that the methods developed in this paper lead to sufficiently accurate wave functions to make meaningful comparisons of theory and experiment possible.

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Dynamic Theory of the Nuclear Collective Model*

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The rotation-vibration model and the hydrodynamic dipole-oscillation model are unified. A coupling between the dipole oscillations and the quadrupole vibrations is introduced in the adiabatic approximation. The dipole oscillations act as a "driving force" for the quadrupole vibrations and stabilize the intrinsic nucleus in a nonaxially symmetric equilibrium shape. The higher dipole resonance splits into two peaks separated by about 1.5–2 MeV. On top of the several giant resonances occur bands due to rotations and vibrations of the intrinsic nucleus. The dipole operator is established in terms of the collective coordinates and the γ-absorption cross section is derived. For the most important 1^- levels the relative dipole excitation is estimated. It is found that some of the dipole strength of the higher giant resonance states is shared with those states in which one surface vibration quantum is excited in addition to the giant resonance.

I. INTRODUCTION

IT has been emphasized by Faessler and Greiner[1] that the anharmonicity of the nuclear surface vibrations implies in a certain sense a triaxiality of the nucleus. In this paper we show that this triaxiality manifests itself in the photonuclear giant resonance. The degeneracy of the upper peak, which in the static model[2,3] is due to the equality of the two minor axes, disappears. Thus, in a dynamic treatment, there appear three dipole peaks which, however, overlap due to the damping of the giant resonance. In the remainder of the Introduction we first give some background material, and then we describe the contents of this paper.

The collective model of surface vibrations and rotations has been spectacularly successful in explaining the nuclear low-energy spectrum.[4–6] Similarly, the collective model predictions of the most important electric dipole transitions have been quite well confirmed by the experiments within the region of applicability, and significantly, the agreement has improved with the increase of details of the theory and with the improvement of the experimental accuracy.[7,8] In the present paper we intend to unify these two aspects of the collective model of the nucleus, namely the unified model and the dipole giant resonance model. In other words, we would like to develop the complete quantum-mechanical collective model of the nucleus, treating all collective degrees of freedom as quantum-mechanical variables. However, we consider in this paper only even-even nuclei. We should emphasize that our treatment is phenomenological in that we do not attempt to derive the collective Hamiltonian from the nuclear many-body problem. Instead, we assume the model and determine its consequences in as consistent a way as possible. By comparing our results with experiment one can then decide the limits of the validity of the model.

We just note for completeness that a considerable amount of work on a "fundamental" level has been

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done since the introduction of the collective models, namely, by the Copenhagen School on the low-energy part[9–12] and by the Hole-Particle School on the high-energy part.[13–15] For this work the reader is referred to the original papers.

We now summarize the results of this paper. In Sec. II we write down the Hamiltonian, i.e., we define the model. For the low-energy part we follow closely the work of Faessler and Greiner.[6] We should therefore expect the treatment to be applicable for nuclei heavier than, say, $A \approx 50$. For lighter nuclei the assumption of, e.g., an infinite number of rotational states is not well justified, and, in fact, the number of such states can be quite limited.[16] However, within the region of applicability, the rotation-vibration part of the Hamiltonian is fully determined except for four adjustable parameters which are taken from low-energy data: moment of inertia, deformation, frequencies of the β and γ vibrations; in β vibrations the nuclear deformation oscillates around the equilibrium deformation leaving the nucleus axially symmetrical, while in γ vibrations the deformation stays essentially constant and the ratio of the lengths of the shorter axes of the nucleus oscillates about the equilibrium value, which here is assumed to equal unity [see Eq. (4) below and footnote 26]. For the giant resonance part, we take the hydrodynamic model. This again limits the region of applicability. The model is expected to be good for medium and heavy nuclei except for a small number of nuclei around doubly closed shells; the region of validity thus complements the region where particle-hole calculations are feasible. This additional limitation is due to the requirement of high-level densities for the validity of the hydrodynamic model. In order for the idiosyncrasies of the different nuclei to average out and thus approach a classical situation, a sufficiently large number of roughly equally important configurations must participate in the dipole state. This is evidently not true for light nuclei and it is not really true for Pb^{208} where, again, the number of important configurations[17] is only about 10.

The position of the resonance energies depends in the hydrodynamic model on one adjustable parameter which, however, is fixed by the integrated cross section,[18,19] and which is due to the nuclear exchange forces. According to one's taste, one may or may not consider this parameter to be a free parameter of the theory. This point has been more fully discussed in Ref. 19.

The complete Hamiltonian contains, in addition to the terms describing the rotations, vibrations and dipole oscillations, interaction terms between all these degrees of freedom. We limit ourselves to the linear approximation in which the different amplitudes are considered to be small compared with the appropriate "unit amplitudes" and we treat some of the interaction terms as perturbations since they contain products of the amplitudes and thus are smaller than the main terms.

Our treatment goes beyond the treatment of this problem within classical mechanics by Araujo.[20] He does not discuss the coupling between the different modes. His treatment was, however, more general in that he considered also the nuclear compressibility. We neglect this nuclear property and as a result lose the states corresponding to compression waves (ordinary sound). We feel justified in doing so because the compressibility of nuclear matter seems to be relatively small and the sound-wave states would thus lie at a much higher energy than the dipole states.

