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Effects of Collisions on Electron Density Fluctuations in Plasmas*

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The spectrum of electron density fluctuations has previously been calculated for a collisionless plasma. In this paper, the theory is extended to include collisions as represented by a Fokker-Planck equation such as the one used in the theory of Brownian motion. Numerical calculations have been carried out to show graphically the transition of the spectrum from the collisionless to the collision-dominated case. The results of the present analysis involve some thus far untabulated functions. These functions have been numerically evaluated and the results are presented in graphical form.

I. INTRODUCTION

RECENTLY, considerable interest has been shown in the determination of the spectrum of the electron density fluctuations in plasma. This interest has been mainly stimulated by the experiment of Bowles[1] on the incoherent scattering of electromagnetic waves by the ionosphere and by the possible extensions of his experiment to laboratory plasmas.

Several independent calculations have been presented for the quantity of interest $\langle|n^e(\mathbf{k},\omega)|^2\rangle$, which represents the spectrum of the spatial Fourier transform of the electron density. The spectrum of the scattered power $P(\omega)$, for the incident wave characterized by the frequency ω_0 and the wave number $\mathbf{k}_0$, is proportional to $\langle|n^e(\mathbf{k},\omega)|^2\rangle$, when ω is measured from ω_0 and $\mathbf{k}$ is equal to $|k_0|(\mathbf{e}_r-\mathbf{e}_0)$, where $\mathbf{e}_r$ and $\mathbf{e}_0$ are unit vectors in the direction of the scattered and incident wave. Dougherty and Farley,[2] Salpeter,[3] and Rosenbluth and Rostoker[4] have presented the results when the plasma can be considered collisionless for the entire range of the parameter hk, where h is the Debye shielding length. Fejer[5] has presented results for the limiting situations of a collisionless and a collision-dominated plasma and in both cases for the two limiting values of the parameter hk, $hk\to 0$, and $hk\to\infty$.[6]

In view of possible applications of incoherent scattering of electromagnetic waves by plasma as a diagnostic technique in the laboratory, where one may encounter a wide range of the parameter hk and collision frequency, we have considered electron density fluctuations over the entire range of collision effects, i.e., from the collisionless to the collision-dominated case, embracing the whole range of the parameter hk. In Sec. II, expressions have been derived for the quantity $\langle|n^e(\mathbf{k},\omega)|^2\rangle$, which are valid for all values of collision frequency and parameter hk. These expressions reduce to the formulas presented by the other authors (Refs. 2–5) when suitable limits are taken. In Sec. III, results of the present analysis are presented in graphical form and are discussed. The analysis is carried out for thermodynamic equilibrium except that the electrons and ions may have different temperatures. Only singly ionized ions are considered, although the extension to a multiply ionized plasma is straightforward.

II. ELECTRON DENSITY FLUCTUATIONS

Fluctuations of electron density in plasma may be conveniently written as the sum of two parts. One part pertains to the fluctuations caused by the random thermal motion of electrons themselves modified by the accompanying electric fields; and the second part pertains to the fluctuations induced in the electron density by the random thermal motion of ions through the interaction of electric fields associated with these two species. In the following analysis, the first part will be denoted by n_e^e and the second part by n_i^e. Similar notation will be used to represent the ion density fluctuations.

To compute the first part n_e^e, we proceed as follows. When the electron density fluctuates about its mean value due to the random thermal motion of electrons, it produces a fluctuating electrostatic field. This electric field, in turn, modifies the random thermal motion of electrons and also induces fluctuations in ion number density. Let n^{er} denote the hypothetical electron density fluctuations which would have existed had the spontaneous random motion of electrons not been modified by the accompanying electric field. The actual electron density fluctuations n_e^e can now be written as the sum of n^{er} and a quantity n^{ec}, which represents the correction to n^{er}. The quantity n^{ec} can be evaluated in terms of n^{er} by considering it a perturbation in electron density caused by an electric field E^e, the source of which, partly external and partly self-consistent, is given by charge density equal to $\{n_e^i-(n^{er}+n^{ec})\}$, where n_e^i, as previously stated, is the change in ion number density induced by the electric field E^e.

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[1] K. L. Bowles, Phys. Rev. Letters **1**, 454 (1958); National Bureau of Standards, Boulder, Colorado, Report 6070, 1959 (unpublished).

[2] J. P. Dougherty and D. T. Farley, Proc. Roy. Soc. (London) **A259**, 79 (1960).

[3] E. E. Salpeter, Phys. Rev. **120**, 1528 (1960).

[4] M. N. Rosenbluth and N. Rostoker, Phys. Fluids **5**, 776 (1962).

[5] J. A. Fejer, Can. J. Phys. **38**, 1114 (1960).

[6] The author would like to acknowledge receipt of the paper by E. C. Taylor and G. G. Comisar [Phys. Rev. **132**, 2379 (1963)]. This paper also deals with the effect of collisions on the plasma density fluctuations but it is from a different point of view.

