state of protonium.

Recently Desai[2] has made calculations which strongly favor the S-state annihilation. In addition to the radiative decay to the ground state, the protonium, since it is much smaller than a hydrogen atom, is able to penetrate other atoms and thus feel the strong electric fields of other protons; thus there are also Stark transitions. As in the case of K-meson capture,[3] Desai finds that the Stark effects are very important, and have the effect of producing rapid transitions to the S states.

Still more recently, with the accumulation of thousands of annihilation events at CERN, there is further evidence for the S state from the absence of $K_1{}^0+K_1{}^0$ decays.[4] The reaction

$$p+\bar{p} \to K_1{}^0+K_1{}^0 \text{ (from } l=0 \text{ state)}$$

is forbidden from the $^1S_0$ state because of parity conservation and from the $^3S_1$ state because of charge conjugation invariance.

We shall therefore assume in what follows that the initial $p\bar{p}$ system has $J=0$ or 1.

Since the proton has isotopic spin $\frac{1}{2}$ and component $+\frac{1}{2}$ along the Z axis in isospace, the antiproton has isotopic spin $\frac{1}{2}$ and z component $-\frac{1}{2}$. Hence, the initial $p\bar{p}$ system has $I_Z=0$, and $I=0$ or 1.

The parity of a $p\bar{p}$ system in an S state is odd. The G parity[5] of a $p\bar{p}$ system in an S state is $+1$ for $I=J$ and (-1) for $I\neq J$.

III. THE FINAL STATE

In this section we shall consider the final state of proton-antiproton annihilation into two neutral ρ mesons. We consider only the 2π decay mode of the ρ, and we shall assume the annihilation takes place from an S state of protonium. We are interested in neutral ρ's because we wish to see four charged pions in the final state.

We shall find that $p+\bar{p} \to \rho^0+\rho^0$ from an S state gives rise to only one possible final state, and we

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[1] K. Breuckner, R. Serber, and K. M. Watson, Phys. Rev. **81**, 575 (1951).

[2] B. Desai, Phys. Rev. **119**, 1385 (1960).

[3] T. B. Day, G. A. Snow, and J. Sucher, Phys. Rev. Letters **3**, 61 (1959).

[4] R. Armenteros *et al.*, in *Proceedings of the International Conference on High-Energy Physics at CERN, 1962* edited by J. Prentki (CERN, Geneva, 1962), pp. 351–356.

[5] T. D. Lee and C. N. Yang, Nuovo Cimento **3**, 749 (1956).

shall construct this state explicitly in the helicity representation.[6,7]

We first mention some properties of Lorentz transformations on pion state vectors.

1. Single-Pion States and Lorentz Transformations

Let $\pi(\mathbf{p},t)$ denote the state vector of a single pion of momentum $\mathbf{p}$ and charge t. The scalar product of two such states is given by

$$(\pi(\mathbf{p},t)\,|\,\pi(\mathbf{p}',t'))=2\omega\delta^{(3)}(\mathbf{p},\mathbf{p}')\delta(t,t')\,,$$

where the δ function is understood to be a Kronecker delta for the discrete variable t, and a Dirac δ function for the continuous variable $\mathbf{p}$. ω is the energy of the pion and in units where the pion mass is 1, is given by $\omega=(1+p^2)^{1/2}$. The quantity $\delta(\mathbf{0})$ is to be understood as a very large but finite volume V which cancels out in the calculation of any physical quantity. The limit $V\rightarrow\infty$, is to be taken after the calculation is completed.

The components of a space-time point are denoted by x_μ ($\mu=0$, 1, 2, 3) with $x_0=t$. The velocity of light is taken as unity.

To each Lorentz transformation which is continuously connected to the identity

$$x_\mu'=L_{\mu\lambda}x_\lambda \tag{2}$$

there corresponds a unitary transformation of the state vectors

$$\begin{aligned} L\pi(\mathbf{p},t)&=\pi(\mathbf{p}',t)\,,\\ p_\mu'&=L_{\mu\lambda}p_\lambda\,. \end{aligned} \tag{3}$$

For the parity operator P, we have, since the pion is pseudoscalar,

$$P\pi(\mathbf{p},t)=-\pi(-\mathbf{p},\,t)\,. \tag{4}$$

Let P_μ be the energy momentum operator; then we have defined the Lorentz operator L such that

$$L^{-1}P_\mu L=L_{\mu\lambda}P_\lambda\,. \tag{5}$$

It will be convenient in the following to have a special notation for specific Lorentz operators. In particular, $L(v)$ will correspond to a Lorentz transformation along the z axis;

$$L(v):\quad \begin{aligned} x'&=x\,,\\ y'&=y\,,\\ z'&=\gamma(z+vt)\,,\\ t'&=\gamma(t+vz)\,, \end{aligned} \tag{6}$$

and $\gamma=1/(1-v^2)^{1/2}$.

Let J_1, J_2, J_3, be the components of the total angular momentum operator. A rotation about the kth coordinate axis through a positive angle φ will be denoted by $R_k(\varphi)$. An explicit representation is

$$R_k(\varphi)=\exp(i\varphi J_k)\,. \tag{7}$$

The rotation with Euler angles α, β, φ, will be denoted by

$$R(\alpha,\beta,\varphi)=R_3(\alpha)R_2(\beta)R_3(\varphi) \tag{8}$$

or

$$R(\alpha,\beta,\varphi)=\exp(i\alpha J_2)\,\exp(i\beta J_3)\,\exp(i\varphi J_2)\,. \tag{9}$$

In the following, where states of many particles are concerned, we shall use superscripts on the operators to tell which particles are affected. Thus, $L^{12}(v)$ will denote the operator for a Lorentz transformation along the z axis for particles "1" and "2" only.

2. Two-Particle States: The ρ Meson

The general two-particle state is a sum of elements $\pi(\mathbf{p}_1,t_1)\pi(\mathbf{p}_2,t_2)$ of the direct product space. Thus, if F is a two-particle state,

$$F=\sum_{t_1,t_2}\int\frac{d^3p_1d^3p_2}{\omega_1\omega_2}F(\mathbf{p}_1,t_1\colon\mathbf{p}_2,t_2)\pi_1(\mathbf{p}_1,t_1)\pi_2(\mathbf{p}_2,t_2)\,. \tag{10}$$

We shall treat particles "1" and "2" as distinguishable and take care of Bose statistics by symmetrizing the amplitude $F(\mathbf{p}_1,t_1\colon\mathbf{p}_2,t_2)$.

The ρ meson has spin one and isotopic spin one. Its mass M is approximately 5.4 pion masses (750 MeV), and its width Γ is about 0.7 pion masses (100 MeV). It decays rapidly into two pions.[8] We are concerned only with the final two-pion state, so that henceforth, when the term "ρ meson" is used, it is to be understood that a two-pion state with the above quantum numbers is meant.

Let $\rho_0{}^{q\lambda}(1,2)$ denote the state of a ρ meson made up of particles "1" and "2," with z component of angular momentum λ, and charge q, and with zero linear momentum. In accordance with the above, we may write

$$\rho_0{}^{q\lambda}(1,2)=\sum_t s_{1t,q-t}^{1\;\;1}\int\frac{d^3p_1}{\omega_1}\frac{d^3p_2}{\omega_2}F_0{}^\lambda(\mathbf{p}_1,\mathbf{p}_2)\times\pi_1(\mathbf{p}_1,t)\pi_2(\mathbf{p}_2,\,q-t)\,, \tag{11}$$

where $s_{1t,q-t}^{1\;\;1}$ is a Clebsch-Gordon coefficient which couples the two isospin one pions to a state of isospin 1.

Since the ρ is at rest, its total angular momentum is equal to its spin and is therefore unity. Also, because

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[6] We follow closely the methods of M. Jacob and G. C. Wick [Ann. Phys. (N. Y.) **7**, 404 (1959)]; however, our conventions are somewhat different. Our spherical harmonics and Clebsch-Gordan coefficients are those of E. Wigner, *Group Theory and its Applications to the Quantum Mechanics of Atomic Spectra*, 1959.

[7] Our rotation matrices are those of A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957).