We include formally a term in the Hamiltonian in order to describe the damping of the giant resonance. It is supposed to lead to off-diagonal matrix elements connecting the dipole states with other states which do not have a dipole moment. We assume that this term can be treated by the Wigner-Weisskopf procedure.[21] We do so in order to have a Hermitian Hamiltonian. An alternative procedure would be to assume that the energy of the dipole states is complex. We prefer the first of these alternatives because of mathematical simplicity.

In Sec. III we discuss the spectrum of the Hamiltonian. The low-energy part of the spectrum remains unchanged from previous treatments.[4,6] In the high-energy part in essence two new features arise.

First, the higher peak of the giant resonance is connected with angular momentum ± 1 along the intrinsic axis of the nucleus. The coupling between the dipole mode and the surface leads to γ vibrations of the nuclear shape which in turn causes a split of this higher giant resonance. This can be explained as follows: It has been shown that the γ vibrations are anharmonic. As a consequence, the probability distribution of γ has maxima at $\gamma \neq 0$ and it vanishes for $\gamma = 0$, even in the ground state. Therefore, the nucleus is in effect "asymmetric," though not, however, in the sense of Davydov

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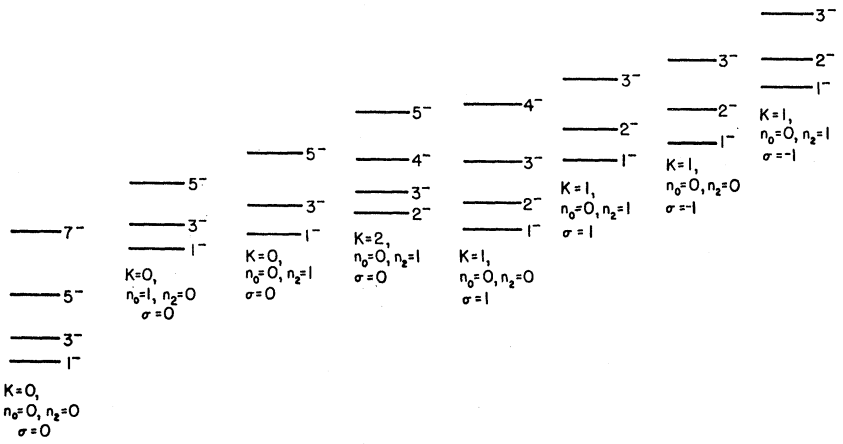

FIG. 1. Schematical level diagram of the lowest giant resonance bands. $\sigma=0$ indicates the excitation of the η_0-mode; $\sigma=+1$ indicates the lower and $\sigma=-1$ indicates the higher minimum of $V(a_2')$, Fig. 6. There are still further bands with $K=2$, 3 in this energy region.

et al.,[5,22,23] who assume a permanently deformed triaxial shape. Upon excitation of a "transverse" dipole mode the nuclear shape becomes stabilized in its axially asymmetric form. This way the degeneracy of the upper dipole resonance is removed and two resonances will appear corresponding to the two axes perpendicular to the "symmetry" axis. We would just like to mention that this degeneracy remains intact if one neglects the anharmonicity of the γ vibrations and treats them as being harmonic.

Second, each dipole state forms the basis of vibrational and rotational bands (Fig. 1). The dipole states thus behave in a way completely analogous to that of the odd-particle states in the low-energy region,[24] a not very surprising result.

In Sec. IV we calculate the dipole operator and derive expressions for the dipole transition probabilities. We find the rather surprising result that, in addition to the dipole states, also those states have an appreciable photon absorption cross section where a vibrational state is excited in addition to the transverse dipole states. The transverse dipole states thus each acquire a "satellite" at an energy about 0.8 MeV higher (Fig. 2). Offhand, one would expect that transitions to these satellite states should be inhibited by a selection rule analogous to the K selection rule in the low-energy spectrum, since in these states more than one degree of freedom makes a transition and the dipole operator is a one-body operator. The breakdown of this selection rule is again associated with the anharmonicity of the γ vibrations and their strong coupling to the "transverse" dipole modes.

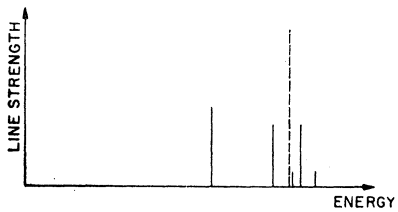

FIG. 2. Schematic picture of the dipole strength distribution. The higher resonance of the static model (dashed line) splits into two lines which each has a weak satellite at a somewhat higher energy. The total energy spread of these lines is about 2 MeV. All the indicated lines are broadened by "friction" to a width of 2–3 MeV and thus overlap.