To compute the response of electrons and ions to the field E^e, we employ a Fokker-Planck equation with collision terms of the type used in the theory of Brownian motion (e.g., see Ref. 7). If $F_j(\mathbf{r},\mathbf{v},t)$ denotes the joint position and velocity distribution at time t of electrons and ions, with subscript j equal to e and i, respectively, then the Fokker-Planck equation under consideration may be written as

$$\left[\frac{\partial}{\partial t}+\mathbf{v}\cdot\frac{\partial}{\partial \mathbf{r}}+\frac{e_j}{m_j}\mathbf{E}^e\cdot\frac{\partial}{\partial \mathbf{v}} -\beta_j\frac{\partial}{\partial \mathbf{v}}\left(\mathbf{v}+\frac{\theta_j}{m_j}\frac{\partial}{\partial \mathbf{v}}\right)\right]F_j(\mathbf{r},\mathbf{v},t)=0\,, \quad (1)$$

where β_j is the effective collision frequency, and e_j, m_j, and θ_j denote the charge, mass, and temperature, respectively. Linearizing $F_j(\mathbf{r},\mathbf{v},t)$ around a Maxwellian distribution and rearranging, we obtain from Eq. (1)

$$\left[\frac{\partial}{\partial t}+\mathbf{v}\cdot\frac{\partial}{\partial \mathbf{r}}-\beta_j\frac{\partial}{\partial \mathbf{v}}\left(\mathbf{v}+\frac{\theta_j}{m_j}\cdot\frac{\partial}{\partial \mathbf{v}}\right)\right]f_j(\mathbf{r},\mathbf{v},t) =-\frac{e_j}{m_j}\mathbf{E}^e\cdot\frac{\partial f_{j0}}{\partial \mathbf{v}}, \quad (2)$$

where we have introduced $f_j=F_j-f_{j0}$, with $f_{j0}(\mathbf{v}) =n_0(m_j/2\pi\theta_j)^{3/2}\exp(-m_jv^2/2\theta_j)$.

One observes that the operator on the left-hand side of Eq. (2) is identical with the one in Eq. (1) except that the term involving E^e is absent. The Green's function for this operator has been previously derived (Ref. 7). Using this Green's function, we construct the solution for Eq. (2) to obtain for n^{ec} and $n_e{}^i$ (calculations are shown in Appendix A)

$$n^{ec}=\int f_e(\mathbf{k},\omega,\mathbf{v})dv=-\frac{k^2}{4\pi e}\phi_{\mathbf{k},\omega}G_e(\omega)\,, \quad (3)$$

$$n_e{}^i=\int f_i(\mathbf{k},\omega,\mathbf{v})dv=\frac{k^2}{4\pi e}\phi_{\mathbf{k},\omega}G_i(\omega)\,, \quad (4)$$

where we have taken time and space Fourier transforms of the functions depending on $\mathbf{r}$ and t; $\phi_{\mathbf{k},\omega}$ is the potential representing the electric field $E^e(\mathbf{k},\omega)$, and $G_j(\omega)$ is given by

$$G_j(\omega)=\frac{4\pi e^2n_0}{m_j}\int_0^\infty \frac{1}{\beta_j}[\exp(-\beta_jt)-1] \times\exp\left[-\frac{k^2\theta_j}{m_j\beta_j{}^2}[\exp(-\beta_jt)+\beta_jt-1]\right] \times\exp(-i\omega t)dt. \quad (5)$$

[7] S. Chandrasekhar, Rev. Mod. Phys. **15**, 1 (1943).

Equations (3) and (4) along with Poissons equation

$$\phi_{\mathbf{k},\omega}=\frac{4\pi e}{k^2}\{n_e{}^i(\mathbf{k},\omega)-[n^{er}(\mathbf{k},\omega)+n^{ec}(\mathbf{k},\omega)]\} \quad (6)$$

can be solved to obtain

$$n_e{}^e=\frac{n^{er}[1-G_i(\omega)]}{1-G_i(\omega)-G_e(\omega)} \quad (7)$$

and

$$n_e{}^i=-\frac{n^{er}[G_i(\omega)]}{1-G_i(\omega)-G_e(\omega)}. \quad (8)$$

By arguments and the analysis analogous to the one that led to Eqs. (7) and (8), one obtains for $n_i{}^i$, the ion density fluctuation due to their own random thermal motion, and $n_i{}^e$, the fluctuations induced in the electron density by the ion density fluctuations

$$n_i{}^i=\frac{n^{ir}[1-G_e(\omega)]}{1-G_i(\omega)-G_e(\omega)}, \quad (9)$$

$$n_i{}^e=\frac{n^{ir}[-G_e(\omega)]}{1-G_i(\omega)-G_e(\omega)}, \quad (10)$$

where $G_j(\omega)$ is the same as defined in Eq. (5).

Summing Eqs. (7) and (10), we arrive at the total electron density fluctuations

$$n^e(\mathbf{k},\omega)=n_e{}^e+n_i{}^e=\frac{n^{er}(\mathbf{k},\omega)[1-G_i(\omega)]}{1-G_i(\omega)-G_e(\omega)} +\frac{n^{ir}(\mathbf{k},\omega)[-G_e(\omega)]}{1-G_i(\omega)-G_e(\omega)}. \quad (11)$$

[Fluctuations in ion number density, if desired, can be obtained by summing Eqs. (8) and (9).]

One notes that n^e as expressed in Eq. (11) could also have been obtained without separating it into two parts $n_e{}^e$ and $n_i{}^e$. To this end, we would have solved Eq. (1) for n^{ec} and n^{ic} with the source of the field E given by the charge density equal to $\{(n^{ir}+n^{ic})-(n^{er}+n^{ec})\}$; in this case, n^{ec} would have included $n_i{}^e$ as part of it. However, it is felt that the partition of n^e into $n_e{}^e$ and $n_i{}^e$ gives a better insight into the problem as will be seen in the results. This partition of n^e is made possible by the linearity of the governing equations.