[8] A. R. Erwin, R. March, W. D. Walker, and E. West, Phys. Rev. Letters **6**, 628 (1961); J. Alitti, J. P. Baton, A. Berthelot, A. Daudin, B. Deler *et al.*, Nuovo Cimento **25**, 365 (1962); E. Pickup, D. K. Robinson, and E. O. Salant, Phys. Rev. Letters **7**, 192 (1961); D. Stonehill, C. Baltay, H. Courant, W. Fickinger, E. C. Fowler *et al.*, *ibid.* **6**, 625 (1961), and C. Alff, D. Berley, D. Colley, N. Gelfand, U. Nauenberg *et al.*, *ibid.* **9**, 322 (1962).

we work in the rest frame we have $\mathbf{p}_2=-\mathbf{p}_1$, and because the width Γ is neglected we have $\omega_1=\omega_2\equiv\omega=M/2$. Taking all this into account we see that the amplitude $F_0{}^\lambda(\mathbf{p}_1,\mathbf{p}_2)$ must be given by

$$F_0{}^\lambda(\mathbf{p}_1,\mathbf{p}_2)=N[\omega_1\omega_2/p_1p_2]^{1/2}\delta(\mathbf{p}_1+\mathbf{p}_2)\times\delta(\omega_1+\omega_2-M)Y_{1\lambda}(\mathbf{p}_1), \quad (12)$$

where $N[\omega_1\omega_2/(p_1p_2)]^{1/2}$ is a conveniently chosen normalization constant, and $Y_{1\lambda}(\mathbf{p})$ is a spherical harmonic of first order.

If we integrate over d^3p_2, and write $d^3p_1=p_1{}^2dp_1d\Omega_1=p_1{}^2d\Omega_1(dp_1/d\omega_1)d\omega_1=p_1\omega_1d\Omega_1d\omega_1$ and then integrate over $d\omega_1$, we obtain

$$\rho_0{}^{q\lambda}(1,2)=N\sum_t s_{1\,t,\,q-t}^{\;\,1\,\,1}\times\int d\Omega_1 Y_{1\lambda}(\mathbf{p})\pi_1(\mathbf{p},t)\pi_2(-\mathbf{p},q-t), \quad (13)$$

where $\mathbf{p}$ is a vector of magnitude $p=(\omega^2-1)^{1/2}=[(M/2)^2-1]^{1/2}$, and $d\Omega_1=\sin\theta d\theta d\varphi$, where θ, φ are the polar angles of the vector $\mathbf{p}$.

We define the state of a ρ meson moving along the positive z axis with momentum k and velocity v, and with *helicity* λ, by means of a Lorentz transformation $L(v)$ along the z axis. Thus,

$$\rho_k{}^{q\lambda}(1,2)\equiv L(v)\rho_0{}^{q\lambda}(1,2). \quad (14)$$

To obtain a state where the ρ is moving with momentum k along a direction with polar angles θ, φ, we shall perform a rotation $R(-\varphi,-\theta,\varphi)$ upon the state in which it is moving along the z axis. Denoting the former state by $\rho_{\mathbf{k}}{}^{q\lambda}(1,2)$ we have

$$\rho_{\mathbf{k}}{}^{q\lambda}(1,2)\equiv R(-\varphi,-\theta,\varphi)\rho_k{}^{q\lambda}(1,2) \quad (15)$$

or

$$\rho_{\mathbf{k}}{}^{q\lambda}(1,2)\equiv\exp(-iJ_3\varphi)\exp(-iJ_2\theta)\times\exp(iJ_3\varphi)\rho_k{}^{q\lambda}(1,2). \quad (16)$$

Let k_i be the ith component of the vector $\mathbf{k}$. Then it follows from Eq. (5) that

$$P_i\rho_{\mathbf{k}}{}^{q\lambda}=k_i\rho_{\mathbf{k}}{}^{q\lambda}, \quad (17)$$

where P_i is the ith component of the energy-momentum operator for particles "1" and "2." Again using Eq. (5) one may show that the helicity operator $\mathbf{J}\cdot\mathbf{P}/|\mathbf{P}|$ commutes with any spatial rotation operator and in particular with $R(-\varphi,-\theta,\varphi)$. Furthermore, the third component of angular momentum commutes with a Lorentz transformation $L(v)$ along the z axis. Using these commutation properties, together with the defining equations (13), (14), and (15), one may show that

$$\frac{\mathbf{J}\cdot\mathbf{P}}{|\mathbf{P}|}\rho_{\mathbf{k}}{}^{q\lambda}=\lambda\rho_{\mathbf{k}}{}^{q\lambda}; \quad (18)$$

that is, the state $\rho_{\mathbf{k}}{}^{q\lambda}$ is an eigenstate of the helicity operator (for particles "1" and "2") belonging to the eigenvalue λ.

Because the first spherical harmonic is an odd function, and $s_{1,t_1,t_2}^{\;\,1\,\,1}$ is antisymmetric in t_1 and t_2, we may show from (11) and (12) that $\rho_0{}^{q\lambda}$ is symmetric in the particle labels "1" and "2," so that the requirements of Bose statistics for the pions are met. Since the Lorentz operators, $L(v)$ and $R(\alpha,\beta,\varphi)$, act symmetrically on particles "1" and "2," it is clear that the states $\rho_{\mathbf{k}}{}^{q\lambda}(1,2)$ are also symmetric in "1" and "2."

We shall now discuss the action of the parity operator P, on states of the ρ.

From Eqs. (4) and (13), we have

$$P\rho_0{}^{q\lambda}=N\sum_t s_{1\,t,\,q-t}^{\;\,1\,\,1}\times\int d\Omega Y_{1\lambda}(\mathbf{p})\pi_1(-\mathbf{p},t)\pi_2(\mathbf{p},q-t). \quad (19)$$

Upon changing the variable of integration from $\mathbf{p}$ to $\mathbf{p}'=-\mathbf{p}$, we see that

$$P\rho_0{}^{q\lambda}=-\rho_0{}^{q\lambda}. \quad (20)$$

In order to discuss the action of P on $\rho_k{}^{q\lambda}$ it will be convenient to denote the state with helicity λ, charge q, and momentum k along the *negative* z axis by $\rho_{-k}{}^{q\lambda}$. Thus,

$$\rho_{-k}{}^{q\lambda}\equiv R_2(-\pi)\rho_k{}^{q\lambda}\equiv R_2(-\pi)L(v)\rho_0{}^{q\lambda}. \quad (21)$$

We shall prove the following useful relation.

$$P\rho_k{}^{q\lambda}=(-1)^\lambda\rho_{-k}{}^{q\,-\lambda}. \quad (22)$$

The phase factor $(-1)^\lambda$ is a direct result of our conventions and further holds only for motion along the z axis. The proof of (22) follows. Since the operators P, $L(v)$, $R_2(-\pi)$ belong to a (operator) representation of the Lorentz group, they obey the relations

$$R_2(-\pi)L(v)=L(-v)R_2(-\pi),\quad PL(v)=L(-v)P. \quad (23)$$

The rest frame state has total angular momentum unity and z component λ. Hence, for any spatial rotation R, we have

$$R\rho_0{}^{q\lambda}=\sum_\mu\rho_0{}^{q\mu}\mathfrak{D}^{(1)}(R)_{\mu\lambda}, \quad (24)$$

where the $\mathfrak{D}^{(1)}(R)$ are the representation matrices. Furthermore,[7]

$$\mathfrak{D}^{(1)}(R_2(-\pi))_{\mu\lambda}=\delta_{\mu,-\lambda}(-1)^{1+\lambda}. \quad (25)$$

The proof of Eq. (22) now proceeds from Eqs. (20), (21), (23), and (24) immediately.

$$\begin{aligned}P\rho_k{}^{q\lambda}&=PL(v)\rho_0{}^{q\lambda}=-L(-v)\rho_0{}^{q\lambda},\\&=-R_2(-\pi)L(v)R_2(-\pi)\rho_0{}^{q\lambda},\\&=(-1)R_2(-\pi)L(v)\rho_0{}^{q,-\lambda}(-1)^{1+\lambda}=(-1)^\lambda\rho_{-k}{}^{q,-\lambda},\end{aligned}$$

where we have used the relation (valid only in the case of integral spin) $R_2(-\pi)R_2(-\pi)=1$.

