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Time-Dependent Correlations in a Solvable Ferromagnetic Model*

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The exact equation of state, two-time correlation function, and linear response function, are calculated in the limit of infinite N for a classical N-spin system with a ferromagnetic phase transition, in a particular nonuniform magnetic field. The correlation function can be analytically continued in temperature (or magnetic-field strength) from the nonferromagnetic to the ferromagnetic region of the T-H plane; the result of such continuation is not, however, the correlation function for the ferromagnetic region, but a function which grows exponentially in time. The frequency-dependent linear response function has a pole at zero frequency throughout the ferromagnetic region due to a broken symmetry; the corresponding function in the nonferromagnetic region developes a pole at zero frequency as the ferromagnetic region is approached, but when the function is continued in temperature (or field strength) into the ferromagnetic region, the pole detaches itself from the origin and moves up into the complex frequency plane, signifying an exponential growth in time of the linear response. The purpose of the model is to demonstrate that this kind of behavior does not contradict any general structural properties of equilibrium thermodynamic correlation or response functions. The possible general significance of such behavior for a theory of metastable states is discussed.

I. INTRODUCTION

In this paper the exact equation of state, two-time correlation functions, and linear response functions are calculated, in the limit of infinite N, for a simple N-particle system which undergoes a phase transition. The model establishes the possibility that, in a system capable of a phase transition, instabilities—i.e., exponentially growing time dependence—can be associated with exact equilibrium correlation and response functions.

This conclusion will be put in a mathematically precise way in Sec. II, but I would first like to give a less formal description of the result, since it is a simple one which might be obscured by the number of definitions required to state it with care. The system we will examine has the following properties:

(a) It possesses a phase transition; i.e., in the limit as $N\rightarrow\infty$, some derivatives of the free energy become discontinuous at certain values of the temperature (and other parameters necessary to determine the thermodynamic state). (We call such values of the thermodynamic parameters transition points.)

(b) The equilibrium two-time correlation functions and linear response functions, considered for fixed time as a function of the thermodynamic parameters, are, in the limit of infinite N, analytic at all real values of the parameters except the transition points.

(c) The unique result of analytically continuing a correlation or linear response function in a thermodynamic parameter through a transition point, is not the equilibrium function on the other side of the transition point; instead, such a procedure leads to a function which grows exponentially in time.

It is the last property that interests us. Since the model is strikingly unlike anything to be found in nature, this result would appear to be of little consequence. Its importance lies in the fact that it establishes the *mathematical possibility* of such behavior. For although there is no reason to reject *a priori* the occurrence of instabilities in analytic continuations of correlation or response functions through transition points, when found in approximate calculations, they have generally been blamed on the inadequacy of the approximation.[1,2] In a sense this is correct, since an unstable response function signifies that the approximation may be giving the response of a state that is not the true equilibrium state. On the other hand, the possibility has not, to my knowledge, been considered, that the dynamically unstable response or correlation functions associated with this thermodynamically unstable (or possibly metastable) state, may be found from the exact equilibrium functions by analytic continuation. This is probably due to the valid belief that an exact correlation function describes only thermodynamically stable states with stable linear response.[3] This view obscures but does not prohibit the possibility that dynamic instabilities may nevertheless be implicit in the exact functions, as described in (c).

This paper might therefore be regarded as an existence proof. We shall produce a Hamiltonian which leads to exact equations of state and two-time functions having properties (a)–(c), thereby demonstrating that the association of exponential growth with such functions is not in contradiction to any of their general structural properties.

There are two kinds of reasons for suspecting that this kind of behavior may be a general feature of phase transitions. There is first the experience gained through approximate calculations. We mention three examples:

(1) If the pair correlation function is calculated (in the ladder approximation) for a Fermion system which

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[1] L. Kadanoff and P. C. Martin, Phys. Rev. **124**, 670 (1961).
[2] N. D. Mermin, Ann. Phys. (N. Y.) **18**, 421 and 454 (1962).
[3] And also because one is usually not interested in the nonequilibrium states.

has a superconducting phase transition (in the BCS approximation), the analytic continuation of its Fourier transform from above to below the transition temperature, has poles with nonvanishing imaginary parts on the physical sheet, i.e., the analytic continuation of the time dependent function grows exponentially.[1]

(2) In a crude random-phase approximation theory of the classical gas-liquid transition, it is found that the analytic continuation in volume (for fixed temperature) of the time-dependent density autocorrelation function through the transition point into the metastable region, develops an exponential time dependence as the metastable region is left and the unstable region entered.[2]

(3) Consider as a very crude model of an electron gas, a set of spin-$\frac{1}{2}$ Fermions with very short-range repulsive interactions. In the Hartree-Fock approximation such a gas is ferromagnetic. If time-dependent spin autocorrelation functions are calculated in a linearized time-dependent Hartree-Fock approximation, it is found that the imaginary part of the spin-wave poles found in the nonferromagnetic region, moves, as one analytically continues in temperature through a transition point, from the unphysical sheet in the lower half-plane to the physical sheet in the upper half-plane. This means that the nonferromagnetic correlation function, when continued into the ferromagnetic region, grows exponentially in time.[4]

In all three cases the approximate correlation functions have properties (a)–(c). It is possible to prove in each case that at values of the parameters where one finds unstable response functions, there must exist additional solutions to the same (approximate) equations which are stable.[5–7] One may therefore always reject unstable functions in favor of stable ones, without going beyond the original approximation. Nevertheless the unstable functions are implicit in the stable ones, can be recovered by analytic continuation in the thermodynamic parameters, and, within the approximation, describe the response of a definite nonequilibrium state.

A second reason for establishing the possibility of property (c) is more speculative. It is generally held that the analytic continuation (or some smooth extrapolation) of the equation of state through a transition point, although it no longer describes the stable equilibrium state, may still describe a physical nonequilibrium state of the system. There seems to be no fundamental theoretical basis for this belief, but it is convincingly supported experimentally.[8] If one accepts it, one may ask whether analytic continuations through transition points of more complicated properties of the equilibrium state, will describe the corresponding properties of this nonequilibrium state. If this is so, then a linear response function for the equilibrium state, provided it describes the response to a perturbation capable of destroying the nonequilibrium state, should, when continued through a transition point, develop an unbounded growth in time.

No answers will be given here to the difficult question of whether analytic continuations through transition points do, in general, describe physical nonequilibrium states, or to the question of whether realistic systems have property (c). What I wish to offer is a first step toward the consideration of these problems: a model that establishes the consistency of exponential growth in analytically continued two-time functions, and which is suggestive of further problems that will have to be faced in deciding whether such analytic continuations are of more general significance.

II. THE MODEL

The model consists of a set of N spins (N even), half of which are in a magnetic field $\mathbf{H}$ directed along the positive z axis, and the other half, in a field $-\mathbf{H}$. Their interaction energy is to be negative and proportional to the square of the total spin. Thus the Hamiltonian is[9]

$$\mathcal{H}=-\sum_{i=1}^{N}\mathbf{H}\cdot\mathbf{s}^i+\sum_{i=\frac{1}{2}N+1}^{N}\mathbf{H}\cdot\mathbf{s}^i-\frac{g}{2N}(\sum_{i=1}^{N}\mathbf{s}^i)^2,\quad(2.1)$$

and the equations of motion,

$$\dot{\mathbf{s}}^i=\mp\mathbf{H}\times\mathbf{s}^i-\frac{g}{N}(\sum_{j=1}^{N}{}'\,\mathbf{s}^j)\times\mathbf{s}^i,\quad\begin{pmatrix}i=1,\cdots\frac{1}{2}N\\ i=\frac{1}{2}N+1,\cdots N\end{pmatrix}.\quad(2.2)$$

(The interaction strength must be proportional to $1/N$ in order that the mean energy per spin be independent of N, as $N\rightarrow\infty$.) This is just the Hamiltonian for a Weiss-model ferromagnet in a particular nonuniform magnetic field.[10] The customary way of finding its equilibrium behavior, via a self-consistent molecular field, is exact only in the limit of infinite N; since we shall need to study the behavior for large but finite N, a more thorough analysis is necessary.

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[4] The static Hartree-Fock stability of this model is considered at zero temperature by D. J. Thouless, *The Quantum Mechanics of Many Body Systems* (Academic Press Inc., New York, 1961). I know of no discussions in the literature of the spin-wave stability at nonzero temperatures.

[5] D. J. Thouless, Nucl. Phys. **22**, 78 (1961).

[6] D. J. Thouless, Ann. Phys. (N. Y.) **10**, 553 (1960).

[7] N. D. Mermin, Ann. Phys. (N. Y.) **21**, 99 (1963).

[8] An obvious example of this is an equation of state of the van der Waals type, which can describe a metastable supercooled gas. The equilibrium equation of state, obtained from the van der Waals equation by applying the Maxwell construction, makes no reference to the metastable states, but they can be recovered from it by extrapolation through a transition point.

[9] We measure H in units such that the energy of a spin $\mathbf{s}$ in the magnetic field is just $-\mathbf{H}\cdot\mathbf{s}$.

[10] The Weiss model in a uniform magnetic field does not lead to growing correlation and response functions. The explanation for this is mentioned in Sec. VI, part E.

An understanding of the relevant (for our purposes) properties of the system is made much easier by taking the spins to be classical variables. Thus the $\mathbf{s}^i$ are vectors of fixed magnitude σ, and orientation determined by their initial values and the equations of motion. [The fixed magnitude is, of course, consistent with (2.2).] We take the spin system to be in thermal equilibrium, so the mean value of any function $F(\mathbf{s}^i)$ is given by the canonical average

$$\langle F(\mathbf{s}^i)\rangle = \int \prod_{i=1}^{N} d\Omega_i \exp[-\beta\mathcal{H}(\mathbf{s}^i)]F(\mathbf{s}^i) \Big/ \int \prod_{i=1}^{N} d\Omega_i \exp[-\beta\mathcal{H}(\mathbf{s}^i)]; \quad (2.3)$$

$\beta = 1/KT$, and the integrations are over all orientations of each spin. The time-dependent spin autocorrelation function is defined to be

$$\mathcal{G}^{ij}(t) = \lim_{N\to\infty} N\langle(\mathbf{s}^i(\{\mathbf{s}^k\},t) - \mathbf{s}^i)\mathbf{s}^j\rangle. \quad (2.4)$$

Several remarks should be made about (2.4):

(a) $\mathbf{s}^i(\{\mathbf{s}^k\},t)$ is the ith spin at time t, given that at time zero the initial spin values were $\{\mathbf{s}^k\}$. We have written it as an explicit function of the initial values to emphasize that it is these initial values that are being averaged over in (2.4). Unless there is a particular reason to emphasize this dependence, we shall use the shorter form, $\mathbf{s}^i(t)$.

(b) We shall see that the factor N is necessary to give a nonzero result. It also arises naturally if we consider not the correlation function, but the linear response function.

(c) A more conventional definition would have replaced the average in (2.4) by

$$\langle\mathbf{s}^i(t)\mathbf{s}^j\rangle - \langle\mathbf{s}^i\rangle\langle\mathbf{s}^j\rangle. \quad (2.5)$$

However the two differ only by a time-independent term which can be calculated from the equilibrium thermodynamics. In an exact calculation (2.4) seems easier to work with.

(d) As defined in (2.4) each $\mathcal{G}^{ij}$ is a 3×3 tensor. We will be interested only in particular components, e.g., $\mathcal{G}_{xy}{}^{ij}$—but it seems desirable to use the tensorial form whenever possible to keep indices to a minimum.