II. THE COUPLING OF ROTATIONS, QUADRUPOLE VIBRATIONS, AND DIPOLE OSCILLATIONS

A. The Rotation-Quadrupole Vibration

We describe the excited states of a deformed nucleus in the extended Bohr-Mottelson model, where the assumption is made that the nucleus in its ground state consists of a rotating intrinsic nucleus with an axially symmetric equilibrium shape which can perform a_0 and a_2 vibrations (see below) about this equilibrium.[6] With this model the low-energy collective spectra can be very well understood. The parameters entering this description are determined from the low-energy excitations. In addition, we now have the "internal" degree of freedom of giant resonance dipole oscillations, i.e., vibrations of the proton and neutron fluids against each other. Let the angular momentum of the dipole oscillations be **j** and that of the rotations be **R**. Then the total angular momentum is

$$\mathbf{M}=\mathbf{R}+\mathbf{j} \tag{1}$$

and we have, therefore, for the rotational energy[25] T

$$T=\sum_\nu \frac{\hbar^2}{2J_\nu(a_\mu)}(M_\nu-j_\nu)^2;$$
$$J_{1,2}=\frac{B}{4}(8a_2^2+12a_0^2\pm 8(6)^{1/2}a_2a_0), \quad J_3=8Ba_2^2, \tag{2}$$

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where $J_\nu(a_\mu)$ are the moments of inertia and a_μ are the parameters specifying the intrinsic shape given by

$$R=R_0(1+a_0Y_{2,0}+a_2(Y_{2,2}+Y_{2,-2})). \tag{3}$$

The energy, Eq. (2), contains the energy of the pure rotations, of the rotation-vibration interaction and of the Coriolis coupling (RPC terms). We introduce the quadrupole vibrations by writing[26]

$$\begin{aligned} &a_0=\beta_0+a_0', \\ &a_2=\beta_2+a_2', \quad \beta_2=0, \quad a_\nu'/\beta_0\ll 1, \end{aligned} \tag{4}$$

where β_0 and β_2 denote the equilibrium values and the primed quantities are the vibrational coordinates, which are assumed to be small compared with the equilibrium deformation. The quadrupole-vibrational Hamiltonian is[1,6]

$$H_{\rm vib}=-\frac{\hbar^2}{2B}\left[\frac{\partial^2}{\partial a_0'^2}+\frac{1}{2}\frac{\partial^2}{\partial a_2'^2}\right]+\frac{C_0}{2}a_0'^2+C_2a_2'^2, \tag{5}$$

where B is the parameter occurring in the moments of inertia in Eq. (2) and C_0, C_2 characterize the vibrational energies.

Expanding Eq. (2) about the equilibrium values, Eq. (4), and using Eq. (5), we obtain for the rotation-vibration Hamiltonian

$$T+H_{\rm vib}=H_{\rm rot}+H_{\rm vib}+H_{\rm vib\ rot}+H_{\rm rot\ dip}, \tag{6}$$

with

$$H_{\rm rot}=[M^2-M_3^2-j_3^2]\frac{\hbar^2}{2J_0}+[(M_3-j_3)^2-1]\frac{\hbar^2}{16Ba_2'^2}. \tag{7}$$

One sees in this equation that the a_2 vibrations cannot be harmonic. The name one gives the terms of the Hamiltonian which depend on more than one dynamical variable is quite arbitrary. We have written the last term of Eq. (7) together with the rotational part of the Hamiltonian, and not with the term $H_{\rm vib\ rot}$, because this term cannot be treated as a perturbation[6] since the range of a_2' includes the value zero. We are going to use the adiabatic approximation with the justification that both the vibrations and the dipole oscillations have much higher energy than the rotations. Then the last term of Eq. (7) will contribute a "potential" term to Eq. (5) making the vibrational Hamiltonian anharmonic. For $(M_3-j_3)^2\neq 0$ this term has simply the character of a centrifugal potential. The (-1) in this term has, however, a different origin; it comes from the phase space of the a_2 vibrations.[1] It appears via the Jacobian when one chooses the volume element to be $d\tau=da_0da_2d\Omega$ with $d\Omega$ the differential of the Euler angles. Further,

$$H_{\rm vib\ rot}=\frac{\hbar^2}{2J_0}[M^2-(M_3-j_3)^2]\left[-\frac{2a_0'}{\beta_0}\right]-\frac{\hbar^2}{4J_0}(M_+^2+M_-^2)\left[\frac{2}{3}(6)^{1/2}\frac{a_2'}{\beta_0}\right], \tag{8}$$

$$H_{\rm dip\ rot}=-(\hbar^2/2J_0)[M_+j_-+M_-j_+]. \tag{9}$$

Here $M_\pm=M_1\pm iM_2$, $j_\pm=j_1\pm ij_2$. Equation (8) couples the vibrations with the rotations and Eq. (9) couples the rotations with the dipole oscillation (Coriolis term).