3. The Final State of $p+\bar{p}\rightarrow 2\rho^0\rightarrow 2\pi^+ + 2\pi^-$

Since we wish to consider $2\rho^0$ the total isotopic spin must be either 0 or 2. But since the initial $p-\bar{p}$ system has isotopic spin 0 or 1, it follows from conservation of isotopic spin that $I=0$. Thus the final state is an isoscalar.

The G quantum number for an even number of pions is $+1$, and so the final state has $G=+1$. Since G is conserved in the strong interactions, the initial $p-\bar{p}$ system must have $G=+1$. We are assuming that the proton-antiproton system decays from an S state, but the only $G=+1$, $I=0$, S state of protonium is the singlet S state[5] so that the final state has total angular momentum $J=0$. Thus, the final state is a scalar.

The parity of a proton-antiproton system in an S state is odd so that $P=-1$.

The final state is thus seen to have $I=J=0$, $P=-1$, and because of Bose statistics for the pions, must be totally symmetric in all of the pions. We shall construct such a state in the over-all center-of-mass frame. Indeed, we shall find that the state is unique.

We shall henceforth suppress the charge variable q in $\rho_k{}^{q\lambda}(1,2)$ since we are restricting ourselves to neutral ρ's, for which $q=0$.

Define the four-pion state Φ_1 by

$$\Phi_1\equiv\rho_k{}^\lambda(1,2)\rho_{-k}{}^\lambda(3,4). \quad (26)$$

Φ_1 is a state of zero linear momentum, in which two (distinguishable) ρ mesons, both of helicity λ move along the positive and negative z axis, respectively. Because the total angular momentum is to be zero, both ρ's must have the same helicity.

Since ρ mesons obey Bose statistics, we must really consider the state

$$\rho_k{}^\lambda(1,2)\rho_{-k}{}^\lambda(3,4)+\rho_{-k}{}^\lambda(1,2)\rho_k{}^\lambda(3,4),$$

which is symmetric in the two ρ's. However, from the definition (21), this state is just

$$(1+R_2(\pi))\Phi_1.$$

Since we wish to have odd parity, we must further multiply this by $(1-P)$.

Since we wish $J=0$, we must integrate over all rotations R, $\int RdR$, where dR is the invariant volume element of the rotation group (see below).

Since the *pions* obey Bose statistics, we introduce the states

$$\begin{aligned}\Phi_2&=\rho_k{}^\lambda(1,3)\rho_{-k}{}^\lambda(2,4),\\ \Phi_3&=\rho_k{}^\lambda(1,4)\rho_{-k}{}^\lambda(2,3).\end{aligned} \quad (27)$$

The totally symmetric, $J=0$, odd parity, final states of two ρ mesons are therefore given by

$$\Psi=\int RdR[1+R_2(\pi)][1-P]\{\Phi_1+\Phi_2+\Phi_3\}, \quad (28)$$

where $R=R(\alpha,\beta,\gamma)$, and $dR=d\alpha\sin\beta d\beta d\gamma$, and $\lambda=0,\pm1$.

The region of integration is

$$0\leq\alpha<2\pi,\quad 0\leq\beta<\pi,\quad 0\leq\gamma<2\pi.$$

Using the invariance of the group volume element[6] in the form $\int dR'=\int dR$ with $R'=R(\alpha,\beta,\gamma)R_2(\pi)$, we immediately obtain

$$\Psi=2\int dRR[1-P]\{\Phi_1+\Phi_2+\Phi_3\}. \quad (29)$$

It appears that we have three possible final states Ψ, corresponding to the three choices: $\lambda=0,\pm1$ in (26) and (27). However, the state Ψ corresponding to λ, is just the negative of the state corresponding to $-\lambda$. To show this, consider

$$\begin{aligned}P\Phi_1(\text{for }\lambda)&\equiv P\rho_k{}^\lambda(1,2)\rho_{-k}{}^\lambda(3,4)\\ &=\rho_{-k}{}^{-\lambda}(1,2)\rho_k{}^{-\lambda}(3,4),\end{aligned} \quad (30)$$

$$P\Phi_1(\text{for }\lambda)=R_2(\pi)\Phi_1(\text{for }-\lambda), \quad (31)$$

using (26), (21), and (22). Obviously the same relation holds for Φ_2 and Φ_3, so that using the invariance of the volume element gives $\Psi(\text{for }\lambda)=-\Psi(\text{for }-\lambda)$. Hence, there is no odd-parity state for $\lambda=0$, while the choices $\lambda=\pm1$ give the same odd parity state except for sign. Therefore, there is only one odd-parity state. [In the same way one may show that there are two even-parity states, by replacing $(1-P)$ by $(1+P)$.]

We now have, choosing $\lambda=+1$, for the unique final state

$$\Psi=2\int dRR[1-P]\{\Phi_1+\Phi_2+\Phi_3\}. \quad (32)$$

We have shown that there is only one possible final state for $p+\bar{p}\rightarrow 2\rho^0$ from an S state, and it must be given by (32).

IV. MATRIX ELEMENTS $(\Psi|V_{12}|\Psi)$

We shall now discuss the expectation value of an operator V_{12} which: (a) depends only on particles "1" and "2," (b) commutes with spatial rotations $R(\alpha,\beta,\gamma)$, (c) commutes with parity P, and, (d) commutes with the momentum operators $P_\mu{}^{(i)}$ of the ith pion.

Since R is a unitary operator, and commutes with V_{12} and P, we have from Eq. (32)

$$\begin{aligned}(\Psi|V_{12}|\Psi)&=\sum_{i,j=1}^{3}4\int dR_1dR_2([1-P]\Phi_i,\,V_{12}R_2{}^{-1}R_1[1-P]\Phi_j)\\ &=\sum_{i,j=1}^{3}4\int dR_1\\ &\quad\times\int dR_2'([1-P]\Phi_i,\,V_{12}R_2'[1-P]\Phi_j),\end{aligned} \quad (33)$$

or, dropping the prime and integrating over dR_1

$$(\Psi|V_{12}|\Psi)$$
$$=8\pi^2\sum_{i,j=1}^{3}4\int dR_2([1-P]\Phi_i,\ V_{12}R_2[1-P]\Phi_j)\,,\quad (34)$$

where we have once more used the fact that dR, the rotation-group volume element, is an invariant and

$$8\pi^2=\int dR=\int_0^{2\pi}d\alpha\int_{-1}^{1}d(\cos\beta)\int_0^{2\pi}d\gamma \qquad (35)$$

is the total volume of group space.

We use the same method of removing the operator from the left for the parity operator; thus, we have

$$(-P\Phi_i|V_{12}R[1-P]|\Phi_j)=(\Phi_i|V_{12}R[1-P]|\Phi_j) \quad (36)$$

since P is unitary, commutes with R, and since $P^2=1$.

Using (36) in (34) we obtain

$$(\Psi|V_{12}|\Psi)=\sum_{i,j=1}^{3}(8\pi)^2\int dR(\Phi_i|V_{12}R[1-P]|\Phi_j)\,. \quad (37)$$

Now recall that $R=R_3(\alpha)R_2(\beta)R_3(\gamma)$. Noting that Φ_i has zero z component of angular momentum,

$$R_3(\gamma)\Phi_j=\exp(i\gamma J_3)\Phi_j=\Phi_j\,, \quad (38)$$

while, denoting the Hermitian conjugate of A by $A^\dagger$,

$$\Phi_i{}^\dagger R_3(\alpha)=\Phi_i{}^\dagger R_3(-\alpha)^\dagger=\Phi_i{}^\dagger\,. \quad (39)$$

Substituting (38) and (39) into (37) we can immediately do the integrations over α and γ, whence

$$(\Psi|V_{12}|\Psi)$$
$$=(4\pi)^4\sum_{i,j=1}^{3}\int_0^{\pi}\sin\beta d\beta(\Phi_i|V_{12}R_2(\beta)[1-P]|\Phi_j)\,. \quad (40)$$

In the Appendix we present an argument to show that the terms for which $i\neq j$ in the right-hand side of (40) vanish in the limit of zero width of the ρ, in comparison with the terms for which $i=j$. Thus, assuming this, we have

$$(\Psi|V_{12}|\Psi)$$
$$=(4\pi)^4\sum_{i=1}^{3}\int_0^{\pi}\sin\beta d\beta(\Phi_i|V_{12}R_2(\beta)[1-P]|\Phi_i)\,. \quad (41)$$

Now since V_{12} depends only upon "1" and "2," it commutes with the transposition $P_{(34)}$ of particles "3" and "4." Looking at Eq. (27) we find

$$\Phi_3=P_{(34)}\Phi_2\,. \quad (42)$$

The property $P_{(34)}{}^\dagger=P_{(34)}{}^{-1}=P_{(34)}$ now easily implies that the term with Φ_3 is identical to that with Φ_2 in the right-hand side of (41).