(e) $\mathcal{G}^{ij}$ depends on which group of spins—those in the field $\mathbf{H}$ or those in the field $-\mathbf{H}$—the ith and jth spins belong to, but not on the particular choice of spins within each group. If we define $\mathbf{s}^{(1)}$ to be the contribution to the mean spin per particle from all spins in the first group,

$$\mathbf{s}^{(1)} = \frac{1}{N}\sum_{i=1}^{\frac{1}{2}N} \mathbf{s}^i, \quad (2.6)$$

and similarly,

$$\mathbf{s}^{(2)} = \frac{1}{N}\sum_{i=\frac{1}{2}N+1}^{N} \mathbf{s}^i, \quad (2.7)$$

then we need only deal with

$$G^{\alpha\gamma}(t) = \lim_{N\to\infty} \langle(\mathbf{s}^{(\alpha)}(t) - \mathbf{s}^{(\alpha)})\mathbf{s}^{(\gamma)}\rangle, \quad \alpha,\gamma = 1,2, \quad (2.8)$$

in terms of which

$$\mathcal{G}^{ij}(t) = 4G^{11}(t), \quad 1\le i,\ j,\ \le \tfrac{1}{2}N, \text{ etc.} \quad (2.9)$$

We would also like to calculate the linear response functions. If the system, initially in thermal equilibrium, is subsequently perturbed by a weak magnetic field $\mathbf{h}^i(t)$ (which in general may vary from spin to spin), then to lowest order in $\mathbf{h}$ the change in $\langle\mathbf{s}^i\rangle$ from its equilibrium value will have the form

$$\delta\langle\mathbf{s}^i(t)\rangle = \int_{-\infty}^{t} dt' \sum_j L^{ij}(t-t')\cdot\mathbf{h}^j(t'). \quad (2.10)$$

It is a consequence of the classical fluctuation dissipation theorem[11] that the linear response tensor L is given by

$$L^{ij}(t-t') = \beta(\partial/\partial t)\langle\mathbf{s}^i(t)\mathbf{s}^j(t')\rangle. \quad (2.11)$$

It follows that the lowest order changes in $\langle\mathbf{s}^{(1)}\rangle$ or $\langle\mathbf{s}^{(2)}\rangle$ in the limit of an infinite system are

$$\delta\langle\mathbf{s}^{(\alpha)}(t)\rangle = \beta\int_{-\infty}^{t} dt' \sum_{\gamma=1}^{2} \frac{\partial}{\partial t} G^{\alpha\gamma}(t-t')\cdot\mathbf{h}^{(\gamma)}(t'), \quad (2.12)$$

where $\mathbf{h}^{(1)}$ and $\mathbf{h}^{(2)}$ are the average magnetic fields perturbing each group:

$$\mathbf{h}^{(1)}(t) = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{\frac{1}{2}N} \mathbf{h}^i(t);$$
$$\mathbf{h}^{(2)}(t) = \lim_{N\to\infty} \frac{1}{N}\sum_{i=\frac{1}{2}N+1}^{N} \mathbf{h}^i(t). \quad (2.13)$$

Evidently if the linear response function grows exponentially in time so will the correlation function, and vice versa.[12]

The function that usually arises in practical calcu-

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[11] We indicate how it can be derived for the peculiar case of a classical spin system in Appendix A. A general discussion of fluctuation dissipation theorems can be found in H. B. Callen and T. A. Welton, Phys. Rev. **83**, 34 (1951).

[12] This becomes rather puzzling if one wishes to take exponentially growing functions seriously, for although a growing response function has a simple interpretation, it is not immediately clear what one should make of an exponentially growing correlation function. The problem does not arise for nonequilibrium states since there is no fluctuation dissipation theorem to connect the two. If, however, we wish to interpret the analytic continuation of the correlation and response functions through a transition point as describing the properties of some physical nonequilibrium state, then, if the continuations are unique (i.e., if the transition point is not a branch point), they will continue to be related by the fluctuation dissipation theorem. This puzzle has a simple resolution in our model. It is discussed in Sec. VI, part B.

lations is

$$L^{\alpha\gamma}(z)=\int_0^\infty dt e^{izt}L^{\alpha\gamma}(t)=\beta\int_0^\infty dt e^{izt}\frac{\partial}{\partial t}G^{\alpha\gamma}(t), \quad (2.14)$$

where z is a complex variable in the upper half-plane. Ordinarily L is analytic in the upper half-plane, with possible poles only on the real axis (giving the frequencies of undamped resonances) or in its analytic continuation into the lower half-plane (representing damped resonances). If, however, $L(t)$ increases exponentially, then $L(z)$ will have a pole in the upper half-plane. It is through such complex poles that the type of instability we shall find in the model has appeared in the approximate calculations described in the Introduction.

Our task is to calculate $G^{\alpha\gamma}(t)$ for all temperatures, and to demonstrate that it can be analytically continued in β (or H) through a ferromagnetic transition point to a function which has exponentially growing time dependence; alternatively, we wish to show that the exact $L(z)$ can be analytically continued in β (or H) to a function of z which has poles in the upper-half z plane.

Without going to the limit of infinite N we can simplify things considerably. It follows from (2.1) that $\mathbf{s}^{(1)}$ and $\mathbf{s}^{(2)}$ satisfy equations of motion involving only themselves:

$$\begin{aligned}\dot{\mathbf{s}}^{(1)}&=-\mathbf{H}\times\mathbf{s}^{(1)}-g\mathbf{s}^{(2)}\times\mathbf{s}^{(1)},\\ \dot{\mathbf{s}}^{(2)}&=\mathbf{H}\times\mathbf{s}^{(2)}-g\mathbf{s}^{(1)}\times\mathbf{s}^{(2)}.\end{aligned} \quad (2.15)$$

They thus depend only on their own initial values and not on the detailed spin configuration within each group. This reduces the ensemble average (2.8) to

$$G^{\alpha\gamma}(t)=\lim_{N\to\infty} N\int d\mathbf{s}^{(1)}d\mathbf{s}^{(2)}P(\mathbf{s}^{(1)},\mathbf{s}^{(2)}) \times([\mathbf{s}^{(\alpha)}(\mathbf{s}^{(1)},\mathbf{s}^{(2)},t)-\mathbf{s}^{(\alpha)}]\mathbf{s}^{(\gamma)}) \Big/ \int d\mathbf{s}^{(1)}d\mathbf{s}^{(2)}P(\mathbf{s}^{(1)},\mathbf{s}^{(2)}), \quad (2.16)$$

where

$$\begin{aligned}P(\mathbf{s}^{(1)},\mathbf{s}^{(2)})&=\int\prod_{i=1}^{N}d\Omega_i \exp[-\beta\mathcal{H}(\mathbf{s}^i)] \times\delta\left(\mathbf{s}^{(1)}-\frac{1}{N}\sum_{j=1}^{\frac{1}{2}N}\mathbf{s}^j\right) \times\delta\left(\mathbf{s}^{(2)}-\frac{1}{N}\sum_{j=\frac{1}{2}N+1}^{N}\mathbf{s}^j\right)\\ &=\exp\{N\beta[\mathbf{H}\cdot(\mathbf{s}^{(1)}-\mathbf{s}^{(2)}) +\tfrac{1}{2}g(\mathbf{s}^{(1)}+\mathbf{s}^{(2)})^2]\}W(s^{(1)})W(s^{(2)}),\end{aligned} \quad (2.17)$$

and

$$W(s)=\int\prod_{i=1}^{\frac{1}{2}N}d\Omega_i\delta\left(\mathbf{s}-\frac{1}{N}\sum_{i=1}^{\frac{1}{2}N}\mathbf{s}^i\right). \quad (2.18)$$

If one inserts the Fourier representation of the δ function,

$$\delta(\mathbf{x})=\int d\mathbf{u}e^{-i\mathbf{u}\cdot\mathbf{x}}\Big/(2\pi)^3,$$

into (2.18), then the integrations over spin directions factor into $N/2$ identical elementary integrations, after which the integration over directions of $\mathbf{u}$ is equally trivial. Up to an irrelevant constant factor which disappears from the normalized P,

$$W(s)=\frac{\sigma}{s}\int_{-\infty}^{\infty}\zeta\sin\left(\frac{Ns}{\sigma}\zeta\right)\left(\frac{\sin\zeta}{\zeta}\right)^{\frac{1}{2}N}d\zeta. \quad (2.19)$$

The problem for finite N has therefore been reduced to evaluating the integral (2.19), placing the general solution[13] of (2.15) into (2.16), and performing the remaining six integrations over the initial values. For our purposes, however, this still formidable calculation is unnecessary, since we are ultimately interested in G only for strictly infinite N. It is only in this limit that the singular behavior going with a phase transition can occur, and hence that the analytic continuation of an equilibrium correlation function from one value of β to another can lead to something which is not the equilibrium correlation function for the new value of β.[14] We shall therefore evaluate the integrals by steepest descent methods, retaining only those terms which continue to contribute to G as $N\to\infty$.

As it turns out, the infinite N limit not only simplifies the integrations in (2.16) and (2.19), but also makes a knowledge of the general solution of (2.15) unnecessary. This is because P is very sharply peaked (in the limit of infinite N, completely concentrated in) values of $\mathbf{s}^{(1)}$ and $\mathbf{s}^{(2)}$ which are time-independent solutions. As a result, if one considers $\mathbf{s}^{(\alpha)}(\mathbf{s}^{(1)},\mathbf{s}^{(2)},t)$ as a function of its initial values, only the linear term in its expansion about the set of initial values giving a stationary solution contributes in the limit of infinite N. Therefore if one wants G only for infinite N one need only solve a version of (2.15) linearized about the stationary solutions which maximize P.

Although our result will be exact only in the limit of infinite N, many features of the correlation functions for large but finite N will be illuminated in the course of discussion. The maxima of P and the equation of

[13] It can be found in terms of elliptic functions.

[14] Since any physical system is finite it must be possible to state our conclusions, if the are of any relevance, for finite N. They could be put something life this: there exists a function of t and β which is analytic for all positive real β, and which, above the transition temperature agrees to within terms of order $1/N$ with the equilibrium correlation function provided the time is less than of order N; but below the transition temperature this function grows exponentially in time.

state are found in Sec. III. In Sec. IV certain features of the exact equations of motion are examined; these justify the subsequent linearization and reveal the sort of behavior one would expect for large but finite N. In Sec. V the infinite N correlation functions are found; their properties are discussed in Sec. VI, along with some puzzles and speculations suggested by the model.