B. The Dipole-Oscillation Hamiltonian

We construct the Hamiltonian for the dipole oscillations in the adiabatic approximation, i.e., with the assumption that the quadrupole vibrations ($\sim$1 MeV) and the rotations ($\sim$100 keV) are slow compared with the giant dipole oscillations (15–20 MeV). Thus, 15–20 dipole oscillations occur during one quadrupole vibration. Therefore, the dipole oscillations "see" the nuclear shape fixed at each moment, so to say. Thus, we have at every moment a triaxial nucleus. In such an ellipsoid three eigenvibrations exist along the three axes which are

$$\begin{aligned} \Psi_1&=j_1(k_1r)\sin\theta\cos\varphi, \\ \Psi_2&=j_1(k_2r)\sin\theta\sin\varphi, \\ \Psi_3&=j_1(k_3r)\cos\theta, \end{aligned} \tag{10}$$

with the wave numbers[2,27]

$$k_i=\frac{2.08}{R_i}\left(1+0.08\frac{\Delta R_i}{R_0}\right), \qquad \Delta R_i=R_i-R_0, \quad i=1,2,3 \tag{11}$$

where R_i are the radii of the three axes given according to Eq. (3) by

$$\begin{aligned} R_1&=R(\pi/2,0)=R_0(1-\tfrac{1}{2}(5/4\pi)^{1/2}a_0+(3/2)^{1/2}(5/4\pi)^{1/2}a_2), \\ R_2&=R(\pi/2,\pi/2)=R_0(1-\tfrac{1}{2}(5/4\pi)^{1/2}a_0-(3/2)^{1/2}(5/4\pi)^{1/2}a_2), \\ R_3&=R(0,0)=R_0(1+(5/4\pi)^{1/2}a_0). \end{aligned} \tag{12}$$

Inserting Eq. (12) into Eq. (11), using Eq. (4) and expanding in the small quantities a_ν', we get

$$\begin{aligned} k_{1,2}&=k_{1,2}(0)[1+G_0a_0'\pm G_2a_2'], \\ k_3&=k_3(0)[1+Fa_0'], \end{aligned} \tag{13}$$

[26] Usually the intrinsic parameters are given by β and γ which are connected with the a_ν by $a_0=\beta\cos\gamma$, $a_2=\frac{1}{2}\beta\sin\gamma$. Therefore, a_0 vibrations are essentially β vibrations and a_2' vibrations are γ vibrations.

[27] E. V. Inopin, Zh. Eksperim. i Teor. Fiz. **38**, 992 (1960) [English transl.: Soviet Phys.—JETP **11**, 714 (1960)].

where

$$k_{1,2}(0)=\frac{2.08[1-0.04(5/4\pi)^{1/2}\beta_0]}{R_0[1-0.5(5/4\pi)^{1/2}\beta_0]},$$

$$k_3(0)=\frac{2.08[1+0.08(5/4\pi)^{1/2}\beta_0]}{R_0[1+(5/4\pi)^{1/2}\beta_0]},$$

$$G_0=\frac{0.5(5/4\pi)^{1/2}}{1-0.5(5/4\pi)^{1/2}\beta_0}-\frac{0.04(5/4\pi)^{1/2}}{1-0.04(5/4\pi)^{1/2}\beta_0}, \quad (14)$$

$$G_2=-\frac{(15/8\pi)^{1/2}}{1-0.5(5/4\pi)^{1/2}\beta_0}+\frac{0.08(15/8\pi)^{1/2}}{1-0.04(5/4\pi)^{1/2}\beta_0},$$

$$F=-\frac{(5/4\pi)^{1/2}}{1+(5/4\pi)^{1/2}\beta_0}+\frac{0.08(5/4\pi)^{1/2}}{1+0.08(5/4\pi)^{1/2}\beta_0}.$$

The frequencies of the three eigenvibrations, Eq. (10), are given in the hydrodynamical model by

$$\omega_\nu=uk_\nu, \quad \nu=1, 2, 3, \quad (15)$$

where

$$u^2=(8\kappa/M^*)(NZ/A^2). \quad (16)$$

κ is the symmetry-energy parameter, M^* is the effective nucleon mass[18,19]; N, Z, and A the number of neutrons, protons, and nucleons, respectively. Inserting Eq. (13) into Eq. (15) we get

$$\omega_{1,2}=\omega_{1,2}(0)[1+G_0a_0'\pm G_2a_2'],$$
$$\omega_3=\omega_3(0)[1+Fa_0'], \quad (17)$$

where

$$\omega_{1,2}(0)\equiv\omega_1\equiv\omega_{-1}=uk_{1,2}(0),$$
$$\omega_3(0)\equiv\omega_0=uk_3(0). \quad (18)$$

Assuming harmonic giant dipole oscillations and introducing the appropriate annihilation and creation operators b_κ, b_κ^+ for the modes, Eq. (10), we can write

$$\tilde{H}=H_{\rm dip}+H_{\rm dip\ vib}=\sum_\kappa \hbar\omega_\kappa b_\kappa^+ b_\kappa. \quad (19)$$

This ansatz contains the pure dipole oscillations as well as their coupling to quadrupole vibrations via the terms with a_ν' in Eq. (17). We have in Eq. (19) no zero-point dipole energy. The states $b_\kappa^+|0\rangle$ which correspond to the Ψ_k of Eq. (10) have the disadvantage that they are, according to Eq. (10), a superposition of states with different angular momenta along the intrinsic axis. We therefore introduce new states with definite components of angular momentum along the intrinsic axis by the canonical transformation

$$b_1^+=\frac{\beta_1^+-\beta_{-1}^+}{\sqrt{2}}, \quad b_2^+=i\frac{\beta_1^+-\beta_{-1}^+}{\sqrt{2}}, \quad (20)$$
$$b_3^+=\beta_0^+.$$