Let us consider the four terms $(\Phi_i|V_{12}R_2(\beta)|\Phi_i)$, $(\Phi_i|V_{12}R_2(\beta)|P\Phi_i)$, for $i=1$, 2 of Eq. (41). We shall show that

$$(\Phi_1|V_{12}R_2(\beta)|\Phi_1)=0 \quad\text{for}\quad \beta\neq 0\,, \quad (43a)$$
$$(\Phi_2|V_{12}R_2(\beta)|\Phi_2)=0 \quad\text{for}\quad \beta\neq 0\,, \quad (43b)$$
$$(\Phi_1|V_{12}R_2(\beta)P|\Phi_1)=0\,, \quad (43c)$$
$$(\Phi_2|V_{12}R_2(\beta)P|\Phi_2)=0 \quad\text{for}\quad \beta\neq\pi\,. \quad (43d)$$

The first of these equations is obtained from the fact that Φ_1 is an eigenstate of the (Hermitian) *momentum*[9] operator $\mathbf{P}^{(34)}$ of particles "3" and "4" with vector eigenvalue along the negative z axis, while $V_{12}R_2(\beta)\Phi_1$ is an eigenstate belonging to a different vector eigenvalue unless $\beta=0$. The second equation is proved with the identical argument but one must use the momentum $\mathbf{P}^{(2)}+\mathbf{P}^{(4)}$ of particles "2" and "4" as the Hermitian operator.

Equation (43c) is proved in the same manner again, except that now we observe that Φ_1 and $V_{12}R_2(\beta)P\Phi_1$ belong to eigenvalues $+1$ and -1, respectively, of the helicity operator for particles "3" and "4." [See Eqs. (26) and (18) above.]

To prove (43d) we use the operator $P_z{}^{(1)}+P_z{}^{(3)}$ for the z component of linear momentum. Note that we must use the fact that $[V_{12},P_z{}^{(1)}]=0$.

Equations (43a)–(43d) state that the integrand in Eq. (41) is equal to zero except at $\beta=0$ or π. If the integrand were bounded, then the integral would vanish; however, more careful consideration of the normalizations involved shows that there are indeed δ-function singularities in the integrand on the right-hand side of (41). Thus, instead we obtain

$$(\Psi,V_{12}\Psi)=C[(\Phi_1|V_{12}|\Phi_1)+2(\Phi_2|V_{12}|\Phi_2)$$
$$-2(\Phi_2|V_{12}R_2(\pi)P|\Phi_2)]\,, \quad (44)$$

where we have lumped all constants into the master constant C.

Equation (44) will be our starting point in subsequent discussions.

Note that if one were to ignore the total angular momentum and not integrate over all rotations dR, but rather choose $\Psi'=(1-P)[1+R_2(\pi)][\Phi_1+\Phi_2+\Phi_3]$ one could get exactly Eq. (44) more quickly. Physically, this means that since V_{12} is a scalar, the result of a measurement of V_{12} is the same for two ρ mesons moving along the z axis, as for them moving along any other direction in space, and there are no interference effects between different spatial directions, in our approximation.

V. ANGULAR CORRELATIONS

We shall now use the result of the previous section to compute the mean value of $\langle\cos\theta_{12}\rangle$ for $\pi^+\pi^-$ pairs,

[9] Recall that the momentum operators $\mathbf{P}^{(ij)}$ are defined by $\mathbf{P}^{(ij)}=\mathbf{P}^{(i)}+\mathbf{P}^{(j)}$ and the angular-momentum operators by $\mathbf{J}^{(ij)}=\mathbf{J}^{(i)}+\mathbf{J}^{(j)}$ in terms of the single-particle operators $\mathbf{P}^{(i)}$ and $\mathbf{J}^{(i)}$.

and $\pi^+\pi^+$ or $\pi^-\pi^-$ pairs.[10] We shall denote the former by $\langle\cos\theta_{12}{}^u\rangle$, the correlation between unlike pions, and the latter by $\langle\cos\theta_{12}{}^l\rangle$, the correlation between like pions. If we look for $\langle\cos\theta_{12}\rangle$ without charge correlations, then we may use as a guide the following argument due to R. Serber.[11]

If we square the equation of momentum conservation

$$\sum_{i=1}^{n} \mathbf{P}_i = 0,$$

we obtain

$$\sum_{i=1}^{n} P_i^2 + \sum_{\substack{i\neq j \\ i,j=1}}^{n} \mathbf{P}_i\cdot\mathbf{P}_j = 0,$$

or, taking mean values, we have

$$n\langle p^2\rangle = -n(n-1)\langle p_1 p_2 \cos\theta_{12}\rangle,$$

whence, roughly

$$\langle\cos\theta_{12}\rangle \approx -1/(n-1). \tag{45}$$

We shall see that this is a very good estimate in our case.

Returning now to the problem with charge correlations, we define projection operators $\Lambda_{12}{}^u$, $\Lambda_{12}{}^l$ which select those states of proper charge correlation:

$$\begin{aligned}\Lambda_{12}{}^u \pi_1{}^{t_1}\pi_2{}^{t_2}\pi_3{}^{t_3}\cdots &= \pi_1{}^{t_1}\pi_2{}^{t_2}\pi_3{}^{t_3}\cdots, \quad \text{if} \quad t_1=-t_2=\pm 1,\\ &=0, \quad \text{otherwise}; \\ \Lambda_{12}{}^l \pi_1{}^{t_1}\pi_2{}^{t_2}\pi_3{}^{t_3}\cdots &= \pi_1{}^{t_1}\pi_2{}^{t_2}\pi_3{}^{t_3}\cdots, \quad \text{if} \quad t_1=t_2=\pm 1,\\ &=0, \quad \text{otherwise},\end{aligned} \tag{46}$$

where $\pi_1{}^{t_1}\pi_2{}^{t_2}\pi_3{}^{t_3}\cdots$ is an n-pion state in which particle "i" has charge t_i. The mean value of $\cos\theta_{12}$ for unlike (like) pions in the state Ψ will be given by

$$\langle\cos\theta_{12}{}^u\rangle = \frac{(\Psi|\Lambda_{12}{}^u C_{12}|\Psi)}{(\Psi|\Lambda_{12}{}^u|\Psi)}, \tag{47}$$

$$\langle\cos\theta_{12}{}^l\rangle = \frac{(\Psi|\Lambda_{12}{}^l C_{12}|\Psi)}{(\Psi|\Lambda_{12}{}^l|\Psi)}, \tag{48}$$

where the operator C_{12} depends only upon the momenta of "1" and "2":

$$C_{12} \equiv C(P_\mu{}^{(1)},P_\nu{}^{(2)}) \equiv \frac{\mathbf{P}^{(1)}\cdot\mathbf{P}^{(2)}}{|\mathbf{P}^{(1)}||\mathbf{P}^{(2)}|}.$$

The denominators in (47) and (48) come from the fact that observations are really being performed on the state $\Lambda_{12}{}^u\Psi$ (or $\Lambda_{12}{}^l\Psi$), whose norm is given by $(\Lambda_{12}{}^u\Psi|\Lambda_{12}{}^u\Psi)$. Since $\Lambda_{12}{}^u$ is Hermitian and idempotent, this norm is just

$$(\Psi|\Lambda_{12}{}^{u+}\Lambda_{12}{}^u|\Psi) = (\Psi|\Lambda_{12}{}^u\Lambda_{12}{}^u|\Psi) = (\Psi|\Lambda_{12}{}^u|\Psi).$$

[10] Angular correlation in $p+\bar{p}\to 2\pi^+ + 2\pi^- + n\pi^0$ have been calculated by G. Goldhaber, S. Goldhaber, W. Lee and A. Pais, Phys. Rev. **120**, 300 (1960).