III. EQUILIBRIUM THERMODYNAMIC PROPERTIES

To find P for a large system we must first know the large N behavior of the density of states factors W appearing in (2.17). Since W is of the form

$$W(s)=\frac{\sigma}{is}\int_{-\infty}^{\infty}\zeta d\zeta e^{-N\phi_s(\zeta)}, \tag{3.1}$$

where

$$\phi_s(\zeta)=-i(s/\sigma)\zeta-\tfrac{1}{2}\ln(\sin\zeta/\zeta), \tag{3.2}$$

an asymptotic expansion can be found by the method of steepest descent. Saddle points occur at roots of

$$0=\phi_s'(\zeta)=-i(s/\sigma)-\tfrac{1}{2}(\cot\zeta-1/\zeta). \tag{3.3}$$

When $\zeta=i\eta$, η real, this becomes

$$s/\sigma=\tfrac{1}{2}(\coth\eta-1/\eta)=f(\eta), \tag{3.4}$$

which has a unique positive root $\eta(s)$, when s/σ is between 0 and $\frac{1}{2}$.[15] One easily verifies that this is the only saddle point on the line $\zeta=\xi+i\eta(s)$, $-\infty<\xi<\infty$, and that this line passes through the saddle point in the direction of steepest descent. The contour in (3.1) may be displaced to run along this line, and the resulting asymptotic form is

$$W(s)=C(s)\exp\{-N[(s/\sigma)\eta(s)-\tfrac{1}{2}\ln(\sinh\eta(s)/\eta(s))]\}, \tag{3.5}$$

where, up to an irrelevant normalization constant,

$$C(s)=\frac{\sigma\eta(s)}{s}\left(\frac{1}{\eta(s)^2}-\frac{1}{\sinh^2\eta(s)}\right)^{-1/2}+o\left(\frac{1}{N}\right).$$

Therefore

$$P(\mathbf{s}^{(1)},\mathbf{s}^{(2)})=C(s^{(1)})C(s^{(2)})\exp[-N\Phi(\mathbf{s}^{(1)},\mathbf{s}^{(2)})], \tag{3.6}$$

where

$$\Phi(\mathbf{s}^{(1)},\mathbf{s}^{(2)})=-\beta\mathbf{H}\cdot(\mathbf{s}^{(1)}-\mathbf{s}^{(2)})-\tfrac{1}{2}\beta g(\mathbf{s}^{(1)}+\mathbf{s}^{(2)})^2+s^{(1)}\eta^{(1)}/\sigma-\tfrac{1}{2}\ln(\sinh\eta^{(1)}/\eta^{(1)})+s^{(2)}\eta^{(2)}/\sigma-\tfrac{1}{2}\ln(\sinh\eta^{(2)}/\eta^{(2)}), \tag{3.7}$$

and $\eta^{(\alpha)}$ is defined by

$$s^{(\alpha)}=\tfrac{1}{2}\sigma f(\eta^{(\alpha)}),\quad \alpha=1,2. \tag{3.8}$$

Since $C(s)$ remains a slowly varying function of s for large N, in this limit P will be very sharply peaked at the value or values of $\mathbf{s}^{(1)}$ and $\mathbf{s}^{(2)}$ which minimize Φ. If Φ has its absolute minimum at a single point, $\mathbf{s}_0^{(1)}$, $\mathbf{s}_0^{(2)}$, then in the limit of infinite N the ensemble average of any N-independent function, $F(\mathbf{s}^{(1)},\mathbf{s}^{(2)})$ will just be $F(\mathbf{s}_0^{(1)},\mathbf{s}_0^{(2)})$. More generally, if Φ does not have a single minimum but assumes its least value on a family of points, then the ensemble average of any N-independent F which assumes the same value at all such points, is given by this value, in the limit of infinite N. Because of this we can immediately deduce the equation of state for the infinite system from the minima of Φ. These are found in Appendix B, and are of two types, depending on the values of β and H:

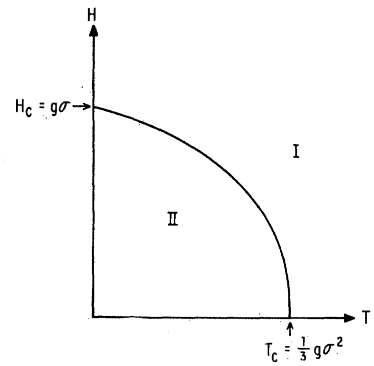

FIG. 1. The nonferromagnetic (I) and ferromagnetic (II) regions of the H-T plane. Transition points lie along the boundary, given by $H=g\sigma\times f(\beta\sigma H)$.

I. $H/g\geqslant\sigma f(\beta\sigma H)$ (nonferromagnetic). Φ is minimum at the single point

$$\mathbf{s}_0^{(1)}=(0,0,\tfrac{1}{2}\sigma f(\beta\sigma H)),\quad \mathbf{s}_0^{(2)}=(0,0,-\tfrac{1}{2}\sigma f(\beta\sigma H)). \tag{3.9}$$

II. $H/g\leq\sigma f(\beta\sigma H)$ (ferromagnetic). Φ assumes its minimum when $\mathbf{s}^{(1)}$ and $\mathbf{s}^{(2)}$ are of the form:

$$\mathbf{s}_0^{(1)}=\tfrac{1}{2}(\mathbf{s}_0+\mathbf{H}/g),\quad \mathbf{s}_0^{(2)}=\tfrac{1}{2}(\mathbf{s}_0-\mathbf{H}/g), \tag{3.10}$$

where $\mathbf{s}_0$ is perpendicular to $\mathbf{H}$, with magnitude determined by

$$(s_0^2+(H/g)^2)^{1/2}=\sigma f(\beta g\sigma[s_0^2+(H/g)^2]^{1/2}). \tag{3.11}$$

The direction of $\mathbf{s}_0$ in the x-y plane is undetermined, i.e., Φ is minimum on a one-dimensional family of points.

The regions of the H-T plane in which the two types of minima are found are indicated in Fig. 1, the boundary of the two regions being given by points satisfying

$$H/g=\sigma f(\beta\sigma H). \tag{3.12}$$

Type II maxima can occur only for low temperatures,

$$T\leqslant T_c=\tfrac{1}{3}g\sigma^2, \tag{3.13}$$

and weak fields,

$$H\leqslant H_c=g\sigma. \tag{3.14}$$

Equations (3.9)–(3.11) give a complete description of the equilibrium state of the infinite system in terms of the variables $\langle s_z^{(1)}\rangle$, $\langle s_z^{(2)}\rangle$, $\langle s_\perp^{(1)}\rangle$ (the magnitude of the projection of $\mathbf{s}^{(1)}$ in the x-y plane), $\langle s_\perp^{(2)}\rangle$, and $\langle s\rangle$ (the magnitude of $\mathbf{s}^{(1)}+\mathbf{s}^{(2)}$, the total spin per par-

[15] s/σ cannot exceed $\frac{1}{2}$, its value when all $\frac{1}{2}N$ spins are parallel.

ticle).[16] In region I:

$$\langle s_z^{(1)}\rangle = -\langle s_z^{(2)}\rangle = \tfrac{1}{2}\sigma f(\beta\sigma H),$$
$$\langle s_\perp^{(1)}\rangle = \langle s_\perp^{(2)}\rangle = \langle s\rangle = 0. \quad (3.15)$$

In region II:

$$\langle s_z^{(1)}\rangle = -\langle s_z^{(2)}\rangle = H/g,$$
$$\langle s_\perp^{(1)}\rangle = \langle s_\perp^{(2)}\rangle = \tfrac{1}{2}s_0, \quad (3.16)$$
$$\langle s\rangle = s_0.$$

In Fig. 2, $\langle s_z^{(1)}\rangle - \langle s_z^{(2)}\rangle$ and $\langle s\rangle$ are plotted against temperature for a typical field strength $H<H_c$. At high temperatures (region I) the total spin is zero, and $\langle s_z^{(1)}\rangle$ and $\langle s_z^{(2)}\rangle$ are oppositely directed along the magnetic field, with magnitude $\frac{1}{2}\sigma f(\beta\sigma H)$. As T decreases, $\langle s\rangle$ remains zero, and $\langle s_z^{(1)}\rangle$ and $\langle s_z^{(2)}\rangle$ grow until their difference equals H/g. At this point region II is entered. Further lowering of the temperature leaves $\langle s_z^{(1)}\rangle$ and $\langle s_z^{(2)}\rangle$ unchanged, but the total spin per particle now developes a nonzero magnitude in the plane perpendicular to **H**, which grows from zero at the transition point to a maximum value of $(\sigma^2-(H/g)^2)^{1/2}$ at $T=0$.

Thus the system is ferromagnetic in region II. The spin alignment is not perfect because the field **H** favors opposite directions of the z components of $\mathbf{s}^{(1)}$ and $\mathbf{s}^{(2)}$. In region I there is no ferromagnetic alignment at all, either because the temperature is above the Curie point, or because the anti-aligning field is too strong.

To calculate correlation functions in the two regions we need to know more about P than the location of its maxima as $N\rightarrow\infty$, since G depends on the fluctuations of $\mathbf{s}^{(1)}$ and $\mathbf{s}^{(2)}$ about their equilibrium values.[17] Still, it is clear that when N is large only initial values in the immediate neighborhood of the maxima will contribute appreciably to G. We therefore turn to an analysis of the solutions of the equations of motion, paying particular attention to those with initial values close to maxima of P.

IV. TIME DEPENDENCE OF $s^{(\alpha)}$

The solutions of (2.15) most important for the calculation of G are those which are independent of time. The general stationary solution either has $\mathbf{s}^{(1)}$ and $\mathbf{s}^{(2)}$ both parallel to **H** and otherwise arbitrary, or $\mathbf{s}^{(1)}-\mathbf{s}^{(2)}=\mathbf{H}/g$, with $\mathbf{s}^{(1)}+\mathbf{s}^{(2)}$ arbitrary. In the limit of infinite N, P is maximum at a stationary solution of the first kind in the nonferromagnetic region, and has a family of maxima of the second kind in the ferromagnetic region. This fact enables us to calculate G for the infinite system (and, for the finite system, to any order in an asymptotic expansion in $1/N$) without having to use the rather complicated form of the general solutions.

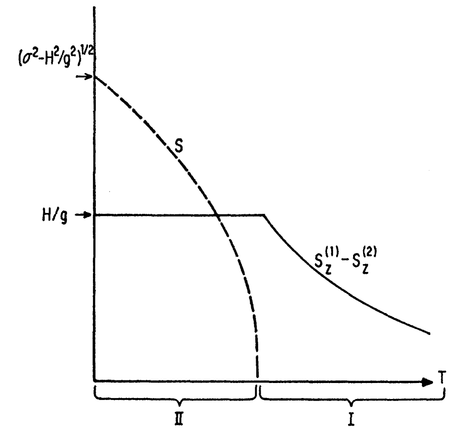

FIG. 2. Equations of state when $H<H_c$. The dashed curve is the magnitude of the total spin, which vanishes in region I. The solid curve is the difference of the z components of $\mathbf{s}^{(1)}$ and $\mathbf{s}^{(2)}$, which is constant in region II.