One can easily see from Eq. (10) that the states $\beta_\nu^+|0\rangle$, which we shall denote by η_ν, are characterized by the angular momentum component ν along the intrinsic axis. Since they describe the annihilation and creation of states with spin 1 the β_ν, β_ν^+ fulfill the boson commutation rules. Introducing Eq. (20) into Eq. (19) we find

$$H_{\rm dip}=\hbar\omega_{1,2}(0)(\beta_1^+\beta_1+\beta_{-1}^+\beta_{-1})+\hbar\omega_0\beta_0^+\beta_0$$
$$+\tfrac{1}{2}(\hbar\omega_1(0)-\hbar\omega_2(0))(\beta_{-1}^+\beta_1+\beta_1^+\beta_{-1}), \quad (21a)$$

$$H_{\rm dip\ vib}=\hbar\omega_{1,2}(0)(\beta_1^+\beta_1+\beta_{-1}^+\beta_{-1})G_0a_0'$$
$$+\hbar\omega_{1,2}(0)(\beta_1^+\beta_{-1}+\beta_{-1}^+\beta_1)G_2a_2'$$
$$+\hbar\omega_0\beta_0^+\beta_0Fa_0'. \quad (21b)$$

Equation (21) shows explicitly the mixing of the a_2 vibrations with the excitations β_1^+, β_{-1}^+. The last term of Eq. (21a) vanishes since we consider the equilibrium of the intrinsic nucleus to be axially symmetric, and therefore $\hbar\omega_1=\hbar\omega_2$ [see Eq. (18)].

We are now able to write down the total Hamiltonian. To the above discussed terms we add a term H_1, which includes all terms omitted in the discussion so far. This is supposed to contain all other nuclear degrees of freedom, e.g., the single-particle coordinates. Formally, it is just $H_1=H_{\rm nuclear}-H_{\rm collective}$, the difference between the actual nuclear Hamiltonian, $H_{\rm nuclear}$, and the model Hamiltonian, $H_{\rm collective}$. It thus describes the nonstationary character of the higher energy states. We shall consider its influence later (Sec. IV) together with the term $H_{\rm rad}$ which describes the interaction with the radiation field. Thus, we have for the total Hamiltonian

$$H=H_{\rm rot}+H_{\rm vib}+H_{\rm dip}+H_{\rm dip\ vib}+H_{\rm vib\ rot}$$
$$+H_{\rm rot\ dip}+H_1+H_{\rm rad}. \quad (22)$$

III. SOLUTIONS OF THE HAMILTONIAN—THE ENERGY SPECTRUM

We are interested in the basic structure of the energy spectrum of Eq. (22); therefore the rotation-vibration interaction can be neglected because for low spins it is one or two orders of magnitude smaller than $H_{\rm rot}$. One might expect that even the rotational energy can be neglected compared with quadrupole vibrations and dipole oscillations. This, however, is not true for states with $K\neq 0$, where K is the component of the total angular momentum along the intrinsic axis. In such states the third moment of inertia may contain an energy which is comparable with that of the quadrupole vibrations. Furthermore, we would lose with the rotations the orientation of the nucleus in space and effects such as nuclear tensor polarizability[27,28] could not be discussed consistently.

We neglect the Coriolis terms; this is reasonable for low spins. They may be of some importance for higher spins. The terms H_1 and $H_{\rm rad}$ are treated later as a perturbation (see Sec. IV). We are then left with the

[28] E. G. Fuller and E. Hayward, Nucl. Phys. **30**, 613 (1962).

Hamiltonian

$$H=H_{\text{rot}}+H_{\text{vib}}+H_{\text{dip}}+H_{\text{dip vib}}. \tag{23}$$

We solve the problem specified by this Hamiltonian for excitations where only one dipole quantum is excited in the intrinsic nucleus.

All the essential approximations we were going to make have been done already at this point. The only place where we shall still use an approximate mathematical method will be in solving the differential equation (35b) below. This is evidently inessential. Furthermore, we feel that our treatment gives a physical insight concerning the mechanism of the vibration-dipole oscillation interaction.

The wave functions are of the type

$$\chi_{\text{vib}}(a_0',a_2')\mathfrak{D}_{MK}{}^I(\theta)\eta_\Omega. \tag{24}$$

They must be invariant under rotations of π around the intrinsic z' axis and x' axis (see Fig. 3).

The first invariance leads to the condition[29]

$$K-\Omega=2v, \quad v=0, \pm1, \pm2\cdots. \tag{25}$$

The second one transforms[24,25]

$$\frac{1}{\sqrt{2}}(\eta_\Omega\pm\eta_{-\Omega})\mathfrak{D}_{MK}{}^I\rightarrow\pm\frac{1}{\sqrt{2}}(\eta_\Omega\pm\eta_{-\Omega})(-)^{I+K+j+\Omega}\mathfrak{D}_{M-K}{}^I.$$