Taking the modulus squared and averaging both sides of Eq. (11), one finds

$$\langle|n_e(\mathbf{k},\omega)|^2\rangle=\frac{\langle|n^{er}(\mathbf{k},\omega)|^2\rangle|1-G_i(\omega)|^2}{|1-G_i(\omega)-G_e(\omega)|^2} +\frac{\langle|n^{ir}(\mathbf{k},\omega)|^2\rangle|G_e(\omega)|^2}{|1-G_i(\omega)-G_e(\omega)|^2}, \quad (12)$$

where the term involving $\langle |n^{er}||n^{ir}|\rangle$ that expresses the correlation between random motion of ions and electrons is neglected, and use is also made of the fact that $G_e(\omega)$ and $G_i(\omega)$ already represent the averaged quantities.

To evaluate $\langle |n^{ir}(\mathbf{k},\omega)|^2\rangle$, we first express $\langle |n(\mathbf{k},\omega)|^2\rangle$ in terms of the transition probability for a single particle as follows:

$$n(\mathbf{r},t)=\sum_i^{n_0}\delta\{\mathbf{r}-\mathbf{r}_i(t)\}\,, \tag{13}$$

where $r_i(t)$ is the position of the ith particle at time t. Taking the Fourier transform, one obtains

$$n_{\mathbf{k}}(t)=\sum_i \exp\{-i\mathbf{k}\cdot\mathbf{r}_i(t)\}\,. \tag{14}$$

Forming the product of $n_{\mathbf{k}}(t)$ with its complex conjugate $n_{\mathbf{k}}^*(t+\tau)$ at a time τ later, we obtain after separating the diagonal and off-diagonal terms

$$n_{\mathbf{k}}(t)n_{\mathbf{k}}^*(t+\tau)=\sum_i \exp\{i\mathbf{k}\cdot[\mathbf{r}_i(t+\tau)-\mathbf{r}_i(t)]\}+\sum_{\substack{i,j\\ i\neq j}}\exp\{-i\mathbf{k},\mathbf{r}_i(t)\}\times\exp[i\mathbf{k}\cdot\mathbf{r}_j(t+\tau)]\,. \tag{15}$$

Taking the average of the modulus of both sides, neglecting the correlation between the ith and jth particles, and taking account of the identical nature of the particles, one finds that Eq. (15) yields

$$\langle |n_{\mathbf{k}}(t)n_{\mathbf{k}}^*(t+\tau)|\rangle=n_0\langle |\exp\{i\mathbf{k}\cdot[\mathbf{r}(t+\tau)-\mathbf{r}(t)]\}|\rangle\,. \tag{16}$$

Making use of the fact that $n_{\mathbf{k}}(t)$ is a stationary random function and applying the Wiener-Khinchin relation to the left-hand side of Eq. (16), the correlation function for $n_{\mathbf{k}}(t)$, one obtains for the spectral density

$$\langle |n(\mathbf{k},\omega)|^2\rangle=\frac{n_0}{2\pi}\int_{+\infty}^{+\infty}\exp(i\omega\tau)d\tau\times\langle |\exp\{i\mathbf{k}\cdot[\mathbf{r}(t+\tau)-\mathbf{r}(t)]\}|\rangle\,. \tag{17}$$

Using the transition probability for a single particle, one finds that the right-hand side of Eq. (16) can be rewritten as

$$\langle |n(\mathbf{k},\omega)|^2\rangle=\frac{n_0}{2\pi}\int\exp(i\omega\tau)d\tau\times\int d\mathbf{r}\exp(i\mathbf{k}\cdot\mathbf{r})W_{\mathbf{u}_0}(\mathbf{r},t+\tau;\mathbf{r}_0,t)\times\exp(-i\mathbf{k}\cdot r_0)W(\mathbf{r}_0)dr_0W(\mathbf{u}_0)d\mathbf{u}_0\,, \tag{18}$$

where $W_{\mathbf{u}_0}(\mathbf{r},t+\tau;\mathbf{r}_0,t)$ denotes the probability of finding the particle at time $t+\tau$ at $\mathbf{r}$ when it was initially at $\mathbf{r}_0$ with velocity $\mathbf{u}_0$; $W(\mathbf{r}_0)$ and $W(\mathbf{u}_0)$ denote the position and velocity probability distributions at time t.

The expression $\langle |n^{ir}(\mathbf{k},\omega)|^2\rangle$ can now be evaluated using Eq. (18), with the transition probability given by that of a field-free Brownian particle [consistent with our use of Eq. (11)]. As shown in Appendix B, after the required integrals are carried out, $\langle |n^{ir}(\mathbf{k},\omega)|^2\rangle$ is given by

$$\langle |n^{ir}(\mathbf{k},\omega)|^2\rangle=\frac{n_0}{\pi}\int_0^\infty\cos\omega t\times\exp\left\{-\frac{k^2\theta_j}{m_j\beta_j^2}[\exp(-\beta_jt)+\beta_jt-1]\right\}dt\,. \tag{19}$$

Equation (12) along with Eqs. (5) and (19) expresses the final result for the spectrum of the electron density fluctuations.

III. RESULTS AND DISCUSSION

To find numerical values for $\langle |n^e(\mathbf{k},\omega)|^2\rangle$ from Eq. (12), one must evaluate the integrals given in Eqs. (5) and (19). The integrals can be evaluated in terms of tabulated functions only for the two limiting cases; (a) $\beta\to 0$ (collisionless case), and (b) $\beta\to\infty$ (collision dominated case). For intermediate cases the integrals have to be evaluated numerically. To simplify the expressions, the following integrals are defined:

$$I_1{}^j=\int_0^\infty\cos\omega t\,\psi^j(t)dt\,,\quad I_2{}^j=\int_0^\infty\sin\omega t\,\psi^j(t)dt\,,$$
$$I_3{}^j=\int_0^\infty\cos\omega t\,\phi^j(t)dt\,,\quad I_4{}^j=\int_0^\infty\sin\omega t\,\phi^j(t)dt\,, \tag{20}$$