It is nevertheless worth carrying an exact analysis of the time-dependent solutions of (2.15) to a point which clearly shows why we are entitled to the simplifications we shall eventually make. In terms of the total spin per particle, $\mathbf{s}=\mathbf{s}^{(1)}+\mathbf{s}^{(2)}$, and the deviation of the spin difference from the stationary value $\mathbf{H}/g$, $\mathbf{\Delta}=\mathbf{s}^{(1)}-\mathbf{s}^{(2)}-\mathbf{H}/g$, (2.15) becomes

$$\dot{\mathbf{s}}=-\mathbf{H}\times\mathbf{\Delta},\quad d\mathbf{\Delta}/dt=-g\mathbf{s}\times\mathbf{\Delta}. \quad (4.1)$$

Taking advantage of the fact that s_z is a constant of the motion, we transform (4.1) to a coordinate system rotating about the z axis with angular frequency

$$\Omega=\tfrac{1}{2}gs_z. \quad (4.2)$$

In the rotating frame,

$$\dot{\mathbf{s}}=\mathbf{\Omega}\times\mathbf{s}-\mathbf{H}\times\mathbf{\Delta},\quad d\mathbf{\Delta}/dt=\mathbf{\Omega}\times\mathbf{\Delta}-g\mathbf{s}\times\mathbf{\Delta}, \quad (4.3)$$

where $\mathbf{\Omega}=\Omega\mathbf{H}/H$. It follows that

$$\ddot{\mathbf{s}}=\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{s})-\mathbf{H}\times[(2\mathbf{\Omega}-g\mathbf{s})\times\mathbf{\Delta}],$$

which, by virtue of the particular value of Ω, simplifies to

$$\ddot{\mathbf{s}}_\perp=-\Omega^2\mathbf{s}_\perp+g(\mathbf{H}\cdot\mathbf{\Delta})\mathbf{s}_\perp,$$

for the components of **s** in the x-y plane. Furthermore,

$$\mathbf{H}\cdot\frac{d\mathbf{\Delta}}{dt}=-g\mathbf{s}\cdot\dot{\mathbf{s}}=-\tfrac{1}{2}g\frac{d}{dt}s_\perp^2,$$

so

$$\mathbf{H}\cdot\mathbf{\Delta}=\mathbf{H}\cdot\mathbf{\Delta}(0)-\tfrac{1}{2}g(s_\perp^2-s_\perp(0)^2). \quad (4.4)$$

In the rotating coordinate system $\mathbf{s}_\perp$ therefore satisfies

$$\ddot{\mathbf{s}}_\perp=-\tfrac{1}{2}g^2s_\perp^2\mathbf{s}_\perp-\alpha\mathbf{s}_\perp, \quad (4.5)$$

where

$$\alpha=\Omega^2-gH\Delta_z(0)-\tfrac{1}{2}g^2s_\perp(0)^2. \quad (4.6)$$

[16] The missing sixth variable, φ, the angle of **s** in the x-y plane, is of no interest, since in region I all components perpendicular to **H** are zero and in region II all directions are equally likely.

[17] The argument that enabled us to deduce the equation of state from the maxima of P does not apply to G because of the factor N appearing in its definition.

This is just the equation of motion for a particle of unit mass moving in two dimensions under the influence of a central force given by the potential

$$U(s_\perp)=\alpha s_\perp^2/2+g^2 s_\perp^4/8. \qquad (4.7)$$

The initial velocity of the particle is

$$\dot{\mathbf{s}}_\perp(0)=\mathbf{\Omega}\times\mathbf{s}_\perp(0)-\mathbf{H}\times\mathbf{\Delta}(0), \qquad (4.8)$$

and its conserved angular momentum is

$$l=(\mathbf{s}_\perp(0)\times\dot{\mathbf{s}}_\perp(0))_z=\Omega s_\perp(0)^2-H(\mathbf{\Delta}_\perp(0)\cdot\mathbf{s}_\perp(0)). \qquad (4.9)$$

The magnitude of $\mathbf{s}_\perp$ therefore satisfies the radial equation

$$\ddot{s}_\perp=-\alpha s_\perp-\tfrac{1}{2}g^2 s_\perp^3+l^2/s_\perp^3, \qquad (4.10)$$

and its angle, ψ, in the x-y plane is required by conservation of angular momentum to obey

$$\dot{\psi}=l/s_\perp^2. \qquad (4.11)$$

The initial velocity necessary for the integration of (4.10) is

$$\dot{s}_\perp(0)=\dot{\mathbf{s}}_\perp(0)\cdot\mathbf{s}_\perp(0)/s_\perp(0)$$
$$=\mathbf{H}\cdot(\mathbf{s}_\perp(0)\times\mathbf{\Delta}_\perp(0))/s_\perp(0). \qquad (4.12)$$

Evidently (4.9)–(4.12) all remain valid in the original stationary coordinate system except that the angular velocity φ now satisfies

$$\dot{\varphi}=l/s_\perp^2-\Omega. \qquad (4.13)$$

Equation (4.10) can be solved in terms of elliptic functions, but all relevant features of the motion can be understood by considering how a particle would move in two dimensions in the potential U.[18] Since the potential depends, through α, on the initial conditions, not all particle orbits in a given U are possible orbits for $\mathbf{s}_\perp$; however for given initial conditions U is determined and the subsequent motion of $\mathbf{s}_\perp$ is given by the orbit in that particular U followed by a particle with the appropriate initial position and velocity. The nature of this orbit depends critically on the sign of α. When α is positive the motion is in a simple potential well with a minimum at $s_\perp=0$; for negative α, U is minimum at $s_\perp=(-2\alpha/g^2)^{1/2}$, the force is repulsive for smaller $s_\perp$, and the origin is a local maximum of U.

Consider now initial values of the form

$$\mathbf{s}^{(1)}(0)=(0,0,\tfrac{1}{2}\sigma f(\beta\sigma H))+o(\epsilon),$$
$$\mathbf{s}^{(2)}(0)=(0,0,-\tfrac{1}{2}\sigma f(\beta\sigma H))+o(\epsilon). \qquad (4.14)$$

In the nonferromagnetic region these are very close to the maximum of P. In the potential that they determine,

$$\alpha=gH(H/g-\sigma f(\beta\sigma H))+o(\epsilon),$$

which, for sufficiently small ϵ, is positive within the nonferromagnetic region. The initial position and velocity, $\mathbf{s}_\perp(0)$ and $\dot{\mathbf{s}}_\perp(0)$ are also of order ϵ. Therefore within the nonferromagnetic region, for small enough ϵ, $\mathbf{s}_\perp$, and $d\mathbf{s}_\perp/dt$ will move along orbits in the neighborhood of zero. This is enough to guarantee that if $\mathbf{s}^{(1)}(0)$ and $\mathbf{s}^{(2)}(0)$ are within $o(\epsilon)$ of the nonferromagnetic maximum of P, then $\mathbf{s}^{(1)}(t)$ and $\mathbf{s}^{(2)}(t)$ stay within $o(\epsilon)$ of this maximum at all times.

On the other hand, if the initial values are of the form

$$\mathbf{s}^{(1)}(0)=\tfrac{1}{2}(s_x(0),s_y(0),H/g)+o(\epsilon),$$
$$\mathbf{s}^{(2)}(0)=\tfrac{1}{2}(s_x(0),s_y(0),-H/g)+o(\epsilon), \qquad (4.15)$$

which puts them in the neighborhood of a ferromagnetic maximum of P, then $\alpha=-\frac{1}{2}g^2 s_\perp(0)^2+o(\epsilon)$, and U is minimum at $s_\perp$ within ϵ of $s_\perp(0)$. The initial velocity is still of order ϵ so $\mathbf{s}_\perp$ can only undergo small oscillations in the neighborhood of its initial value. The angular velocity (in the stationary coordinate system) will be

$$\dot{\varphi}=-H(\mathbf{\Delta}_\perp(0)\cdot\mathbf{s}_\perp(0))/s_0^2+o(\epsilon^2), \qquad (4.16)$$

which is of order ϵ, and, to this order, time-independent. There will thus be a slow uniform precession of $\mathbf{s}_\perp$ about $\mathbf{H}$ superimposed on the small oscillations in its magnitude. As $\epsilon\to 0$ the period of precession becomes infinite.

We shall draw further on this helpful reduction of the two-spin problem to a problem of two-dimensional motion in a central force, in the concluding discussion in Sec. VI.

V. CALCULATION OF CORRELATION AND RESPONSE FUNCTIONS

We shall calculate correlation functions involving components of $\mathbf{s}=\mathbf{s}^{(1)}+\mathbf{s}^{(2)}$ and $\mathbf{\Sigma}=\mathbf{s}^{(1)}-\mathbf{s}^{(2)}$, since these follow more naturally from the equations of motion. The $\mathbf{s}^{(1)}$, $\mathbf{s}^{(2)}$ correlation functions can then be found by taking simple linear combinations of the $\mathbf{s}$, $\mathbf{\Sigma}$ functions; the time dependence of the latter follows from a knowledge of the two functions

$$G_{zz}^{\Sigma\Sigma}(t)=\lim_{N\to\infty} N\langle(\Sigma_z(t)-\Sigma_z)\Sigma_z\rangle, \qquad (5.1)$$

and

$$G_{+-}^{\Sigma\Sigma}(t)=\lim_{N\to\infty} N\langle(\Sigma_+(t)-\Sigma_+)\Sigma_-\rangle,$$
$$\Sigma_\pm=\Sigma_x\pm i\Sigma_y. \qquad (5.2)$$

This is because:

(a) Correlation functions pairing a z component with a $\pm$ component, a plus with a plus, or a minus with a minus, vanish even for finite N due to the symmetry under rotations about the z axis;

(b) $s_z(t)\equiv s_z(0)$;

(c) $ds_\pm(t)/dt=\mp iH\Sigma_\pm(t)$;

(d) the $-+$ functions are complex conjugates of the $+-$ ones;

(e) $\langle(\Sigma_+(t)-\Sigma_+)s_-\rangle=\langle(s_+(-t)-s_+)\Sigma_-\rangle^*$.

[18] The motion of $\mathbf{s}_\perp$ immediately determines the motion of everything else: Δ_z, by (4.4), and $\mathbf{\Delta}_\perp$, since $\mathbf{\Delta}_\perp=\mathbf{H}\times\dot{\mathbf{s}}_\perp/H^2$.

Due to (a) and (b), $G_{zz}^{\Sigma\Sigma}(t)$ is the only nonvanishing $s-\Sigma$ correlation function involving z components; using (c)–(e) we can obtain the remaining nonvanishing $s-\Sigma$ correlation functions from $G_{+-}(t)$ by time integration, complex conjugation, and a knowledge of the equal time correlation functions

$$\lim_{N\to\infty} N\langle\Sigma_+\Sigma_-\rangle \quad\text{and}\quad \lim_{N\to\infty} N\langle s_+\Sigma_-\rangle.$$

It is also convenient to write the correlation functions in terms of the deviation of $\Sigma_z(t)$ from its (stationary) value at the maximum of P, $\Sigma_{0_z}=H/g$. We can replace (5.1) by

$$G_{zz}^{\Sigma\Sigma}(t)=\lim_{N\to\infty} N\langle(\Sigma_z(t)-\Sigma_z)(\Sigma_z-\Sigma_{0_z})\rangle,$$

since this only adds a term which vanishes even for finite N due to the translational invariance in time of canonical ensemble averages. Furthermore, if we define

$$\hat{G}_{zz}^{\Sigma\Sigma}(t)=\lim_{N\to\infty} N\langle(\Sigma_z(t)-\Sigma_{0_z})(\Sigma_z-\Sigma_{0_z})\rangle, \quad (5.3)$$

then

$$G_{zz}^{\Sigma\Sigma}(t)=\hat{G}_{zz}^{\Sigma\Sigma}(t)-\hat{G}_{zz}^{\Sigma\Sigma}(0), \quad (5.4)$$

so it suffices to find $\hat{G}$. Similarly, if

$$\hat{G}_{+-}^{\Sigma\Sigma}(t)=\lim_{N\to\infty} N\langle\Sigma_+(t)\Sigma_-\rangle, \quad (5.5)$$

($\Sigma_{0_\pm}$ vanishes in both regions), then

$$G_{+-}^{\Sigma\Sigma}(t)=\hat{G}_{+-}^{\Sigma\Sigma}(t)-\hat{G}_{+-}^{\Sigma\Sigma}(0). \quad (5.6)$$

The analysis now depends on which region β and H lie in.