Hence, a properly symmetrized solution is of the form

$$\begin{aligned}\chi_K{}^I(a_0',a_2')\Big[&\mathfrak{D}_{MK}{}^I\frac{1}{\sqrt{2}}(\eta_\Omega\pm\eta_{-\Omega})\\&\pm(-)^{I+K+j+\Omega}\mathfrak{D}_{M-K}{}^I\frac{1}{\sqrt{2}}(\eta_\Omega\pm\eta_{-\Omega})\Big]\\=\chi_K{}^I(a_0',a_2')&\frac{1}{\sqrt{2}}(\eta_\Omega\pm\eta_{-\Omega})\\&\times[\mathfrak{D}_{MK}{}^I\pm(-)^{I+K+j+\Omega}\mathfrak{D}_{M-K}{}^I],\end{aligned} \tag{26}$$

where K and Ω have to fulfill Eq. (25). In our case $j=1$ and $\Omega=\pm1$, 0. Therefore, two types of solution are possible:

$$\chi_{K,\pm1}{}^I(a_0',a_2')\frac{1}{\sqrt{2}}(\eta_1\pm\eta_{-1})[\mathfrak{D}_{MK}{}^I\pm(-)^{I+K}\mathfrak{D}_{M-K}{}^I],$$

$$I=K, K+1, K+2, \cdots$$

$$K=1, 3, 5, \cdots \tag{27a}$$

and

$$\begin{aligned}&\chi_{K,0}{}^I(a_0',a_2')\eta_0[\mathfrak{D}_{MK}{}^I+(-)^{I+1}\mathfrak{D}_{M-K}{}^I],\\&I=1, 3, 5, 7, \cdots \quad \text{for} \quad K=0,\\&I=K, K+1, K+2, \cdots \quad \text{for} \quad K=2, 4, 6, \cdots.\end{aligned} \tag{27b}$$

[29] A. Bohr, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **26**, No. 14 (1952).

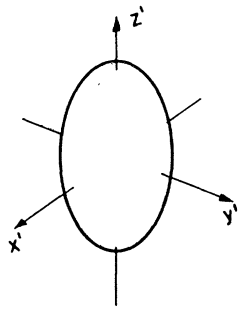

FIG. 3. Intrinsic coordinate system of the nucleus; Z' is the axis of symmetry of the equilibrium shape.

The ansatz $(\eta_\Omega\pm\eta_{-\Omega})$ for the dipole wave function is suggested by the structure of the Hamiltonian Eq. (21a) and it can be seen immediately that Eq. (27) is diagonal in this part of the wave function and in the rotational part. We look now for the quadrupole vibrations in both cases [Eq. (27)] separately. Inserting Eq. (27a) into Eq. (23) one obtains a differential equation for $\chi_K(a_0',a_2')$, if one multiplies with

$$[\eta_1\pm\eta_{-1}][\mathfrak{D}_{MK}{}^I+(-1)^{I+1}\mathfrak{D}_{M-K}{}^I]$$

from the left hand and integrates,

$$\begin{aligned}\Big\{-\frac{\hbar^2}{2B}\Big[\frac{\partial^2}{\partial a_0'^2}+\frac{1}{2}\frac{\partial^2}{\partial a_2'^2}\Big]+\frac{\hbar^2K^2}{16Ba_2'^2}+C_2a_2'^2\\\pm\hbar\omega_1G_2a_2'+\frac{C_0}{2}a_0'^2+\hbar\omega_1G_0a_0'\Big\}\chi_{K,\pm1}{}^I(a_0',a_2')\\=E_{K,\pm1}{}^I\chi_{K,\pm1}{}^I(a_0',a_2').\end{aligned} \tag{28a}$$

Similarly, using Eq. (27b) we get a differential equation for $\chi_{K,0}{}^I$.

$$\begin{aligned}\Big\{-\frac{\hbar^2}{2B}\Big[\frac{\partial^2}{\partial a_0'^2}+\frac{1}{2}\frac{\partial^2}{\partial a_2'^2}\Big]+\frac{\hbar^2(K^2-1)}{16Ba_2'^2}\\+C_2a_2'^2+\frac{C_0}{2}a_0'^2+\hbar\omega_0Fa_0'\Big\}\chi_{K,0}{}^I(a_0',a_2')\\=E_{K,0}{}^I\chi_{K,0}{}^I(a_0',a_2').\end{aligned} \tag{28b}$$

Here E denotes the total energy of the system and

$$\begin{aligned}E_{K,\pm1}{}^I&=E-\frac{I(I+1)-(K^2+1)}{2J_0}\hbar^2-\hbar\omega_1,\\E_{K,0}{}^I&=E-\frac{I(I+1)-K^2}{2J_0}\hbar^2-\hbar\omega_0.\end{aligned} \tag{29}$$

We see that the interaction $H_{\text{dip vib}}$ occurs in Eq. (28) as an additional potential for the quadrupole vibrations. The fast dipole oscillations act, so to say, as a "driving force" for the shape of the intrinsic nucleus. Equations (28) can be separated immediately into a_0' and a_2' vibrations.

$$\chi_{K,v}{}^I(a_0',a_2')=u_{K,v}{}^I(a_0')\varphi_{K,v}{}^I(a_2'), \quad v=0, \pm1. \tag{30}$$