[11] Professor R. Serber (private communication).

We therefore have to compute the four expectation values which appear in the right hand side of (47) and (48). These operators are true scalars, depend only on particles "1" and "2" and commute with the momentum operator $P_\mu{}^{(i)}$ of the ith particle, and so their expectation values are given by formula (44).

According to Eq. (44) we must compute the mean value of an operator in the states Φ_1, Φ_2. In the two cases we shall consider, V_{12} is diagonal in the momenta $P_\mu{}^{(1)}$, $P_\mu{}^{(2)}$ of pions "1" and "2." The states Φ_1 and Φ_2 may be written as

$$\begin{aligned}\Phi_1 &= L^{(12)}(v)R_2{}^{(34)}(-\pi)L^{(34)}(v)\rho_0{}^{0,1}(1,2)\rho_0{}^{0,1}(3,4),\\ \Phi_2 &= L^{(13)}(v)R_2{}^{(24)}(-\pi)L^{(24)}(v)\rho_0{}^{0,1}(1,3)\rho_0{}^{0,1}(2,4),\end{aligned} \tag{50}$$

where $L^{(ij)}(v)$ is a Lorentz operator which acts only on particles "i" and "j." Making use of (21), (24), and (29), we can rewrite this as

$$\begin{aligned}\Phi_1 &= L^{(12)}(v)L^{(34)}(-v)\rho_0{}^{01}(1,2)\rho_0{}^{0,-1}(3,4),\\ \Phi_2 &= L^{(13)}(v)L^{(24)}(-v)\rho_0{}^{01}(1,3)\rho_0{}^{0,-1}(2,4).\end{aligned} \tag{51}$$

Thus, Φ_1, Φ_2 may be written as Lorentz operators acting on states of two ρ mesons at rest. We shall, in computing expectation values, take the operator off the rest frame states, and compute the "transformed V_{12}" in the rest frame. To be more specific, we have

$$\begin{aligned}\Phi_1 &= L_a\rho_0{}^{01}(1,2)\rho_0{}^{0,-1}(3,4),\\ \Phi_2 &= L_b\rho_0{}^{01}(1,3)\rho_0{}^{0,-1}(2,4),\end{aligned} \tag{52}$$

with

$$\begin{aligned}L_a &= L^{(12)}(v)L^{(34)}(-v),\\ L_b &= L^{(13)}(v)L^{(24)}(-v),\end{aligned} \tag{53}$$

and the "transformed V_{12}" is just $L_a{}^{-1}V_{12}L_a$ (and $L_b{}^{-1}V_{12}L_b$), and these are to be computed in the states $\rho_0{}^{01}(1,2)\rho_0{}^{0,-1}(3,4)$ [and $\rho_0{}^{0,1}(1,3)\rho_0{}^{0,-1}(2,4)$].

From the fact that $\Lambda_{12}{}^u$, $\Lambda_{12}{}^l$ both commute with Lorentz transformations, and from the definitions of the rest states $\rho_0{}^{q\lambda}(1,2)$ in Eq. (13), we obtain

$$\begin{aligned}(\Phi_1|\Lambda_{12}{}^u|\Phi_1) &= \|\Phi\| \equiv (\Phi_1,\Phi_1),\\ (\Phi_2|\Lambda_{12}{}^u|\Phi_2) &= \tfrac{1}{2}\|\Phi\|,\\ (\Phi_1|\Lambda_{12}{}^l|\Phi_1) &= 0,\\ (\Phi_2|\Lambda_{12}{}^l|\Phi_2) &= \tfrac{1}{2}\|\Phi\|,\\ (\Phi_2|\Lambda_{12}{}^u R_2(\pi)P|\Phi_2) &= 0,\\ (\Phi_2|\Lambda_{12}{}^l R_2(\pi)P|\Phi_2) &= 0,\end{aligned} \tag{54}$$

and if we use the *operator* equations

$$L_a{}^{-1}\sum_{i=1}^{3} P_i{}^{(1)}P_i{}^{(2)}L_a = \sum_{i=1}^{3} P_i{}^{(1)}P_i{}^{(2)} - P_0{}^{(1)}P_0{}^{(2)} + L_a{}^{-1}P_0{}^{(1)}P_0{}^{(2)}L_a \tag{55}$$

$$L_a{}^{-1}\Big[\sum_{i=1}^{3} P_i{}^{(1)}P_i{}^{(1)}\Big]^{-1/2}L_a = \big[-1 + L_a{}^{-1}P_0{}^{(1)}P_0{}^{(1)}L_a\big]^{-1/2}, \tag{56}$$

and similar equations for L_b, and use Eq. (5), as well as note that in our space the invariant pion mass operator $\sum_{i=1}^{3} P_i^{(1)}P_i^{(1)}-P_0^{(1)}P_0^{(1)}$ is just the negative of the identity, -1, we finally obtain, after some manipulation,

$$(\Phi_1|\Lambda_{12}{}^uC_{12}|\Phi_1)=\|\Phi_1\|\int d\Omega\frac{|Y_{11}(\theta,\varphi)|^2[\gamma^2(\omega^2-v^2p^2\cos^2\theta)-(p^2+\omega^2)]}{\{[\gamma^2(\omega+vp\cos\theta)^2-1][\gamma^2(\omega-vp\cos\theta)^2-1]\}^{1/2}} \tag{57}$$

$$(\Phi_2|\Lambda_{12}{}^uC_{12}|\Phi_2)=\tfrac{1}{2}\|\Phi_1\|\int\frac{d\Omega_1d\Omega_2|Y_{11}(\theta_1\varphi_1)|^2|Y_{1-1}(\theta_2,\varphi_2)|^2[\gamma^2(p\cos\theta_1+v\omega)(p\cos\theta_2-v\omega)]}{\{[\gamma^2(\omega+vp\cos\theta_1)^2-1][\gamma^2(\omega-vp\cos\theta_2)^2-1]\}^{1/2}}, \tag{58}$$

where p, ω are the magnitude of the momentum and energy, respectively, of a pion in the rest frame of a ρ meson; v is the velocity of the ρ in the proton-antiproton rest frame and $\gamma=1/(1-v^2)^{1/2}$. The numerators of the above equations are just the Lorentz transforms of $\mathbf{p}_1\cdot\mathbf{p}_2$, the denominators are the Lorentz transforms of $|\mathbf{p}_1||\mathbf{p}_2|$, and we have omitted the Lorentz transforms of $p_{1x}p_{2x}+p_{1y}p_{2y}$ in (58), since their azimuthal integrals vanish.

In exactly the same way, one finds expressions for the other necessary matrix elements

$$\begin{aligned}&(\Phi_1|\Lambda_{12}{}^lC_{12}|\Phi_1)=0,\\&(\Phi_2|\Lambda_{12}{}^lC_{12}|\Phi_2)=(\Phi_2|\Lambda_{12}{}^uC_{12}|\Phi_2).\end{aligned} \tag{59}$$

The integrals which describe $(\Phi_2|\Lambda_{12}C_{12}R_2(\pi)P|\Phi_2)$ vanish for $\Lambda_{12}{}^u$ and $\Lambda_{12}{}^l$ because of azimuthal integrations in the ρ-meson rest frames.