A. Nonferromagnetic Region

Because P is very sharply peaked at $\mathbf{s}_0$, $\mathbf{\Sigma}_0$, for fixed t we expand $\mathbf{\Sigma}(\mathbf{s},\mathbf{\Sigma},t)$ about this point and examine the contribution of each term to (5.3) and (5.5). The leading term is just $\mathbf{\Sigma}_0=(0,0,\Sigma_{0_z})$, and therefore gives no contribution. The linear term gives a contribution that goes as N times a mean-square fluctuation, and is therefore of order unity.[19] All higher order terms give no contribution, since the mth order terms give N times a mean mth power deviation, and are therefore no larger than $N^{-\frac{1}{2}m+1}$.

We may therefore replace $\mathbf{\Sigma}(\mathbf{s},\mathbf{\Sigma},t)$ by its lowest order term in $\mathbf{s}-\mathbf{s}_0$ and $\mathbf{\Sigma}-\mathbf{\Sigma}_0$. But this is found simply by solving the equations of motion linearized about the stationary solutions $\mathbf{s}(t)\equiv\mathbf{s}_0=0$, $\mathbf{\Sigma}(t)\equiv\mathbf{\Sigma}_0=(0,0,\sigma f(\beta\sigma H))$. The exact equation (2.15) in terms of $\mathbf{s}$ and $\mathbf{\Sigma}$ is

$$\dot{\mathbf{s}}=-\mathbf{H}\times\mathbf{\Sigma}, \quad d\mathbf{\Sigma}/dt=-\mathbf{H}\times\mathbf{s}+g\mathbf{\Sigma}\times\mathbf{s}, \quad (5.7)$$

which linearizes to

$$\begin{aligned}\dot{\mathbf{s}}&=-\mathbf{H}\times\mathbf{\Sigma},\\ d\mathbf{\Sigma}/dt&=-(1-(g\sigma/H)f(\beta\sigma H))(\mathbf{H}\times\mathbf{s}).\end{aligned} \quad (5.8)$$

[19] The equilibrium fluctuations are discussed in Appendix C.

The required solutions to (5.8) are

$$\Sigma_z(\mathbf{s},\mathbf{\Sigma},t)\equiv\Sigma_z, \quad (5.9)$$

$$\Sigma_+(\mathbf{s},\mathbf{\Sigma},t)=\Sigma_+\cos\omega_0 t-i(\omega_0/H)s_+\sin\omega_0 t, \quad (5.10)$$

where

$$\omega_0=[H(H-g\sigma f(\beta\sigma H))]^{1/2}. \quad (5.11)$$

From (5.9) it follows that $\hat{G}_{zz}^{\Sigma\Sigma}(t)$ is independent of time, so

$$G_{zz}^{\Sigma\Sigma}(t)\equiv 0. \quad (5.12)$$

Equation (5.10) gives $\hat{G}_{+-}^{\Sigma\Sigma}(t)$ in terms of equal time correlation functions. These are evaluated in Appendix C, and the result is

$$\hat{G}_{+-}^{\Sigma\Sigma}(t)=(2\sigma/\beta H)f(\beta\sigma H)\cos\omega_0 t,$$

or

$$G_{+-}^{\Sigma\Sigma}(t)=(2\sigma/\beta H)f(\beta\sigma H)(\cos\omega_0 t-1). \quad (5.13)$$

If we integrate $\hat{G}_{+-}^{\Sigma\Sigma}(t)$ between 0 and t, we find

$$G_{+-}^{s\Sigma}(t)=(-2i\sigma/\beta\omega_0)f(\beta\sigma H)\sin\omega_0 t=G_{+-}^{\Sigma s}(t). \quad (5.14)$$

Again integrating (5.14) and using the fact that $\langle s_+(0)\Sigma_-(0)\rangle=0$ (Appendix C) we have

$$G_{+-}^{ss}(t)=(2\sigma H/\beta\omega_0^2)f(\beta\sigma H)(\cos\omega_0 t-1). \quad (5.15)$$

The linear response functions corresponding to (5.12)–(5.15) can be found at once from (2.14) and are

$$L_{zz}^{\Sigma\Sigma}(z)=0; \quad (5.16)$$

$$L_{+-}^{\Sigma\Sigma}(z)=2\sigma f(\beta\sigma H)\omega_0^2/(z^2-\omega_0^2)H; \quad (5.17)$$

$$L_{+-}^{s\Sigma}(z)=L_{+-}^{\Sigma s}(z)=2\sigma f(\beta\sigma H)z/(z^2-\omega_0^2); \quad (5.18)$$

$$L_{+-}^{ss}(z)=2\sigma f(\beta\sigma H)H/(z^2-\omega_0^2). \quad (5.19)$$

B. Ferromagnetic Region

We can deal with the added complexity due to the family of maxima of $P(\mathbf{s}^{(1)},\mathbf{s}^{(2)})$ by using the cylindrical coordinates $s_\perp^{(1)}$, $s_z^{(1)}$, $\theta^{(1)}$, $s_\perp^{(2)}$, $s_z^{(2)}$, $\theta^{(2)}$. In its angular dependence P is a function only of $\theta=\theta^{(1)}-\theta^{(2)}$:

$$P(\mathbf{s}^{(1)},\mathbf{s}^{(2)})=\hat{P}(s_\perp^{(1)},s_z^{(1)},s_\perp^{(2)},s_z^{(2)},\theta), \quad (5.20)$$

and $\hat{P}$ has a single maximum at

$$\begin{aligned}s_{0_\perp}^{(1)}&=s_{0_\perp}^{(2)}=\tfrac{1}{2}s_0;\\ s_{0_z}^{(1)}&=-s_{0_z}^{(2)}=H/2g, \quad \theta_0=0,\end{aligned} \quad (5.21)$$

[where s_0 is the solution to (3.11)]. Now from Eq. (4.4),

$$G_{zz}^{\Sigma\Sigma}(t)=-(g/2H)\lim_{N\to\infty} N\langle(s_\perp^2(t)-s_\perp^2)\Sigma_z\rangle. \quad (5.22)$$

It is also true that

$$G_{+-}^{\Sigma\Sigma}(t)=(1/H^2)\lim_{N\to\infty} N\langle(\dot{s}_+(t)-\dot{s}_+)\dot{s}_-\rangle. \quad (5.23)$$

In Eqs. (5.22) and (5.23) the quantities to be averaged depend on the initial values $\theta^{(1)}$ and $\theta^{(2)}$ only through θ. We can therefore replace the average over $\mathbf{s}^{(1)}$ and $\mathbf{s}^{(2)}$ by an average over the five variables on which $\hat{P}$

depends, and then, taking advantage of the unique maximum of $\hat{P}$, proceed as in the nonferromagnetic case.

We first define, in analogy to (5.3) and (5.5),

$$\hat{G}_{zz}^{\Sigma\Sigma}(t)=-(g/2H)\lim_{N\to\infty}N\langle(s_\perp^2(t)-s_0^2)\Delta_z\rangle, \quad (5.24)$$

$$\hat{G}_{+-}^{\Sigma\Sigma}(t)=(1/H^2)\lim_{N\to\infty}N\langle\dot{s}_+(t)\dot{s}_-\rangle, \quad (5.25)$$

where $\Delta_z=\Sigma_z-\Sigma_{0_z}=\Sigma_z-H/g$ [and $\dot{\mathbf{s}}_+(\mathbf{s}_0,\mathbf{\Sigma}_0)=\dot{\mathbf{s}}_-(\mathbf{s}_0,\mathbf{\Sigma}_0)=0$], so that

$$G(t)=\hat{G}(t)-\hat{G}(0) \quad (5.26)$$

in both cases. The factors $s_\perp^2(t)-s_0^2$, Δ_z, $\dot{s}_+(t)$, and $\dot{s}_-(t)$ are all, to lowest nonvanishing order, linear in the deviations of their initial values from the stationary values (5.21). It therefore suffices to replace each by the linear term in its expansion about the stationary initial values. Equation (5.24) immediately reduces to

$$\hat{G}_{zz}^{\Sigma\Sigma}(t)=-(gs_0/H)\lim_{N\to\infty}N\langle(s_\perp(t)-s_0)\Delta_z\rangle. \quad (5.27)$$

We may rewrite (5.25):

$$\hat{G}_{+-}^{\Sigma\Sigma}(t)=(1/H^2)\lim_{N\to\infty}N\langle(\dot{s}_\perp(t)+is_\perp(t)\dot{\varphi}(t))\times(\dot{s}_\perp-is_\perp\dot{\varphi})e^{i[\varphi(t)-\varphi]}\rangle. \quad (5.28)$$

Because $\dot{s}_\perp(t)$ and $\dot{\varphi}(t)$ are linear in the deviation of the initial values from equilibrium, for any finite time the phase factor in (5.28) makes no contribution as $N\to\infty$. We may also replace $s_\perp(t)$ and $s_\perp$ by their equilibrium values, to get:

$$\hat{G}_{+-}^{\Sigma\Sigma}(t)=(1/H^2)\lim_{N\to\infty}[\langle\dot{s}_\perp(t)\dot{s}_\perp\rangle+s_0^2\langle\dot{\varphi}(t)\dot{\varphi}\rangle+is_0(\langle\dot{\varphi}(t)\dot{s}_\perp\rangle-\langle\dot{s}_\perp(t)\dot{\varphi}\rangle)]. \quad (5.29)$$

Now from (4.13), (4.9), and (4.2),

$$\dot{\varphi}(t)=-(H/s_0)[(s_\perp^{(1)}(0)-s_{0_\perp}^{(1)})-(s_\perp^{(2)}(0)-s_{0_\perp}^{(2)})], \quad (5.30)$$

to lowest order, and this is independent of time; as a result, the imaginary term in (5.29) vanishes and we are left with

$$\hat{G}_{+-}^{\Sigma\Sigma}(t)=H^{-2}\lim_{N\to\infty}N[\langle\dot{s}_\perp(t)\dot{s}_\perp\rangle+s_0^2\langle\dot{\varphi}^2\rangle]. \quad (5.31)$$

It remains to find the linearized solution, $s_\perp$, of (4.10). Since l^2 and Ω^2 are of second order, we can replace (4.10) by

$$\ddot{s}_\perp(t)=-\tfrac{1}{2}g^2(s_\perp^2(t)-s_\perp^2(0))s_\perp(t)+gH\Delta_z(0)s_\perp(t). \quad (5.32)$$

Further linearization of the remaining terms gives

$$\ddot{s}_\perp(t)=-g^2s_0^2(s_\perp(t)-s_\perp(0))+gH\Delta_z(0)s_0, \quad (5.33)$$

which has as its solution

$$s_\perp(t)=s_\perp(0)+(H\Delta_z(0)/gs_0)(1-\cos gs_0t)+(\dot{s}_\perp(0)/gs_0)\sin gs_0t. \quad (5.34)$$

The initial velocity is, to lowest order,

$$\dot{s}_\perp(0)=\tfrac{1}{2}Hs_0\theta(0). \quad (5.35)$$

If we place these results in (5.27) we find that

$$\hat{G}_{zz}^{\Sigma\Sigma}(t)=\lim_{N\to\infty}N[\langle\Delta_z^2\rangle(\cos gs_0t-1)+\langle\theta\Delta_z\rangle\tfrac{1}{2}s_0\sin gs_0t], \quad (5.36)$$

plus a time-independent term which does not contribute to G_{zz}. The equilibrium ensemble averages are found in Appendix C; the result is

$$G_{zz}^{\Sigma\Sigma}(t)=(1/\beta g)(\cos gs_0t-1). \quad (5.37)$$

To find $G_{+-}^{\Sigma\Sigma}$, we differentiate (5.34) and use (5.35) to get:

$$\dot{s}_\perp(t)\dot{s}_\perp(0)=(\tfrac{1}{2}Hs_0)^2\cos(gs_0t)\theta(0)^2-\tfrac{1}{2}H^2s_0\sin(gs_0t)\theta(0)\Delta_z(0).$$