where

$$\psi^j(t)=\exp\left\{-\frac{k^2\theta_j}{m_j\beta_j^2}[\exp(-\beta_jt)+\beta_jt-1]\right\}$$

and

$$\phi^j(t)=\frac{\exp(-\beta_jt)-1}{\beta_j}\psi^j(t)\,.$$

In terms of the following dimensionless quantities

$$\sigma_j^2=2k^2\theta_j/m_j;\quad x_j=\omega/\sigma_j;\quad p_j=\beta_j/\sigma_j;\quad y=\beta t/p\,, \tag{21}$$

the integrals $I_1{}^j$ and $I_2{}^j$ can be rewritten as

$$\sigma_jI_1{}^j=\int_0^\infty\cos x_jy\exp\left\{-\frac{1}{2p_j^2}[\exp(-p_jy)+p_jy-1]\right\}dy\equiv C^j(x_j)\,, \tag{21a}$$

$$\sigma_jI_2{}^j=\int_0^\infty\sin x_jy\exp\left\{-\frac{1}{2p_j^2}[\exp(-p_jy)+p_jy-1]\right\}dy\equiv S^j(x_j)\,. \tag{21b}$$

Integrals $I_3{}^j$ and $I_4{}^j$ can be expressed in terms of $I_1{}^j$ and $I_2{}^j$ by the following relations (see Appendix C):

$$I_3{}^j = -2/\sigma_j{}^2 + (2x_j/\sigma_j)I_2{}^j, \qquad I_4{}^j = -(2x_j/\sigma_j)I_1{}^j. \tag{22}$$

After some algebra, the expression $\langle |n^e(\mathbf{k},\omega)|^2\rangle$, as given in Eq. (12), may be expressed in terms of C^j and S^j as

$$\frac{\pi}{n_0}\langle |n^e(\mathbf{k},\omega)|^2\rangle = \frac{(1/\sigma_e)C^e(x_e)\{[1+(1-x_iS^i)/h_e{}^2k^2]^2+[x_iC^i/h_i{}^2k^2]^2\}}{[1+(1-x_eS^e)/h_e{}^2k^2+(1-x_iS^i)/h_i{}^2k^2]^2+[x_eC^e/h_e{}^2k^2+x_iC^i/h_i{}^2k^2]^2}$$
$$+\frac{(1/\sigma_i)C^i(x_i)\{[(1-x_eS^e)/h_e{}^2k^2]^2+[x_eC^e/h_e{}^2k^2]^2\}}{[1+(1-x_eS^e)/h_e{}^2k^2+(1-x_iS^i)/h_i{}^2k^2]^2+[x_eC^e/h_e{}^2k^2+x_iC^i/h_i{}^2k^2]^2}, \tag{23}$$

where

$$h_j{}^2 = \theta_j/4\pi e^2 n_0.$$

Results of numerical integration for $C(x)$ and $S(x)$, for various values of the parameter p, are presented in Figs. 1(a) through 2(b). In Figs. 3 and 4, $xC(x)$ and $xS(x)$ are plotted to make the evaluation of the different terms appearing in Eq. (23) more convenient. For the two limiting cases of the parameter p, $S(x)$ and $C(x)$ can be expressed by the following simple relations (see Appendix D):

$$\lim_{p\to 0} S(x) = 2\exp(-x^2)\int_0^x \exp(u^2)du;$$
$$C(x) = \pi^{1/2}\exp(-x^2) \tag{24a}$$

$$\lim_{p\to\infty} S(x) = \frac{4p^2x}{1+4p^2x^2};\quad C(x) = \frac{2p}{1+4p^2x^2}. \tag{24b}$$

[The function $\exp(-x^2)\int_0^x \exp(u)^2du$ has been previously tabulated, see for example, Ref. 8.]

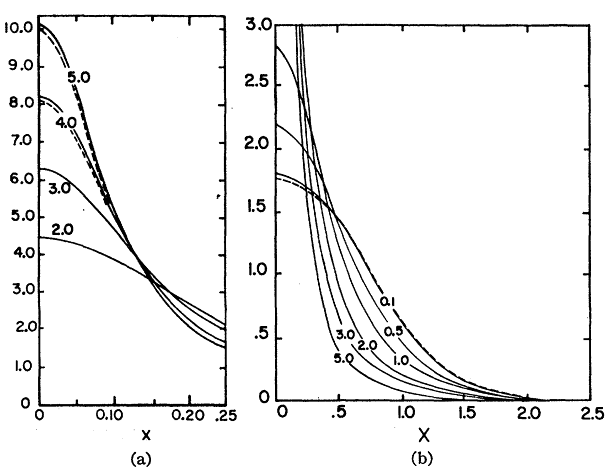

FIG. 1. (a) The function $C(x)$, defined in Eq. (21a), plotted for $p=5$, 4, 3, and 2. Dashed curves show $C(x)$ for $p=5$ and 4 as computed from the approximation to $C(x)$ for $p\to\infty$ given in Eq. (24b). (b) The function $C(x)$, defined in Eq. (21a), plotted for $p=5$, 3, 2, 1, 0.5, and 0.1. Dashed curve shows $C(x)$ for $p=0$ as computed from Eq. (24a).

[8] B. D. Fried and S. D. Conte, *The Plasma Dispersion Function* (Academic Press Inc., New York, 1961).

As can be seen from Figs. 1 and 2, the effects of collisions are negligible for values of p up to about 0.1. For values of p greater than ~ 5, $S(x)$ and $C(x)$ can be approximated by the formulas for the limiting case $p\to\infty$.