Comparing (47), (48), (54), (44), we obtain

$$\langle\cos\theta_{12}{}^u\rangle=\frac{1}{2}\left[\int d\Omega\frac{|Y_{11}(\theta,\varphi)|^2[\gamma^2(\omega^2-v^2p^2\cos^2\theta)-(p^2+\omega^2)]}{\{[\gamma^2(\omega+vp\cos\theta)^2-1][\gamma^2(\omega-vp\cos\theta)^2-1]\}^{1/2}}\right]+\tfrac{1}{2}\langle\cos\theta_{12}{}^l\rangle, \tag{60}$$

$$\langle\cos\theta_{12}{}^l\rangle=\int\frac{d\Omega_1d\Omega_2|Y_{11}(\theta_1\varphi_1)|^2|Y_{1,-1}(\theta_2\varphi_2)|^2[\gamma^2(p\cos\theta_1+v\omega)(p\cos\theta_2-v\omega)]}{\{[\gamma^2(\omega+vp\cos\theta_1)^2-1][\gamma^2(\omega-vp\cos\theta_2)^2-1]\}^{1/2}}. \tag{61}$$

Physically, Eqs. (60) and (61) may be understood by noting that particles "1" and "2" may have like charges only if they "belong" to different ρ mesons. Thus $\langle\cos\theta^l\rangle$ is the expectation value of $\cos\theta$ for pions of different ρ mesons. However "1" and "2" may be oppositely charged if they belong to the same ρ, as well as if they belong to different ρ. The integral in (60) is just the expectation value of $\cos\theta$ for pions emitted from the same moving ρ meson. Note that in the limit $\gamma\rightarrow\infty$, $v\rightarrow 1$ we obtain the value $\langle\cos\theta_{12}{}^l\rangle=-1.00$, and $\langle\cos\theta_{12}{}^u\rangle=(+1-1)/2=0$ which corresponds, as it should, to one $\pi^+\pi^-$ pair moving parallel to each other in one direction and another $\pi^+\pi^-$ pair moving parallel to each other in the opposite direction. We should also like to observe here that the mean value of $\cos\theta_{12}$, without charge correlations is given by $-\frac{1}{3}$ in this limiting case in complete agreement with Serber's argument. If we take the limit $\gamma\rightarrow 1$, $v\rightarrow 0$ in (60) and (61) we again obtain $-\frac{1}{3}$ for $\langle\cos\theta_{12}\rangle$ without charge correlations.

The integral in (60) may be written in terms of standard elliptic integrals and the integral in (61) may be evaluated in terms of elementary functions. For the case of antiproton annihilation at rest, we obtain[12]

$$\langle\cos\theta_{12}{}^l\rangle=-0.29\quad\langle\cos\theta_{12}{}^u\rangle=-0.31. \tag{62}$$

Note that these values too, agree with Serber's argument, although not exactly.

[12] We wish to point out that the values of $\cos\theta_{12}{}^u$ and $\cos\theta_{12}{}^l$ quoted here differ from the values quoted in the original dissertation p. 76 and p. 87. The present values are the correct ones.

VI. MASS CORRELATIONS: TWO-PION EFFECTIVE MASS DISTRIBUTION

In this section we shall compute the probability distribution for the two particle effective mass. The mass operator S_{12} is defined in terms of the momenta by

$$\begin{aligned}S_{12}&=(P_0^{(1)}+P_0^{(2)})^2-\sum_{i=1}^{3}(P_i^{(1)}+P_i^{(2)})^2,\\&=2+2(P_0^{(1)}P_0^{(2)}-\mathbf{P}^{(1)}\cdot\mathbf{P}^{(2)}).\end{aligned} \tag{63}$$

For a ρ meson we shall clearly have

$$(\rho(1,2)|S_{12}|\rho(1,2))=M^2(\rho(1,2)|\rho(1,2)) \tag{64}$$

where M is the mass of the ρ. We shall compute the mass distribution for unlike ($+-$ and $-+$) charges as well as for like ($++$ and $--$) charges.

The mass distributions in a state Ψ are given by

$$f^u(S')=\frac{(\Psi|\delta(S'-S_{12})\Lambda_{12}{}^u|\Psi)}{(\Psi|\Lambda_{12}{}^u|\Psi)} \tag{65}$$

$$f^l(S')=\frac{(\Psi|\delta(S'-S_{12})\Lambda_{12}{}^l|\Psi)}{(\Psi|\Lambda_{12}{}^l|\Psi)} \tag{66}$$

where the $\Lambda_{12}{}^u(\Lambda_{12}{}^l)$ have been defined in Eq. (46) to guarantee that particles "1" and "2" have opposite charge (have the same charge). $\delta(S'-S_{12})$ is an operator since S_{12} is an operator, and is diagonal in the momentum representation. Upon inserting a complete set of states we can show easily that

$$\int_0^\infty f^u(S')dS'=\int_0^\infty f^l(S')dS'=1; \tag{67}$$

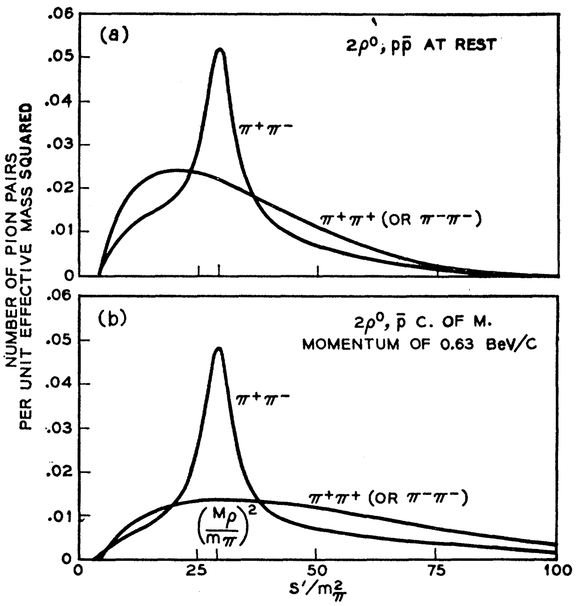

FIG. 1. Calculated di-pion effective mass distributions for $p+\bar{p}\to 2\rho^0\to 2\pi^+ + 2\pi^-$ plotted against the square of the effective mass s', in units where the π^+ mass is unity. The $\pi^+\pi^-$ distribution is one-half of a relativistic Breit-Wigner resonance plus one-half of the $\pi^+\pi^+$ distribution: (a) corresponds to $p\bar{p}$ annihilation at rest; (b) corresponds to a $\bar{p}$ center-of-mass momentum 0.63 BeV/c.

that is, the distributions are normalized.

The operator $V_{12}=\Lambda_{12}\delta(S'-S_{12})$ is a rotational scalar, commutes with parity P, and with the momentum operators, and further depends only on particles "1" and "2." Therefore, by the discussion of Sec. IV above, all of the matrix elements of (65) and (66) satisfy Eq. (44).

As in the case of the angular correlations, we shall "take the Lorentz operators off the state vectors" and put them on $\delta(S'-S_{12})$ and then write the matrix elements as integrals in the ρ-meson rest frames. The first matrix element in Eq. (44) is quite trivial according to (64) and yields

$$(\Phi_1|\Lambda_{12}{}^u\delta(S'-S_{12})|\Phi_1)=\delta(S'-M^2)\|\Phi_1\|, \quad (68)$$

with $\|\Phi_1\|$ being the norm, (Φ_1,Φ_1). As the probability of "1" and "2" having the same charge is zero in Φ_1, we also see that

$$(\Phi_1|\Lambda_{12}{}^l\delta(S'-S_{12})|\Phi_1)=0. \quad (69)$$

For the other two terms, we note that

$$L_b{}^{-1}S_{12}L_b=L^{(1)}(v)^{-1}L^{(2)}(-v)^{-1}S_{12}L^{(1)}(v)L^{(2)}(-v), \quad (70)$$

where L_b is defined in (53) and $L^{(i)}(\pm v)$ is a Lorentz transformation along the z axis for the ith particle, corresponding to velocity v.

Hence, we have

$$\begin{aligned}L_b{}^{-1}S_{12}L_b=&2\gamma^2(P_0{}^{(1)}+vP_3{}^{(1)})(P_0{}^{(2)}-vP_3{}^{(2)})\\&-2\gamma^2(P_3{}^{(1)}+vP_0{}^{(1)})(P_3{}^{(2)}-vP_0{}^{(2)})\\&-2P_1{}^{(1)}P_1{}^{(2)}-2P_2{}^{(1)}P_2{}^{(2)}+2, \quad (71)\end{aligned}$$

where, as usual, $\gamma^2=(1-v^2)^{-1}$ and we have used the fact that the single-particle mass is unity.