The time-independent term in (5.31) does not contribute to G, and, using the ensemble averages in Appendix C, we have

$$G_{+-}^{\Sigma\Sigma}(t)=(1/\beta g)(\cos gs_0t-1). \quad (5.38)$$

The values of the equilibrium correlation functions (Appendix C)

$$\lim_{N\to\infty}N\langle\Sigma_+(0)\Sigma_-(0)\rangle=2/\beta g,$$

$$\lim_{N\to\infty}N\langle\Sigma_+(0)s_-(0)\rangle=0,$$

enable us to find, by time integrations of (5.38),

$$G_{+-}^{s\Sigma}(t)=G_{-+}^{s\Sigma}(t)=-i\frac{H}{\beta g}\left(\frac{\sin gs_0t}{gs_0}+t\right), \quad (5.39)$$

and

$$G_{+-}^{ss}(t)=\frac{H^2}{\beta g}\left(\frac{\cos gs_0t-1}{(gs_0)^2}-\frac{t^2}{2}\right). \quad (5.40)$$

The corresponding linear response functions are

$$L_{zz}^{\Sigma\Sigma}(z)=gs_0^2/(z^2-(gs_0)^2), \quad (5.41)$$

$$L_{+-}^{\Sigma\Sigma}(z)=gs_0^2/(z^2-(gs_0)^2), \quad (5.42)$$

$$L_{+-}^{s\Sigma}(z)=L_{+-}^{\Sigma s}(z)=\frac{H}{g}\left(\frac{z}{z^2-(gs_0)^2}+\frac{1}{z}\right), \quad (5.43)$$

$$L_{+-}^{ss}(z)=\frac{H^2}{g}\left(\frac{1}{z^2}+\frac{1}{z^2-(gs_0)^2}\right). \quad (5.44)$$

VI. CONCLUSIONS

There are several remarks to be made:

(A) The correlation functions have the properties described in the Introduction. In particular ω_0 [Eq. (5.11)] is real in the nonferromagnetic region and becomes imaginary as the ferromagnetic region is entered. Since all the nonferromagnetic correlation functions are even functions of ω_0, as a transition point is approached they remain analytic functions of β and H, and have unique analytic continuations into the ferromagnetic region, which grow exponentially in time. Similarly, as the ferromagnetic region is approached, poles of the response functions, (5.17)–(5.19), move toward the origin, reaching it at the transition point. Beyond the transition point the correct response functions are the ferromagnetic ones, (5.42)–(5.44). These join continuously to the nonferromagnetic ones. The pole at $z=0$ remains there throughout the ferromagnetic region, being just the Goldstone pole associated with the broken symmetry of the ferromagnetic state.[20] However, in the analytic continuation of the nonferromagnetic response function the Goldstone pole does not stay at the origin, but moves up into the complex plane, reflecting the exponential growth of the time-dependent function.

(B) An exponentially growing linear response function signifies the failure of the assumption that a weak disturbance produces a weak response, but what is one to make of a correlation function that grows exponentially? To answer this we first re-emphasize the fact that the infinite N case is not only the only one in which it is easy to calculate exact results, but also the only case in which our conclusions apply. When N is finite there is no mathematical phase transition, and all analytic continuations of equilibrium correlation functions remain equilibrium correlation functions. Exponentially growing correlation functions can therefore be associated only with the mathematically infinite system. But in the infinite system growth from order unity to order N is indistinguishable from unbounded growth.

To see how this accounts for the unbounded growth of the correlation functions, consider, as the simplest example, the function

$$G^{s_\perp s_\perp}(t)=\lim_{N\to\infty} N\langle[s_\perp(t)-s_\perp]s_\perp\rangle$$
$$=(2\sigma H/\beta\omega_0^2)f(\beta\sigma H)(\cos\omega_0 t-1)\,,$$

in the nonferromagnetic region. [Although this is one we have not explicitly calculated it must be the same as

$$G_{+-}^{ss}(t)=\lim_{N\to\infty} N\langle(s_\perp(t)e^{i[\varphi(t)-\varphi]}-s_\perp)s_\perp\rangle\,,$$

since $s_\perp(t)s_\perp$ is already of order $1/N$ in the nonferromagnetic region, and hence the phase factor can be ignored at any finite time.] Its oscillatory behavior is due to the fact that P picks out only initial values very close to the stable stationary point $s_\perp=0$. When continued into the ferromagnetic region it still behaves as if only initial values near $s_\perp=0$ were contributing, but now this is an unstable stationary point. Therefore[21] initial values very near $s_\perp=0$ will lead to oscillations of $s_\perp(t)$ between 0 and a value of order σ, i.e., of order unity with respect to N. Thus $N[s_\perp(t)-s_\perp(0)]$ will be of order N for almost all times when $s_\perp(0)$ is close to, but not equal to, zero.

We have now reversed the problem and must explain why in the limit of infinite N the continuation of $G^{s_\perp s_\perp}$ into the ferromagnetic region is not infinite for all nonzero times. The answer to this is that because both the initial position and velocity of $s_\perp$ become closer and closer to zero with increasing N, the period of the macroscopic oscillations becomes infinite as $N\to\infty$. Thus, when N is enormous, the initial values contributing to the analytic continuation of $G^{s_\perp s_\perp}$ are so close to being stationary that it takes $\mathbf{s}_\perp$ an enormous time to begin its journey from the origin. When N becomes infinite it takes $G^{s_\perp s_\perp}$ an infinite time to grow to order N, and what we see is an exponential growth.

The role of the fluctuation dissipation theorem that this feature of our model reflects leads to an interesting possibility. Suppose it were eventually established that analytic continuations of correlation and linear response functions through transition points described physical nonequilibrium states. If the singularity at the transition point were a branch point, the continued functions might lie on different sheets and would then no longer be related by the fluctuation dissipation theorem. If, however, as in our model, the continuation were unique, then the equilibrium fluctuation dissipation theorem would continue to hold in the nonequilibrium states. This would impose stringent restrictions on the kinds of perturbations capable of destroying the nonequilibrium state: They would have to be such that the corresponding correlation functions would be capable of growing from order unity to order N in describing the relaxation back to an equilibrium state. In our model this growth can be rationalized for all perturbing magnetic fields, but in a realistic system the requirement of only physically sensible growing correlation functions would lead to the stability of the nonequilibrium state under a large class of perturbations. Although it is fun to contemplate the ramifications of this idea, it would be foolish to push it any further before one knows whether the speculations on which it is based, can be put on solid ground.

(C) The secular terms in the ferromagnetic correlation functions (5.39) and (5.40) are correct. They are there as a reflection of the infinitesimal rate of precession of $\mathbf{s}_\perp$ around the z axis, and appear as an unbounded growth in the infinite N correlation functions for reasons essentially the same as those just discussed. Since they are due to a zero frequency (the rate of precession goes to zero as $N\to\infty$) mode of the system that exists in the absence of any perturbation,

[20] Symmetry is broken in the sense that the equilibrium $\langle \mathbf{s}_\perp\rangle$ in the presence of an additional weak magnetic field perpendicular to $\mathbf{H}$ does not approach, as the perpendicular field vanishes, the value $\langle \mathbf{s}_\perp\rangle=0$ which holds in the absence of such a field.

[21] This and subsequent statements about the exact time dependence are easy to prove in terms of the motion of the equivalent particle in two dimensions discussed in Sec. IV.

they lead to the poles at $z=0$ in the linear response functions (5.43) and (5.44). In this context they are much less alarming, being just the Goldstone poles due to the broken symmetry that is present throughout the ferromagnetic region.

(D) If analytically continued correlation functions describe a physical state, is this state metastable or unstable? In our model the nonequilibrium state so reached is one which could be described by an ensemble in which, initially, $\mathbf{s}^{(1)}(t)$ and $\mathbf{s}^{(2)}(t)$ were very close to the unstable stationary points $(0, 0, \pm\frac{1}{2}s_{0_z})$, $s_{0_z}>H/g$. But even though this point is dynamically unstable, the time-dependent[22] $\langle\mathbf{s}^{(\alpha)}(t)\rangle$ will remain close to their initial values, since the spins of each element of the ensemble, although they oscillate between the initial point and a macroscopically differing one, still spend almost all of their time in the neighborhood of their initial values, because of the microscopic difference between their initial values and the stationary ones. If, however, there were some dissipative mechanism present to damp these large oscillations, then after a long time the $\langle\mathbf{s}^{(\alpha)}(t)\rangle$ would assume values appropriate to an equilibrium state in the ferromagnetic region.[23] The time this took would be of order $\ln(N)$ times the number of oscillations possible before the damping became macroscopic. Although this is infinite for strictly infinite N, for large but finite N the state would appear to be unstable in the presence of dissipation.[24]

(E) The nonuniform magnetic field is essential. When the Weiss model is not in a magnetic field, the continuations of nonferromagnetic correlation and linear response functions into the ferromagnetic region remain dynamically stable, even though the corresponding stationary point of the spin distribution function is not a local maximum. This is because the magnitude of the total spin is now conserved. Although a state with total spin zero would like, from thermodynamic considerations, to grow into one with a net macroscopic spin, it is dynamically incapable of doing so. Nor will instabilities appear in the Weiss model in a uniform magnetic field, since s_z is conserved, and hence a state aligned opposite to $\mathbf{H}$ cannot correct itself.

(F) In the limit of infinite N the random-phase approximation gives the exact linear response functions of the model.[25] It is easy, once one understands the structure of $P(\mathbf{s}^{(1)},\mathbf{s}^{(2)})$ and the exact equations of motion to convince oneself that this must be so. It might be instructive in developing further the kind of approach Haag[26] used to prove that the BCS solution was exact in the limit of infinite volume, to try to construct along similar lines a rigorous proof that the random phase approximation is (or is not) exact in the infinite N limit of the quantum version of our model.

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APPENDIX A

The fluctuation dissipation theorem for a classical spin system may be somewhat unfamiliar, so its proof is indicated here.

Suppose we have a classical system with canonical variables $q_1\cdots q_N$, $p_1\cdots p_N$ (which we shall denote collectively simply by x) and with Hamiltonian $\mathcal{H}(x)$. Let the system be perturbed by a potential $U(x(t),t)$ which vanishes in the remote past. To linear order in U, the time dependence of any function $A(x(t))$ is given by

$$A(x(t))=A(x_0(t))+\int_{-\infty}^{t}[A(x_0(t)),U(x_0(t'),t')]dt', \quad \text{(A1)}$$

where $x_0(t)$ is the solution to the equations of motion in the absence of U, and the bracket is a Poisson bracket. We wish to specialize (A1) to the case $A=s_m{}^i(t)$, $U=\sum_{jn}h_n{}^j(t')s_n{}^j(t')$, and average both sides over a canonical ensemble of initial values at some time before the appearance of U. Suitable canonical variables for a classical spin are s_z and φ (its angle in the x-y plane), so the Poisson bracket is

$$[A,U]=\sum_{i=1}^{N}\frac{\partial A}{\partial\varphi^i}\frac{\partial U}{\partial s_z{}^i}-\frac{\partial A}{\partial s_z{}^i}\frac{\partial U}{\partial\varphi^i}$$

$$=\sum_{i=1}^{N}\mathbf{s}^i\cdot\frac{\partial A}{\partial\mathbf{s}^i}\times\frac{\partial U}{\partial\mathbf{s}^i}. \quad \text{(A2)}$$

The second apparently noncanonical form is much easier to use, since it allows us to remain in rectangular coordinates, where, for example, the equations of motion are enormously simplified.