The spectrum of electron density fluctuations as given in Eq. (23) is characterized by the following parameters: p_i and p_e, which represents the ratio of the wavelength of the incoming wave to the mean free paths of the ions and electrons, respectively; h_ek, which represents the ratio of the Debye shielding length to the wavelength and thus governs the extent of the collective behavior of the plasma; and $h_e{}^2/h_i{}^2$, which represents the ratio of the electron and ion temperatures. For any set of values of these parameters, Eq. (23) can

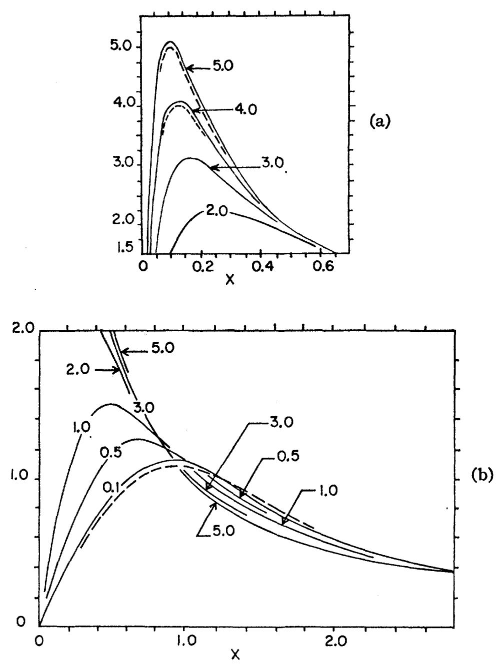

FIG. 2. (a) The function $S(x)$, defined in Eq. (21b), plotted for $p=5$, 4, 3, and 2. Dashed curves show $S(x)$ for $p=5$ and 4 as computed from the approximation to $S(x)$ for $p\to\infty$ given in Eq. (24b). (b) The function $S(x)$, defined in Eq. (21b), plotted for $p=5$, 3, 2, 1, 0.5, and 0.1. Dashed curves shown $S(x)$ for $p=0$ as computed from Eq. (24a).

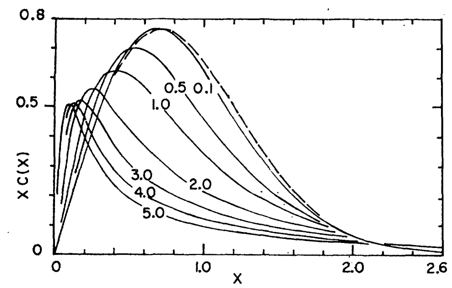

FIG. 3. The function $xC(x)$, defined in Eq. (21a), plotted for various values of p. Dashed curves indicate $xC(x)$ for $p=0$, computed using Eq. (24a).

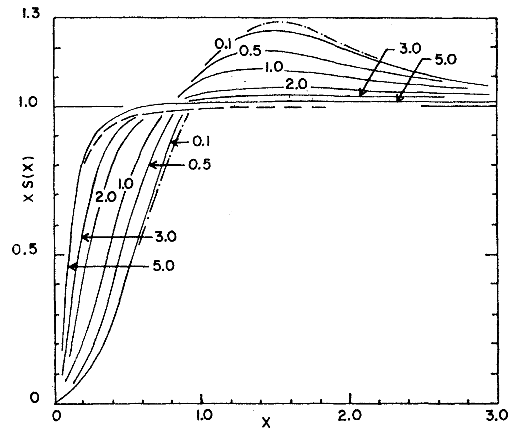

FIG. 4. The function $xS(x)$, defined in Eq. (21b), plotted for various values of p. Dot-dash curve indicates $xS(x)$ for $p=0$ computed using Eq. (24a). Dashed curve indicates $xS(x)$ for $p=5$ computed from the approximation to $S(x)$ for $p\rightarrow\infty$ given in Eq. (24b).

be evaluated with the help of Figs. 1 through 4 and Eq. (24). Since it is not feasible to discuss the results for all possible combinations of the various parameters, the discussion is restricted to the two limiting cases of the parameter hk, when $\theta_i=\theta_e$ and $p_i\sim p_e$.

A. Case (1) $hk\gg 1$

For this limiting case, Eq. (23) reduces to

$$\begin{aligned}(\pi/n_0)\langle|n^e(\mathbf{k},\omega)|^2\rangle d\omega &= C^e(x_e)dx_e\\ &=\pi^{1/2}\exp(-x_e^2)dx_e, \quad \lim p\rightarrow 0\\ &=[2p_e/(1+4p_e^2x_e^2)]dx_e, \quad \lim p\rightarrow\infty\,. \qquad (25)\end{aligned}$$

Equation (25) shows that the spectrum of the electron density fluctuations for wavelengths much smaller than the Debye shielding length is purely due to the random thermal motion of electrons. It reflects the fact that for deviations in electron density over length scales small compared with the Debye shielding length the effects of the self-induced electric fields, as discussed previously are negligible.

B. Case (2) $hk\ll 1$

For this limiting case and for x_i small; i.e., for the central region of the spectrum, after noting that $x_e\ll x_i$, Eq. (23) is seen to simplify to

$$\frac{\pi}{n_0}\langle|n^e(\mathbf{k},\omega)|^2\rangle d\omega=\frac{C^i(x_i)dx_i}{|2-x_iS^i|^2+|x_iC^i|^2} \qquad (26a)$$

$$=\frac{\pi^{1/2}\exp(-x_i^2)dx_i}{\left[\left|2-2x_i\exp(-x_i^2)\int_0^{x_i}\exp u^2du\right|^2+\pi x_i^2\exp(-2x_i^2)\right]}, \quad \text{for} \quad p\rightarrow 0 \qquad (26b)$$

$$=\frac{2p_idx_i}{4+4p_i^2x_i^2}, \quad \text{for} \quad p\rightarrow\infty\,. \qquad (26c)$$

Graphical representation of the results presented in the preceding equation is given in Fig. 5, which shows how the spectrum changes when the value of p increases from zero.