Using (71) and (52), we obtain

$$\begin{aligned}(\Phi_2|\Lambda_{12}{}^u\delta(S'-S_{12})|\Phi_2)&\\=(\Phi_2|\Lambda_{12}{}^l\delta(S'-S_{12})|\Phi_2)&\\=\|\Phi_1\|\int d\Omega_1 d\Omega_2|Y_{11}(\theta_1\varphi_1)|^2&\\\times|Y_{1,-1}(\theta_2\varphi_2)|^2\delta(S'-\bar{S}_{12})& \quad (72)\end{aligned}$$

with the *number* $\bar{S}_{12}$ given by

$$\begin{aligned}\bar{S}_{12}=2+2\gamma^2\{&(\omega+vp\cos\theta_1)(\omega-vp\cos\theta_2)\\&-(p\cos\theta_1+v\omega)(p\cos\theta_2-v\omega)\}\\&-2p^2\sin\theta_1\sin\theta_2\cos(\varphi_1-\varphi_2), \quad (73)\end{aligned}$$

where p, ω are the momentum and energy of a pion in the ρ rest frame.

From Eq. (52) and the fact that $PR_2(\pi)$ commutes with Lorentz transformations along the z axis, we learn that

$$PR_2(\pi)\Phi_2=L_b\rho_0{}^{0,-1}(1,3)\rho_0{}^{0,1}(2,4) \quad (74)$$

and hence

$$\begin{aligned}(\Phi_2|\Lambda_{12}{}^u\delta(S'-S_{12})PR_2(\pi)|\Phi_2)&\\=(\Phi_2|\Lambda_{12}{}^l\delta(S'-S_{12})PR_2(\pi)|\Phi_2)&\\=\tfrac{1}{2}\|\Phi_1\|\int d\Omega_1 d\Omega_2 Y_{11}{}^*(\theta_1\varphi_1)Y_{1,-1}(\theta_1\varphi_1)&\\\times Y_{1,-1}{}^*(\theta_2\varphi_2)Y_{11}(\theta_2\varphi_2)\delta(S'-\bar{S}_{12}),& \quad (75)\end{aligned}$$

with $\bar{S}_{12}$ given as before by (73).

Taking into account (54), (44), (68), (69), (72), (75) we have

$$f^u(S')=\tfrac{1}{2}[\delta(S'-M^2)+f^l(S')]$$

$$\begin{aligned}f^l(S')=&\int d\Omega_1 d\Omega_2\delta(S'-\bar{S}_{12})|Y_{11}(\theta_1\varphi_1)|^2|Y_{1,-1}(\theta_2\varphi_2)|^2\\&-\int d\Omega_1 d\Omega_2\delta(S'-\bar{S}_{12})Y_{11}(\theta_1\varphi_1)^*Y_{1,-1}(\theta_1\varphi_1)\\&\times Y_{1,-1}(\theta_2\varphi_2)^*Y_{11}(\theta_2\varphi_2). \quad (76)\end{aligned}$$

Equations (76) show clearly the effect of the ρ resonance on the two particle mass distribution. The curve $f^l(S')$ for like pions will be some smooth function which is mostly determined by phase space. The curve $f^u(S')$ for $\pi^+\pi^-$ pairs consists of this smooth background function together with the δ-function peak at the mass of the ρ squared. It is clear that if we took into account the width Γ of the ρ, the δ function would appear as a Breit-Wigner resonance curve. Furthermore, if we considered the reaction

$$p+\bar{p}\to\rho^0+\pi^++\pi^-$$

we would have similar results except that whereas the resonance is one-half of f^u for two ρ^0, the resonance would be only one-quarter of f^u for one ρ^0 and 2π. There would also be differences because the interference effects between the π's which belong to the ρ's and the π's which do not, may not be negligible.

Returning to the integrals we introduce the variables

$$\begin{aligned} &x\equiv\cos\theta_1\\ &y\equiv\cos\theta_2\\ &t\equiv(1/2\gamma^2\omega^2)(S'-2)\\ &u\equiv p/\omega=\text{velocity of a pion in }\rho\text{-meson rest frame}\\ &\alpha\equiv t-[(1+v^2)(1-u^2xy)+2vu(x-y)]\\ &\xi\equiv u^2\gamma^{-2}[(1-x^2)(1-y^2)]^{1/2}, \end{aligned} \tag{77}$$

then from (73)

$$\delta(S'-\bar{S}_{12})=(1/2\gamma^2\omega^2)\delta(\alpha+\xi\cos(\varphi_1-\varphi_2)). \tag{78}$$

Clearly, for $|\alpha|>\xi$, $\delta(S'-\bar{S}_{12})$ vanishes, and one may easily show that

$$\begin{aligned} &\int_0^{2\pi}d\varphi_1\int_0^{2\pi}d\varphi_2\delta(\alpha+\xi\cos(\varphi_1-\varphi_2))\\ &\qquad=\frac{4\pi}{(\xi^2-\alpha^2)^{1/2}}\theta(\xi-|\alpha|),\\ &\int_0^{2\pi}d\varphi_1\int_0^{2\pi}d\varphi_2\delta(\alpha+\xi\cos(\varphi_1-\varphi_2))e^{-2i(\varphi_1-\varphi_2)}\\ &\qquad=\frac{4\pi[2(\alpha^2/\xi^2)-1]}{(\xi^2-\alpha^2)^{1/2}}\theta(\xi-|\alpha|), \end{aligned} \tag{79}$$

where $\theta(z)$ is the step function which is unity for positive z and zero for negative z. Using (79) we may perform the azimuthal integrations in (76), and combining the two terms of the integrand, and writing the spherical harmonics out we obtain

$$\begin{aligned} f^l(S')&=\frac{9}{32\pi\gamma^2\omega^2}\int_{-1}^{1}dx\int_{-1}^{1}dy\frac{(1-x^2)(1-y^2)}{(\xi^2-\alpha^2)^{1/2}}\\ &\qquad\times\theta(\xi-|\alpha|)\left[1-\left(2\frac{\alpha^2}{\xi^2}-1\right)\right],\\ &=\frac{9}{16\pi\gamma^2\omega^2}\int_{-1}^{1}dx\int_{-1}^{1}dy\frac{(1-x^2)(1-y^2)}{\xi^2}\\ &\qquad\times\theta(\xi-|\alpha|)(\xi^2-\alpha^2)^{1/2}, \end{aligned} \tag{80}$$

and upon noting the definition of ξ in (77) we get

$$f^l(S')=\frac{9\gamma^2}{16\pi u^4\omega^2}\int_{-1}^{1}dx\int_{-1}^{1}dy(\xi^2-\alpha^2)^{1/2}\theta(\xi-|\alpha|). \tag{81}$$

The quantity $\xi^2-\alpha^2$ occurring in (76) is a quadratic in y which opens downward and which vanishes only in the interval $-1\leq y\leq 1$ for the values of x and S' which are of interest. Thus, one may do the y integration in (81) between the roots (in y) of $\xi^2-\alpha^2$, and ignore the step function. The remaining x integration involves a quartic divided by a quadratic raised to the $\frac{3}{2}$ power and may also be done analytically.

In Fig. 1 we have plotted the distribution $f^l(S')$ for $\pi^+\pi^+$ pairs as well as the $\bar{f}^u(S')$, where $\bar{f}^u(S')$ is defined

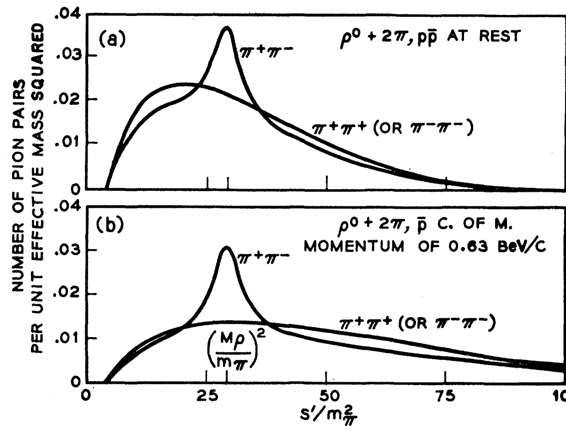

FIG. 2. Calculated di-pion effective mass distribution for $p+\bar{p}\to\rho^0+\pi^++\pi^-\to 2\pi^++2\pi^-$ plotted against the square of the effective mass s', in units where the π^+ mass is unity. The $\pi^+\pi^-$ distribution is one-fourth of a relativistic Breit-Wigner resonance plus three-fourths of the $\pi^+\pi^+$ distribution (which is assumed to be the same as in Fig. 1): (a) corresponds to $p\bar{p}$ annihilation at rest; (b) corresponds to a $\bar{p}$ center of mass momentum of 0.63 BeV/c.

by

$$\bar{f}^u(S')=\tfrac{1}{2}(M\Gamma/\pi)/[(S'-M^2)^2+M^2\Gamma^2]+\tfrac{1}{2}f^l(S') \tag{82}$$

and is obtained from $f^u(S')$ by replacing the δ function by the more realistic Breit-Wigner resonance curve $[M\Gamma/\pi]/[(S'-M^2)^2+M^2\Gamma^2]$.