[22] They are time-dependent since the density matrix is now no longer a function of the Hamiltonian.

[23] This can be seen from considering the behavior, in the presence of a dissipative term, of the two-dimensional particle of Sec. IV.

[24] From a purely thermodynamic point of view we should probably call it unstable, since it is not described by a local maximum of P.

[25] In our model the random-phase approximation reduces to the following procedure: define, as in the theory of the Weiss model, an internal molecular field, $\mathbf{H}_m=g(\langle\mathbf{s}^{(1)}\rangle+\langle\mathbf{s}^{(2)}\rangle)$, find the partition function in the effective field $\mathbf{H}+\mathbf{H}_m$, and use it to determine self-consistently the value of $\mathbf{H}_m$; then calculate the response of $\mathbf{s}^{(\alpha)}$ to a perturbation $\mathbf{h}(t)$ to lowest order in $\mathbf{h}$ and the deviation of $\mathbf{s}^{(\alpha)}$ from a solution to the self-consistent equations. There is a slight complication in the ferromagnetic region, since there are many solutions to the self-consistent equations to use as initial equilibrium states; however, the random-phase approximation response of a given ferromagnetic equilibrium state averaged over all possible orientations of $\mathbf{s}_\perp$ gives the correct result.

[26] R. Haag, Nuovo Cimento **25**, 287 (1962).

The linear response is therefore

$$\delta\langle s_m{}^i(t)\rangle=\int_{-\infty}^{t}dt'\int\prod_{i=1}^{N}d\Omega^i\exp[-\beta\mathcal{H}(\{\mathbf{s}^i\})]\times\Bigg[\sum_{k,n,j}\mathbf{s}^k\cdot\bigg(\frac{\partial}{\partial\mathbf{s}^k}s_m{}^i(\{\mathbf{s}^l\},t)\times\frac{\partial}{\partial\mathbf{s}^k}s_n{}^j(\{\mathbf{s}^l\},t')\bigg)h_n{}^j(t')\Bigg].\quad\text{(A3)}$$

Now the kth term in the summation will contain an integral of the form

$$\sigma\int d\Omega^k\exp(-\beta\mathcal{H})\hat{n}(\Omega^k)\cdot\frac{\partial}{\partial\mathbf{s}^k}s_m{}^i(\{\mathbf{s}^l\},t)\times\frac{\partial}{\partial\mathbf{s}^k}s_n{}^j(\{\mathbf{s}^l\},t').$$

If we transform this to a volume integral, it becomes

$$\sigma\int_{s^k<\sigma}d\mathbf{s}^k\frac{\partial}{\partial\mathbf{s}^k}\cdot\bigg(\exp(-\beta\mathcal{H})\frac{\partial}{\partial\mathbf{s}^k}s_m{}^i(\{\mathbf{s}^l\},t)\times\frac{\partial}{\partial\mathbf{s}^k}s_n{}^j(\{\mathbf{s}^l\},t')\bigg).\quad\text{(A4)}$$

But (A4) is invariant under cyclical permutation of the three functions $\exp(-\beta\mathcal{H})$, $s_m{}^i$, and $s_n{}^j$. Therefore

$$\begin{aligned}\delta\langle s_m{}^i(t)\rangle&=\int_{-\infty}^{t}dt'\int\prod_{i=1}^{N}d\Omega^i\exp(-\beta\mathcal{H})\times\sum_{jn}[s_m{}^i(t),s_n{}^j(t')]h_n{}^j(t')\\&=\int_{-\infty}^{t}dt'\int\prod_{i=1}^{N}d\Omega^i\sum_{jn}s_n{}^j(t')\times[\exp(-\beta\mathcal{H}),s_m{}^i(t)]h_n{}^j(t')\\&=\beta\int_{-\infty}^{t}dt'\sum_{jn}\langle\dot{s}_m{}^i(t)s_n{}^j(t')\rangle h_n{}^j(t').\end{aligned}$$

APPENDIX B

In the limit of large N, the distribution function $P(\mathbf{s}^{(1)},\mathbf{s}^{(2)})$ will be almost entirely concentrated at points where $\Phi(\mathbf{s}^{(1)},\mathbf{s}^{(2)})$ [Eq. (3.7)] is minimum. In finding the minima of Φ we shall need the following properties of the function $f(\eta)=\coth\eta-1/\eta$ [27]:

$$f(\eta)=-f(-\eta);\quad f(\eta)>0,\ \eta>0;\quad f(0)=0;\quad\text{(B1)}$$

$$f'(\eta)>0;\quad f'(0)=\tfrac{1}{3};\quad\text{(B2)}$$

$$f''(\eta)<0,\quad\eta>0;\quad\text{(B3)}$$

$$d/d\eta(f(\eta)/\eta)<0,\quad\eta>0.\quad\text{(B4)}$$

[27] Equations (B1) and (B2) are trivial, and (B3) with $f(0)=0$ implies (B4); the fact that $\sinh 3x-3\sinh x-4x^3\cosh x$ has a Taylor series with positive coefficients establishes (B3).

Minima of Φ can only occur at points satisfying

$$0=\partial\Phi/\partial\mathbf{s}^{(1)}=-\beta(\mathbf{H}+g(\mathbf{s}^{(1)}+\mathbf{s}^{(2)}))+\eta^{(1)}\mathbf{s}^{(1)}/\sigma s^{(1)},\quad\text{(B5)}$$

$$0=\partial\Phi/\partial\mathbf{s}^{(2)}=\beta(\mathbf{H}-g(\mathbf{s}^{(1)}+\mathbf{s}^{(2)}))+\eta^{(2)}\mathbf{s}^{(2)}/\sigma s^{(2)}.\quad\text{(B6)}$$

Such a point will be a local minimum provided the matrix of second derivatives at the point,

$$\begin{aligned}M_{\alpha m,\gamma n}&=\partial^2\Phi/\partial s_m{}^{(\alpha)}\partial s_n{}^{(\gamma)}\\&=\delta_{mn}[(\eta^{(\alpha)}/\sigma s^{(\alpha)})-\beta g]+2(s_m{}^{(\alpha)}s_n{}^{(\alpha)}/s^{(\alpha)2})\\&\quad\times[(1/\sigma^2f'(\eta^{(\alpha)}))-(\eta^{(\alpha)}/2\sigma s^{(\alpha)})],\quad\alpha=\gamma,\\&=-\beta g\delta_{mn},\quad\alpha\neq\gamma,\end{aligned}\quad\text{(B7)}$$

is positive definite.

We first look for stationary points with $\mathbf{s}^{(1)}$ parallel to $\mathbf{s}^{(2)}$. (Consider them parallel when either is 0.) This can only happen if both are parallel to $\mathbf{H}$. If we define

$$\chi^{(\alpha)}=\eta^{(\alpha)}(\sin(s_z{}^{(\alpha)})),\quad\text{(B8)}$$

the (3.8) becomes

$$s_z{}^{(\alpha)}=\tfrac{1}{2}\sigma f(\chi^{(\alpha)}),\quad\text{(B9)}$$

and the stationary conditions are

$$\begin{aligned}\chi^{(1)}&=\beta\sigma(H+gs_z),\\\chi^{(2)}&=-\beta\sigma(H-gs_z),\end{aligned}\quad\text{(B10)}$$

where $s_z=s_z{}^{(1)}+s_z{}^{(2)}$. The general solution to (B9) and (B10) is

$$\begin{aligned}s_{x,y}{}^{(\alpha)}&=0,\quad\alpha=1,2;\\s_z{}^{(1)}&=\tfrac{1}{2}\sigma f[\beta\sigma(H+gs_z)];\\s_z{}^{(2)}&=-\tfrac{1}{2}\sigma f[\beta\sigma(H-gs_z)];\end{aligned}\quad\text{(B11)}$$

where s_z is any solution to

$$s_z=\tfrac{1}{2}\sigma[f(\beta\sigma(H+gs_z))-f(\beta\sigma(H-gs_z))].\quad\text{(B12)}$$

One solution to (B12) is $s_z=0$, which leads to the stationary point given in (3.9). To establish that this point gives the absolute minimum of Φ in region I, we will show that it is a local minimum whenever $H/g>\sigma f(\beta\sigma H)$, that no other solution of (B12) gives a local minimum, and that Φ has no nonparallel stationary points unless $H/g<\sigma f(\beta\sigma H)$.

Let us label the rows and columns of M by $s_x{}^{(1)}$, $s_y{}^{(1)}$, $s_z{}^{(1)}$, $s_x{}^{(2)}$, $s_y{}^{(2)}$, $s_z{}^{(2)}$, in that order. At a parallel stationary point

$$M=\beta g\begin{pmatrix}d^{(1)}&-1\\-1&d^{(2)}\end{pmatrix},\quad\text{(B13)}$$

where 1 is the 3×3 unit matrix, and $d^{(1)}$ and $d^{(2)}$ are diagonal 3×3 matrices with diagonal elements

$$\begin{aligned}d_i{}^{(\alpha)}&=(\chi^{(\alpha)}/\beta gs_z{}^{(\alpha)})-1,\quad i=x,y;\\d_z{}^{(\alpha)}&=[2/\beta g\sigma^2f'(\chi^{(\alpha)})]-1.\end{aligned}\quad\text{(B14)}$$

One easily verifies that

$$(M-\lambda)^{-1}=\frac{1}{\beta g}\begin{pmatrix} d(d^{(2)}-\mu) & d \\ d & d(d^{(1)}-\mu)\end{pmatrix} \quad \text{(B15)}$$

where

$$\mu=\lambda/\beta g\,,\quad d=[(d^{(1)}-\mu)(d^{(2)}-\mu)-1]^{-1}. \quad \text{(B16)}$$

The eigenvalues of M are therefore βg times the roots of

$$(d_i^{(1)}-\mu)(d_i^{(2)}-\mu)-1=0\,,\quad i=x,y,\ \text{or}\ z\,,$$

all of which must be positive for the corresponding stationary point to be a local minimum of Φ. We must therefore have

$$d_i^{(1)}+d_i^{(2)}>0\,,$$

and

$$d_i^{(1)}d_i^{(2)}-1>0;\quad i=x,y,\ \text{or}\ z. \quad \text{(B17)}$$

At the stationary points (3.9)

$$\begin{aligned} d_i^{(1)}&=d_i^{(2)}=(2H/g\sigma f(\beta\sigma H))-1\,,\quad i=x,y;\\ d_z^{(1)}&=d_z^{(2)}=(2/\beta g\sigma^2 f'(\beta\sigma H))-1\,. \end{aligned} \quad \text{(B18)}$$

This reduces (B17) to the conditions

$$g\sigma f(\beta\sigma H)/H<1\,,\quad \beta g\sigma^2 f'(\beta\sigma H)<1. \quad \text{(B19)}$$

The second condition is unnecessary, since $f(x)/x$ always exceeds $f'(x)$, and the first is just the condition that β and H are within region I.