Equation (21) shows, for the central portion of the spectrum, and when the wavelength under consideration is much larger than the Debye shielding length, that the fluctuations induced in the electron density by the ion random thermal motion predominate over the density fluctuations due to the thermal motion of the electrons, themselves. From the point of view of the present analysis, this equation explains the main feature of the results of the Bowles experiment which showed that under the above conditions the linewidth of the scattered signal is characteristic of the thermal velocity of the ions rather than that of the electrons.

As x increases the first term in Eq. (23) starts to become the dominant term. However, the magnitude of $\langle|n_e(\mathbf{k},\omega)|^2\rangle$, for these higher values of ω, is negligibly small compared with the central portion of the spectrum. For sufficiently high values of x, Eq. (23) may be simplified to

$$\frac{\pi}{n_0}\langle|n^e(\mathbf{k},\omega)|^2\rangle=\frac{1}{\sigma_e}C^e(x_e)\frac{[h^2k^2+(1-x_iS^i)]^2+[x_iC^i]^2}{[h^2k^2+(1-x_eS^e)]^2+[x_eC^e]^2}.$$

The right-hand side of the above equation has a maximum at $h^2k^2 = x_e S^e - 1$, where it may be further simplified to

$$\frac{\pi}{n_0}\langle |n^e(\mathbf{k},\omega)|^2\rangle = \frac{1/\sigma_e C^e(x_e)h^4k^4}{[h^2k^2+(1-x_eS^e)]^2+[x_eC^e]^2}.$$

For the collisionless case the condition $h^2k^2 = x_eS^e - 1$, after restoring the dimensionality and using the asymptotic form $S(x) = 1+1/2x^2+\cdots$ for $x \gg 1$, yields $\omega = \omega_p$. The structure of this maximum has been previously discussed; and, as has been noted earlier, the area under this peak is very small compared with that of the central region.

To study the structure of the frequency spectrum around this maximum when collisions are taken into account, one has to numerically evaluate $C(x)$ and $S(x)$ for very high values of x. The formula for the limiting collision dominated case does not yield this maximum, for as can be readily seen, the equation

$$h^2k^2 = x_eS^e - 1 = 4p_e^2x_e^2/(1+4p_e^2x_e^2) - 1$$

[using the formula for $S(x)$ given in Eq. (24b)] does not have any root for a finite value of x. Figure 4 clarifies the point under consideration where the function $xS(x)$ has been plotted using the formula for $p \to \infty$ and the results obtained numerically.

To investigate the structure of this maximum when collisions are present, it may be instructive to use a transport equation with a simpler collision term such as the one given by the relaxation model. It may also be desirable to investigate the central region of the spectrum using both a relaxation model and a more sophisticated one, e.g., that of Rosenbluth, MacDonald, and Judd[9]; in the hope that if and when experiments on the scattering of electromagnetic waves by density fluctuations become sufficiently precise, their results may provide a testing ground for the various collision models.

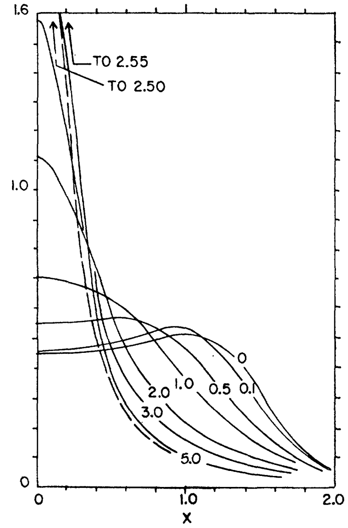

FIG. 5. The transition of the spectrum of electron density fluctuation, as given in Eq. (26a), from collisionless to collision-dominated plasmas. Numbers on the curve indicate the value of parameter p. Dashed curve indicates the spectrum computed from Eq. (26c) for $p=5$ and the curve labelled $p=0$ is computed from Eq. (26b).

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APPENDIX A: CONSTRUCTION OF THE SOLUTION OF EQUATION (2)

If we denote by $\mathbf{r}_0$ and $\mathbf{v}_0$ the position and velocity coordinates of the source at time t_0, then the Green's function for the operator appearing on the left-hand side of Eq. (2) may be written as [Ref. 7, Eq. (280)]

$$G(\mathbf{r},\mathbf{v},t;\mathbf{r}_0\mathbf{v}_0,t_0) = \frac{\exp(3\beta\tau)}{(2\pi)^3\Delta^{3/2}}\exp\left\{-\frac{1}{2\Delta}[a|\boldsymbol{\varrho}-\boldsymbol{\varrho}_0|^2+2h|\boldsymbol{\varrho}-\boldsymbol{\varrho}_0|\cdot|\mathbf{p}-\mathbf{p}_0|+b|\mathbf{p}-p_0|^2]\right\}, \quad \text{(A1)}$$

where

$$\boldsymbol{\varrho}-\boldsymbol{\varrho}_0 = \mathbf{v}\exp(\beta\tau)-\mathbf{v}_0;\quad \mathbf{p}-\mathbf{p}_0 = \mathbf{r}-\mathbf{r}_0+[(\mathbf{v}-\mathbf{v}_0)/\beta]\quad \tau = t-t_0;\quad \Delta = ab-h^2,\quad a=(2\theta/m\beta)\tau$$

$$b=(\theta/m)[\exp(2\beta\tau)-1],\quad h=-(2\theta/m\beta)[\exp(\beta\tau)-1].$$