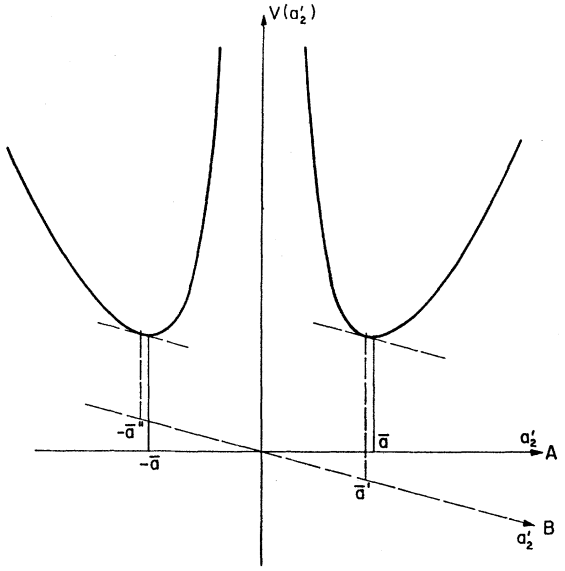

FIG. 4. Potential energy for the a_2 vibrations, transverse dipole modes. (A) Orthogonal coordinate system: For even $K\geq 2$, Eq. (36b) and for odd K without the linear term in Eq. (36a). The minima of the potential lie symmetrically at $\bar{a}$ and $-\bar{a}$. (B) Non-orthogonal coordinate system: For odd K with the linear term in Eq. (36a). The minima of the potential are displaced to the left to the places $\bar{a}'$ and $-\bar{a}''$. They do not lie symmetrically with respect to $a_2=0$. The situation shown corresponds to the excitation of the dipole mode $\eta_1-\eta_{-1}$. For the excitation of the mode $\eta_1+\eta_{-1}$ the minima would shift to the right and lie at $\bar{a}''$ and $-\bar{a}'$. Both dipole modes thus split in the same way, which is a consequence of the symmetry between the x' and y' directions in the intrinsic system.

The $u_{K,v}{}^I(a_0')$ satisfy

$$\left\{-\frac{\hbar^2}{2B}\frac{\partial^2}{\partial a_0'^2}+\frac{C_0}{2}a_0'^2+\hbar\omega_1 G_0 a_0'\right\}u_{K,\pm1}{}^I(a_0')$$
$$=\epsilon_{\pm1}u_{K,\pm1}{}^I(a_0'), \quad (31a)$$

$$\left\{-\frac{\hbar^2}{2B}\frac{\partial^2}{\partial a_0'^2}+\frac{C_0}{2}a_0'^2+\hbar\omega_0 F a_0'\right\}u_{K,0}{}^I(a_0')$$
$$=\epsilon_0 u_{K,0}{}^I(a_0'). \quad (31b)$$

Both lead to harmonic vibrations with a frequency $\omega^2=C_0/B$ around the new equilibrium value of a_0.

$$a_{01}=\beta_0-\frac{\hbar\omega_1}{C_0}G_0\equiv\beta_0+\beta_{01}, \quad (32a)$$
$$a_{00}=\beta_0-\frac{\hbar\omega_0}{C_0}F\equiv\beta_0+\beta_{00},$$
$$\beta_{01}=-\frac{\hbar\omega_1 G_0}{E_\beta}\frac{\epsilon}{E_\beta}-3\beta_0^2, \quad (32b)$$
$$\beta_{00}=-\frac{\hbar\omega_0 F}{E_\beta}\frac{\epsilon}{E_\beta}-3\beta_0^2.$$

Here we have expressed the quantities β_{0v} in terms of the parameters E_β, β_0, and $\epsilon=\hbar^2/J_0=\hbar^2/3B\beta_0^2$ which are known from the low-energy spectra. The energies are given by

$$\epsilon_{\pm1}(n_0)=-\left(\frac{\hbar\omega_1 G_0}{C_0}\right)^2\frac{C_0}{2}+(n_0+\tfrac{1}{2})E_\beta, \quad (33a)$$

$$\epsilon_0(n_0)=-\left(\frac{\hbar\omega_0 F}{C_0}\right)^2\frac{C_0}{2}+(n_0+\tfrac{1}{2})E_\beta, \quad (33b)$$

where $\hbar(C_0/B)^{1/2}=E_\beta$ and the wave functions are Hermite polynomials H_{n_0} about the new equilibrium value. The constant terms in Eq. (33) give the energy change due to the change of the shape of the intrinsic nucleus. Explicitly the wave functions are

$$u_{K,v,n_0}{}^I=\left[\frac{\alpha}{2^{n_0}(\pi)^{1/2}}\right]^{1/2}H_{n_0}[\alpha(a_0'-\beta_{0,v})]\times\exp\left[-\frac{\alpha^2}{2}(a_0'-\beta_{0,v})^2\right]$$
$$\alpha^4=BC_0/\hbar^2, \quad v=0,\pm1. \quad (34)$$

The $\varphi_{K,v,n_2}{}^I(a_2')$ satisfy the Eqs. (35a) for odd K and (35b) for even K.