At a stationary point with $s_z\neq 0$, the second of conditions (B17) always fails when $i=x$ (or y). With (B9) and (B14) it requires that

$$1>\frac{\beta g\sigma^2}{2}\left(\frac{f(\chi^{(1)})}{\chi^{(1)}}+\frac{f(\chi^{(2)})}{\chi^{(2)}}\right). \quad \text{(B20)}$$

On the other hand, (B9) and (B12) give

$$1=(\sigma/2s_z)(f(\chi^{(1)})+f(\chi^{(2)})). \quad \text{(B21)}$$

Subtracting (B21) from (B20) and again applying (B12) gives

$$0>\frac{H}{gs_z}\left(\frac{f(\chi^{(2)})}{\chi^{(2)}}-\frac{f(\chi^{(1)})}{\chi^{(1)}}\right).$$

Now $f(x)/x$ is a decreasing function of the magnitude of x, so either H and s_z have the same sign and $|\chi^{(1)}|<|\chi^{(2)}|$, or H and s_z have opposite sign and $|\chi^{(1)}|>|\chi^{(2)}|$. But either possibility is inconsistent with (B10), so there are no parallel minima of Φ other than (3.9).

To see that there are no nonparallel stationary points in region I, and to establish the nature of the minima of Φ in region II, we return to the general stationary conditions (B5) and (B6). When both hold,

$$\mathbf{s}^{(1)}[(\eta^{(1)}/\sigma s^{(1)})-2\beta g]+\mathbf{s}^{(2)}[(\eta^{(2)}/\sigma s^{(2)})-2\beta g]=0. \quad \text{(B22)}$$

If $\mathbf{s}^{(1)}$ and $\mathbf{s}^{(2)}$ are not parallel, then

$$\eta^{(1)}/\sigma s^{(1)}=\eta^{(2)}/\sigma s^{(2)}=2\beta g. \quad \text{(B23)}$$

Since $f(\eta)$ and $f(\eta)/\eta$ are monotonic for positive η, (B23) and (3.8) imply

$$\eta^{(1)}=\eta^{(2)}\,,\quad s^{(1)}=s^{(2)}. \quad \text{(B24)}$$

Furthermore, (B23) reduces (B5) or (B6) to

$$\mathbf{s}^{(1)}-\mathbf{s}^{(2)}=\mathbf{H}/g\,, \quad \text{(B25)}$$

which requires that (3.10) holds, with $\mathbf{s}_0$ perpendicular to $\mathbf{H}$ because of (B24). Equations (3.10), (B23), and (3.8) now lead to (3.11), which determines the magnitude of $\mathbf{s}_0$, and has a real positive solution if and only if β and H are in region II.

The direction of $\mathbf{s}_0$ is arbitrary, but the value of Φ is independent of this direction. Since (3.10) and (3.11) give the only stationary points in region II, these points must give Φ an absolute minimum.

APPENDIX C

Nonferromagnetic Region

Because $P(\mathbf{s}^{(1)},\mathbf{s}^{(2)})$ has the form (3.6), in a calculation of the second moments,

$$\Delta_{ij}^{\alpha\gamma}=\lim_{N\to\infty} N\langle(s_i^{(\alpha)}-\langle s_i^{(\alpha)}\rangle)(s_j^{(\gamma)}-\langle s_j^{(\gamma)}\rangle)\rangle\,, \quad \text{(C1)}$$

the leading term is given by taking

$$P(\mathbf{s}^{(1)},\mathbf{s}^{(2)})=\text{constant}\times\exp[-\tfrac{1}{2}N\sum M_{\alpha i,\gamma j}\times(s_i^{(\alpha)}-s_{0_i}^{(\alpha)})(s_j^{(\gamma)}-s_{0_j}^{(\gamma)})]\,, \quad \text{(C2)}$$

where M is the matrix of second derivatives, given at the nonferromagnetic maximum by (B13) and (B18). Corrections to the moments given by (C2) due to the expansion of $C(\mathbf{s}^{(\alpha)})$ about $\mathbf{s}_0^{(\alpha)}$, or due to higher order terms in the expansion of $\Phi(\mathbf{s}^{(1)},\mathbf{s}^{(2)})$ all give no contributions to (C1) as $N\to\infty$. Furthermore, higher moments,

$$\lim_{N\to\infty} N\langle(s_{i_1}^{(\alpha_1)}-\langle s_{i_1}^{(\alpha_1)}\rangle)\cdots(s_{i_m}^{(\alpha_m)}-\langle s_{i_m}^{(\alpha_m)}\rangle)\rangle$$

have leading terms which vanish as $N^{-\frac{1}{2}m+1}$.

The second moments are therefore given by

$$\Delta_{ij}^{\alpha\gamma}=(M^{-1})_{\alpha i,\gamma j}. \quad \text{(C3)}$$

The elements of the inverse matrix are found from (B15) (with $\mu=0$), (B16), and (B18), and lead to the moments:

$$\begin{aligned} \Delta_{ii}^{\alpha\gamma}&=\frac{\sigma}{4\beta H}f(\beta\sigma H)\left(2\delta_{\alpha\gamma}+\frac{g\sigma f(\beta\sigma H)}{H-g\sigma f(\beta\sigma H)}\right),\quad i=x,y;\\ \Delta_{zz}^{\alpha\gamma}&=\frac{\sigma^2}{4}f'(\beta\sigma H)\left(2\delta_{\alpha\gamma}+\frac{\beta g\sigma^2 f'(\beta\sigma H)}{1-\beta g\sigma^2 f'(\beta\sigma H)}\right);\\ \Delta_{ij}^{\alpha\gamma}&=0\,,\quad i\neq j. \end{aligned} \quad \text{(C4)}$$

In particular,

$$\lim_{N\to\infty} N\langle\Sigma_+\Sigma_-\rangle$$
$$=\lim_{N\to\infty} N\langle(s_x^{(1)}-s_x^{(2)})^2+(s_y^{(1)}-s_y^{(2)})^2\rangle$$
$$=\Delta_{xx}^{11}+\Delta_{xx}^{22}+\Delta_{yy}^{11}+\Delta_{yy}^{22}-2(\Delta_{xx}^{12}+\Delta_{yy}^{12})$$
$$=2\sigma f(\beta\sigma H)/\beta H\,, \tag{C5}$$

and, similarly,

$$\lim_{N\to\infty} N\langle s_+\Sigma_-\rangle=0\,. \tag{C6}$$

Ferromagnetic Region

The difficulty due to the existence of a family of maxima of P is removed by working in cylindrical coordinates with the function $\hat{P}$ [Eq. (5.20)], which has a single maximum, (5.21), and by taking only moments of the cylindrical variables:

$$s_\perp^{(1)},\quad s_\perp^{(2)},\quad s_z^{(1)},\quad s_z^{(2)},\quad \theta. \tag{C7}$$

The second variation in Φ near a stationary point follows from (B7):

$$d^2\Phi=\tfrac{1}{2}\beta g\{\tfrac{1}{4}s_0^2(d\theta)^2+(ds_\perp^{(1)}-ds_\perp^{(2)})^2+(ds_z^{(1)}-ds_z^{(2)})^2+K[(s_0ds_\perp^{(1)}+(H/g)ds_z^{(1)})^2+(s_0ds_\perp^{(2)}-(H/g)ds_z^{(2)})^2]\}\,, \tag{C8}$$

where

$$K=[2/(s_0^2+(H/g)^2)][(1/\beta g\sigma^2 f'(\eta))-1]\,, \tag{C9}$$

and

$$\eta=\beta g[s_0^2+(H/g)^2]^{1/2}. \tag{C10}$$

Since (3.11) can be written, $1=\beta g\sigma^2 f(\eta)/\eta$, and since $0<f'(\eta)<f(\eta)/\eta$, K is positive and (C8) is positive definite.

We can write (C8) more compactly as

$$d^2\Phi=\tfrac{1}{2}\beta g[\tfrac{1}{4}s_0^2(d\theta)^2+\sum_1^4 M_{\mu\nu}ds_\mu ds_\nu]\,, \tag{C11}$$

where we understand ds_μ to run through $ds_\perp^{(1)}$, $ds_\perp^{(2)}$, $ds_z^{(1)}$, $ds_z^{(2)}$, in that order. The matrix M is

$$M=\begin{bmatrix}1+Ks_0^2 & -1 & Ks_0H/g & 0\\ -1 & 1+Ks_0^2 & 0 & -Ks_0H/g\\ Ks_0H/g & 0 & 1+K(H/g)^2 & -1\\ 0 & -Ks_0H/g & -1 & 1+K(H/g)^2\end{bmatrix}. \tag{C12}$$

Second moments involving the variables in (C7) are given by

$$\lim_{N\to\infty} N\langle\theta^2\rangle=4/\beta g s_0^2\,, \tag{C13}$$

$$\lim_{N\to\infty} N\langle\theta(s_\mu-\langle s_\mu\rangle)\rangle=0\,, \tag{C14}$$

$$\lim_{N\to\infty} N\langle(s_\mu-\langle s_\mu\rangle)(s_\nu-\langle s_\nu\rangle)\rangle=(M^{-1})_{\mu\nu}/\beta g\,. \tag{C15}$$

The inverse matrix can be verified to be

$$M^{-1}=\frac{1}{2K(H/g)^2}\begin{bmatrix}a^2 & a^2 & 0 & 0\\ a^2 & a^2 & 0 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 1 & 1\end{bmatrix}+\frac{1}{4}\begin{bmatrix}a^2+1 & a^2-1 & -a-a^{-1} & a-a^{-1}\\ a^2-1 & a^2+1 & -a+a^{-1} & a+a^{-1}\\ -a-a^{-1} & -a+a^{-1} & a^{-2}+1 & a^{-2}-1\\ a-a^{-1} & a+a^{-1} & a^{-2}-1 & a^{-2}+1\end{bmatrix}, \tag{C16}$$

where $a=H/gs_0$.

The moments required in Sec. V are:

$$\lim_{N\to\infty} N\langle\Delta_z^2\rangle=\lim_{N\to\infty} N\langle(ds_z^{(1)}-ds_z^{(2)})^2\rangle$$
$$=(1/\beta g)((M^{-1})_{33}+(M^{-1})_{34}-(M^{-1})_{34}-(M^{-1})_{43})=1/\beta g\,; \tag{C17}$$

$$\lim_{N\to\infty} N\langle\Sigma_+\Sigma_-\rangle=\lim_{N\to\infty} N\langle(s_\perp^{(1)2}+s_\perp^{(2)2}-2s_\perp^{(1)}s_\perp^{(2)}\cos\theta)\rangle$$
$$=\lim_{N\to\infty} N(\langle(ds_\perp^{(1)}-ds_\perp^{(2)})^2\rangle+\tfrac{1}{4}s_0^2\langle\theta^2\rangle)$$
$$=(1/\beta g)((M^{-1})_{11}+(M^{-1})_{22}-(M^{-1})_{21}-(M^{-1})_{12})+1/\beta g=2/\beta g\,. \tag{C18}$$

Finally, note that

$$\lim_{N\to\infty} N\langle\Sigma_+s_-\rangle=\lim_{N\to\infty} N(\langle s_\perp^{(1)2}-s_\perp^{(2)2}\rangle+i\langle s_\perp^{(1)}s_\perp^{(2)}\sin\theta\rangle)=0 \tag{C19}$$

(even for finite N), since P is a symmetric function of $s_\perp^{(1)}$ and $s_\perp^{(2)}$, and an even function of θ.