With the use of Green's function $f(\mathbf{r},\mathbf{v},t)$, the solution of Eq. (2) can be expressed by the following integral:

$$f(\mathbf{r},\mathbf{v},t) = \int_{\mathbf{v}_0=-\infty}^{+\infty} d\mathbf{v}_0 \int_{\mathbf{r}_0=-\infty}^{+\infty} d\mathbf{r}_0 \int_{t_0=-\infty}^{t} dt_0 G(\mathbf{r},\mathbf{v},t;\mathbf{r}_0,\mathbf{v}_0,t_0)I(\mathbf{r}_0,\mathbf{v}_0,t_0), \quad \text{(A2)}$$

where $I(\mathbf{r}_0,\mathbf{v}_0,t_0)$ is given by the right-hand side of Eq. (2).

Taking time and space Fourier transforms of Eq. (A2), we obtain

$$f(\mathbf{k},\mathbf{v},\omega) \equiv \int_{\mathbf{r}=-\infty}^{+\infty}\int_{t=-\infty}^{+\infty} f(\mathbf{r},\mathbf{v},t)\exp[-i(\omega t-\mathbf{k}\cdot\mathbf{r})]dtd\mathbf{r}$$

$$= \int_{\mathbf{r}=-\infty}^{+\infty} dr \int_{t=-\infty}^{+\infty} dt \exp[-i(\omega t-\mathbf{k}\cdot\mathbf{r})] \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{t} G(\mathbf{r},\mathbf{v},t;\mathbf{r}_0,\mathbf{v}_0,t_0)I(\mathbf{r}_0,\mathbf{v}_0,t_0)d\mathbf{v}_0 d\mathbf{r}_0 dt_0. \quad \text{(A3)}$$

[9] M. N. Rosenbluth, W. M. MacDonald, and D. L. Judd, Phys. Rev. **107**, 1 (1957).

If one introduces new variables $\xi=\mathbf{r}-\mathbf{r}_0$, $\tau=t-t_0$ and notes that G depends only upon ξ and τ, Eq. (A3) becomes after some rearrangement

$$f(\mathbf{k},\mathbf{v},\omega)=\int_{-\infty}^{+\infty}d\mathbf{v}_0\int_{+\infty}^{-\infty}d\xi\int_{+\infty}^{0}d\tau G(\xi,\mathbf{v},\tau,v_0)\exp[-i(\omega\tau-\mathbf{k}\cdot\xi)]\int_{-\infty}^{+\infty}d\mathbf{r}_0\int_{-\infty}^{+\infty}dt_0 I(r_0,v_0,t_0)\exp[-i(\omega t_0-\mathbf{k}\cdot\mathbf{r}_0)],$$

$$f(\mathbf{k},\mathbf{v},\omega)=\int_{-\infty}^{+\infty}d\mathbf{v}_0\int_{+\infty}^{-\infty}d\xi\int_{+\infty}^{0}d\tau G(\xi,\mathbf{v},\tau,v_0)\exp[-i(\omega\tau-\mathbf{k}\cdot\xi)]I(\mathbf{k},\mathbf{v}_0,\omega). \quad \text{(A4)}$$

The required number density $n(\mathbf{k},\omega)$ can now be obtained by integrating both sides of Eq. (A4) over velocity $\mathbf{v}$

$$n(\mathbf{k},\omega)=\int_{-\infty}^{+\infty}d\mathbf{v}f(\mathbf{k},\mathbf{v},\omega)=\int_{-\infty}^{+\infty}d\mathbf{v}\int_{-\infty}^{+\infty}d\mathbf{v}_0\int_{+\infty}^{-\infty}d\xi\int_{+\infty}^{0}d\tau G(\xi,\mathbf{v},\tau,v_0)\exp[-i(\omega\tau-\mathbf{k}\cdot\xi)]I(\mathbf{k},\mathbf{v}_0,\omega).$$

Carrying out the integration over ξ, $\mathbf{v}$, and $\mathbf{v}_0$ with

$$I(\mathbf{k},\mathbf{v}_0,\omega)=+(e/\theta)(i\mathbf{k}\cdot\mathbf{v}_0)n_0(m/2\pi\theta)^{3/2}\phi_{k,\omega}\exp(-mv_0^2/2\theta)$$

(where $\phi_{k,\omega}$ is the time and space Fourier transform of potential representing the field E), we finally arrived at

$$n(\mathbf{k},\omega)=\frac{k^2}{4\pi e}\phi_{k,\omega}\left(\frac{4\pi e^2n_0}{m}\right)\int_0^\infty\frac{[\exp(-\beta t)-1]}{\beta}\exp\left\{-\frac{k^2\theta}{m\beta^2}[\exp(-\beta t)+\beta t-1]\right\}\exp(-i\omega t)dt.$$

APPENDIX B: EVALUATION OF $\langle|n(k,\omega)|^2\rangle$ FOR FIELD-FREE BROWNIAN PARTICLE

Equation (18) may be written

$$\langle|n(\mathbf{k},\omega)|^2\rangle=\frac{n_0}{2\pi}\int_{-\infty}^{+\infty}dt\exp(i\omega\tau)C_k(\tau), \quad \text{(B1)}$$

where

$$C_k(\tau)=\int\int\int\exp(i\mathbf{k}\cdot\mathbf{r})d\mathbf{r}Wu_0(\mathbf{r},t+\tau;\mathbf{r}_0t)\exp(-i\mathbf{k}\cdot\mathbf{r}_0),\quad W(r_0)d\mathbf{r}_0W(u_0)d\mathbf{u}_0. \quad \text{(B2)}$$