If we consider the reaction $p\bar{p}\to\rho^0+\pi^++\pi^-$, then we can estimate that the $+-$ mass distribution would be

$$\tfrac{1}{4}(M\Gamma/\pi)/[(S'-M^2)^2+M^2\Gamma^2]+\tfrac{3}{4}F^l(S') +\text{interference terms}. \tag{83}$$

If we argue that $F^l\sim f^l$ and ignore the interference terms, then the mass distributions would be as shown in Fig. 2.

It is not possible to compare our results with those published by Button *et al.*[13] as their 4π events correspond to only a small fraction of the total events presented. Figures 1 and 2 clearly show how the need to measure all possible pion pairs masks the resonance. We have taken $\Gamma=100$ MeV, but clearly this masking effect would be increased if Γ were larger, as had been reported in the original measurements.

We shall conclude this section with some comments on the relationship of the interference terms to Bose statistics.[14] The "background" terms in f^u would appear even if the state were not symmetric in all of the pion variables, because in computing f^u we use *all* unlike pion pairs, including those in which the pions belong to different ρ mesons. In the case that the state were not symmetrized there would obviously be no interference terms between states where pions "1" and "2" belong to the same ρ, and states where pions "1"

[13] J. Button, G. Kalbfleisch, G. Lynch, B. Maglić, A. Rosenfeld, and L. Stevenson, Phys. Rev. **126**, 1858 (1962).

[14] The author is indebted to Professor R. Serber for the comments which follow.

and "3" belong to the same ρ. Thus, although in our calculation we have postulated a symmetric state, we have neglected these interference terms, which are truly the effect of Bose statistics. Therefore, the difference between our calculated f^u and f^l is not to be attributed to Bose statistics, but rather to the definitions of these distributions which take into account all possible pion pairs. The terms which come from Bose statistics are the interference terms.

VII. CONCLUSION

We have considered the $p\bar{p}$ annihilation into $2\rho^0$ as a means of determining the effect of measuring all possible pion pairs, which is to mask the resonances. We have found that the "background" mass distribution which comes from pions that belong to different ρ's is peaked near the resonance peak and is in fact 42% of the distribution at the resonance peak. Upon extrapolating our results to $p\bar{p}\to\rho^0+\pi^++\pi^-$, by assuming that the $\pi^+\pi^+$ mass distribution does not differ too much from that which we found for $2\rho^0$, the "background" is 59% of the mass distribution. Furthermore, although we have assumed that the $p\bar{p}$ annihilation proceeds from an S state, we believe that the results for the mass distributions would be only slightly changed if some other l value were assumed, because the available energy-momentum phase space is the important factor.

With regard to the angular correlations, we found that Serber's argument, which yields $\langle\cos\theta_{12}\rangle \cong -1/(n-1)$ for an n-particle system in the center of mass, very accurately describes our results for the four-pion final state. For $p-\bar{p}$ annihilation at rest into $2\rho^0$, we found that there is very little difference between the mean value of $\cos\theta_{12}$ for $\pi^+\pi^+$ pairs and the mean value for $\pi^+\pi^-$ pairs.[15]

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APPENDIX

In this Appendix we shall estimate the order of magnitude of the interference terms which were omitted in deriving Eq. (44) of Sec. IV. The argument depends completely upon the size of the regions of the allowed phase space and so we neglect isotopic spin and total angular momentum factors.

Let $F_1(\mathbf{p}_1,\mathbf{p}_2|\mathbf{p}_3,\mathbf{p}_4)$ be the amplitude for a ρ meson of pions "1" and "2" moving in some direction with momentum k and energy E, together with a ρ meson of pions "3" and "4" moving in the opposite direction with momentum k and energy E. Then if the ρ has zero width,

$$F_1(\mathbf{p}_1,\mathbf{p}_2|\mathbf{p}_3,\mathbf{p}_4)\propto\delta^{(3)}(\sum\mathbf{p}_i)\delta(2E-\sum\omega_i)\times\delta(S_{12}-M^2)\delta(S_{34}-M^2),\quad\text{(A1)}$$

where

$$S_{ij}=(\omega_i+\omega_j)^2-(\mathbf{p}_i+\mathbf{p}_j)^2.$$

Let F_2 be a similar amplitude, which differs from F_1 in that pions "2" and "3" have exchanged roles:

$$\begin{aligned}F_2(\mathbf{p}_1,\mathbf{p}_2|\mathbf{p}_3,\mathbf{p}_4)&=F_1(\mathbf{p}_1,\mathbf{p}_3|\mathbf{p}_2,\mathbf{p}_4),\\ F_2(\mathbf{p}_1,\mathbf{p}_2|\mathbf{p}_3,\mathbf{p}_4)&\propto\delta^{(3)}(\sum\mathbf{p}_i)\delta(2E-\sum\omega_i)\times\delta(S_{13}-M^2)\delta(S_{24}-M^2).\end{aligned}\quad\text{(A2)}$$

The region R_i in which $F_i\neq0$ is a six-dimensional hypersurface embedded in the twelve-dimensional space of points $(p_{1x},\cdots,p_{4z})$. Consider the region $R_1\cap R_2$ (the intersection of R_1 and R_2), where simultaneously $F_1\neq0$, and $F_2\neq0$. In addition to the four equations of total energy-momentum conservation, we have

$$S_{12}=M^2,\quad S_{34}=M^2,\quad S_{13}=M^2,\quad S_{24}=M^2,\quad\text{(A3)}$$

so that $R_1\cap R_2$ will be, in general, of dimension four.

If we smear out the ρ mass by an amount Γ, then R_i will become an eight-dimensional region whose volume is proportional to Γ^2 and the intersection $R_1\cap R_2$ will become an eight-dimensional region whose volume is proportional to Γ^4.

As the energy E of each ρ increases, the momenta of the pions emitted from the same ρ are almost parallel and are opposite to the momenta of the pions emitted from the other ρ. For sufficiently large E, there are no values of momenta which satisfy A3 and energy momentum conservation; that is, the volume of $R_1\cap R_2$ tends to zero as E increases. It is therefore plausible that the volume of $R_1\cap R_2$ is proportional to $\Gamma^2(\Gamma/E)^2$ relative to a volume Γ^2 of R_1.[16] Thus, if $V_{12}(\mathbf{p}_i)$ is a nonsingular function of the momenta, then

$$\frac{\int F_1{}^*V_{12}F_2}{\int F_1{}^*V_{12}F_1}=\frac{\int_{R_1R_2}F_1{}^*V_{12}F_2}{\int_{R_1}F_1{}^*V_{12}F_1}\lesssim\left(\frac{\Gamma}{M}\right)^2.\quad\text{(A4)}$$

The ratio of interference terms $(\Phi_i|V|\Phi_j)$ to the terms $(\Phi_i|V|\Phi_i)$ is thus seen to vanish as $\Gamma\to0$, and is probably less than $(\Gamma/M)^2=(100\text{ MeV}/750\text{ MeV})^2\approx0.02$. The neglect of the interference terms in the deduction of Eq. (44) of Sec. IV above is thus justified.

[15] We do not believe it reasonable to extrapolate our results on the angular correlations to the case of $\rho+2\pi$, because in this case there are a large number of possible values of the relative angular momentum between the pions which do not form the ρ, and the angular correlations are sensitive to this.

[16] Upon restricting the ρ mesons to motion along the z axis, we can more clearly compute the volume of R_1 and the volume of $R_1\cap R_2$, and we indeed find a ratio of $(\Gamma/E)^2$. In Eq. A4 we use the more modest upper bound of $(\Gamma/M)^2$.