For the transition probability of a field-free Brownian particle, we take the results presented in Eq. (171) of Ref. 7

$$Wu_0(\mathbf{r},t,+\tau;\mathbf{r}_0,t)=\left[\frac{m\beta^2}{2\pi\theta[2\beta\tau-3+4\exp(-\beta\tau)-\exp(-2\beta\tau)]}\right]^{3/2}\times\exp\left\{-\frac{m\beta^2}{2\theta}\frac{|\mathbf{r}-\mathbf{r}_0-\{\mathbf{u}_0[1-\exp(-\beta\tau)]/\beta\}|^2}{[2\beta\tau-3+4\exp(-\beta\tau)-\exp(-2\beta\tau)]}\right\}\quad\text{for}\quad\tau>0. \quad \text{(B3)}$$

If one introduces the new variable $\xi=\mathbf{r}-\mathbf{r}_0$ and notes that Wu_0 depends only upon ξ, Eq. (B2) can be written as

$$C_k(\tau)=\int\int_{\xi=-\infty}^{+\infty}d\xi\exp(i\mathbf{k}\cdot\xi)Wu_0(\xi,\tau)W(\mathbf{u}_0)d\mathbf{u}_0\int_{\mathbf{r}_0=-\infty}^{+\infty}W(r_0)d\mathbf{r}_0,\quad\text{for}\quad\tau>0$$

$$C_k(\tau)=\int\int d\xi\exp(i\mathbf{k}\cdot\xi)Wu_0(\xi,\tau)W(\mathbf{u}_0)d\mathbf{u}_0, \quad \text{(B4)}$$

where we have used the fact that $\int W(r_0)dr_0=1$.

Evaluating the integral in Eq. (B4) with $W(\mathbf{u}_0)=n_0(m/2\pi\theta)^{3/2}\exp\{-mu_0^2/2\theta\}$, we arrive at

$$C_k(\tau)=n_0\exp\left\{-\frac{k^2\theta}{m\beta^2}[\exp(-\beta\tau)+\beta\tau-1]\right\}\quad\text{for}\quad\tau>0. \quad \text{(B5)}$$

Substituting Eq. (B5) into Eq. (B1) along with $C_k(\tau)$ for $\tau<0$ equal to $C_k(-\tau)$ for $\tau>0$, we obtain

$$\langle|n(\mathbf{k},\omega)|^2\rangle=\frac{n_0}{\pi}\int_\theta^\infty\cos\omega\tau\exp\left\{-\frac{k^2\theta}{m\beta^2}[\exp(-\beta\tau)+\beta\tau-1]\right\}dt. \quad \text{(B6)}$$

APPENDIX C: EVALUATION OF THE INTEGRALS I_3 AND I_4 IN TERMS OF I_1 AND I_2

With the use of the nondimensional quantities defined in Eq. (21), integral I_3 can be written as

$$I_3=\frac{1}{\sigma^2}\int_0^\infty \frac{[\exp(-py)-1]}{p}\times\exp\left\{-\frac{1}{2p^2}[\exp(-py)+py-1]\right\}\cos xy dy.$$

The above integral can be rewritten as

$$I_3=\frac{1}{\sigma^2}\int_0^\infty \left[\frac{d}{dy}\left\{2\exp\left[-\frac{1}{2p^2}\exp(-py)\right]\right\}\right]\times\exp\left[-\frac{1}{2p^2}(py-1)\right]\cos xy dy+\frac{1}{\sigma^2}\left(-\sigma\frac{I_1}{p}\right).$$

Integrating the first term by parts and evaluating the integrals in terms of I_1 and I_2, we obtain

$$I_3=\frac{1}{\sigma^2}\left(-2+2x\sigma I_2+\frac{1}{p}\sigma I_1-\frac{1}{p}\sigma I_1\right)=-\frac{2}{\sigma^2}+\frac{2x}{\sigma}I_2.$$

Following the same procedure, we get for I_4

$$I_4=-(2x/\sigma)I_1.$$

APPENDIX D: EVALUATION OF $S(x)$ AND $C(x)$ FOR THE LIMITING CASES OF PARAMETER p

For $p\to 0$ the exponential function appearing in the integrand of $S(x)$ and $C(x)$ can be reduced as follows:

$$\exp\left\{-\frac{1}{2p^2}[\exp(-py)+py-1]\right\}=\exp\left(-\frac{y^2}{4}+p\frac{y^3}{12}+\cdots\right)\underset{p\to 0}{\longrightarrow}\exp\left(-\frac{y^2}{4}\right).$$

Thus, for limit $p\to 0$

$$S(x)=\int_0^\infty \sin xy\exp\left(-\frac{y^2}{4}\right)dy=2\exp(-x^2)\int_0^x \exp(u^2)du$$

and

$$C(x)=\int_0^\infty \cos xy\exp\left(-\frac{y^2}{4}\right)dy=\pi\exp(-x^2).$$

For $p\to\infty$, we approximate the above function by noticing that for $p\to\infty$, py dominates $\exp(-py)$ and 1. Thus,

$$\exp\left\{-\frac{1}{2p^2}[\exp(-py)+py-1]\right\}\sim\exp\left(-\frac{y}{2p}\right)$$

and we obtain for $S(x)$ and $C(x)$ in this limit,

$$S(x)\sim\int_0^\infty \sin xy\exp\left(-\frac{y}{2p}\right)dy=\frac{4p^2x}{1+4p^2x^2},$$

$$C(x)\sim\int_0^\infty \cos xy\exp\left(-\frac{y}{2p}\right)dy=\frac{2p}{1+4p^2x^2}.$$