$$\left\{-\frac{\hbar^2}{2(2B)}\frac{\partial^2}{\partial a_2'^2}+\frac{\hbar^2K^2}{16Ba_2'^2}+C_2a_2'^2\pm\hbar\omega_1 G_2 a_2'\right\}$$
$$\times\varphi_{K,\pm1,n_2}{}^I(a_2')=\tilde{\epsilon}_{\pm1}(n_2)\varphi_{K,\pm1,n_2}{}^I, \quad (35a)$$

$$\left\{-\frac{\hbar^2}{2(2B)}\frac{\partial^2}{\partial a_2'^2}+\frac{\hbar^2(K^2-1)}{16Ba_2'^2}+C_2a_2'^2\right\}\varphi_{K,0,n_2}{}^I(a_2')$$
$$=\tilde{\epsilon}_0(n_2)\varphi_{K,0,n_2}{}^I. \quad (35b)$$

Remembering from Eq. (14) that G_2 is negative, the effective potential of the a_2' vibrations, is shown in Fig. 4. It is seen that two stable minima occur in the potential energy:

$$V_{K,\pm1}(a_2')=\frac{\hbar^2K^2}{16Ba_2'^2}+C_2a_2'^2\pm\hbar\omega_1 G_2 a_2', \quad K=1,3,5,\cdots \quad (36a)$$

$$V_{K,0}(a_2')=\frac{\hbar^2(K^2-1)}{16Ba_2'^2}+C_2a_2'^2, \quad K=2,4,6,\cdots \quad (36b)$$

which are at the positions

$$\bar{a}_{2,K,\pm1}\approx\pm(\hbar^2K^2/16BC_2)^{1/4}-\frac{\hbar\omega_1 G_2}{8C_2}$$
$$=\pm\frac{(3K)^{1/2}}{2}(\epsilon/E_\gamma)^{1/2}\beta_0-\frac{3}{8}\frac{\hbar\omega_1 G_2}{E_\gamma}\frac{\epsilon}{E_\gamma}\beta_0^2$$
$$K=1,3,5,\cdots, \quad (37a)$$

$$\bar{a}_{2,K,0}\approx\pm(\hbar^2(K^2-1)/16BC_2)^{1/4}$$
$$=\pm\tfrac{1}{2}[\sqrt{3}(K^2-1)^{1/4}](\epsilon/E_\gamma)^{1/2}\beta_0,$$
$$\text{for}\quad K\neq 0$$
$$K=2,\ 4,\ 6,\ \cdots. \tag{37b}$$

$E_\gamma=\hbar(C_2/B)^{1/2}$ is the γ-vibration energy.

Equation (37a) represents the first two terms of a perturbation expansion which considers the shift of the potential minimum due to the linear term (see Fig. 4) to be small. Indeed, the second term of Eq. (37a) is small compared with the first term and is of the order $\frac{1}{4}\sqrt{3}(\hbar\omega_1 G_2/E_\gamma)(\epsilon/E_\gamma)^{1/2}\beta_0\simeq\frac{1}{4}$. No displaced equilibrium value exists for $K=0$. Instead we have a situation as shown in Fig. 5. The frequencies about the new equilibrium values, Eq. (37), are given by the curvature of the potential energy, Eq. (36), at these points:

$$2\bar{C}_2(\pm 1)=\frac{3\hbar^2K^2}{8B\bar{a}_2{}'^2}+2C_2=8C_2,\quad K=1,\ 3,\ 5 \tag{38a}$$

$$2\bar{C}_2(0)=8C_2,\quad \text{for}\quad K\neq 0,\quad K=2,\ 4,\ 6,\ \cdots, \tag{38b}$$

so we see that both minima in Fig. 4 lead to approximately the same vibrational energies given by

$$\hbar\bar{\omega}_\gamma=\hbar(\bar{C}_2/B)^{1/2}\cong 2E_\gamma. \tag{39}$$

However, the ground states have different energies.

We can solve Eq. (35a) approximately. Since we saw that the two minima in the potential energy occur at the same $\bar{a}'_{2K,\pm 1}$ and the linear terms do not influence the curvature, Eq. (38), we can expect that the wave functions are not very sensitive to these linear terms,

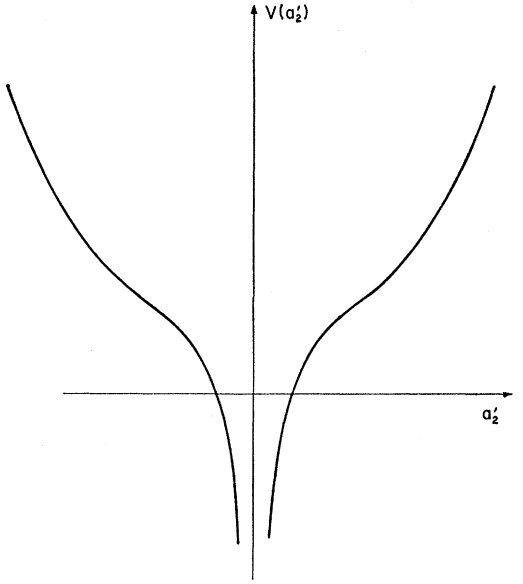

FIG. 5. Potential energy for the a_2 vibrations, η_0 dipole mode, and $K=0$. The nucleus retains its axially symmetric shape.

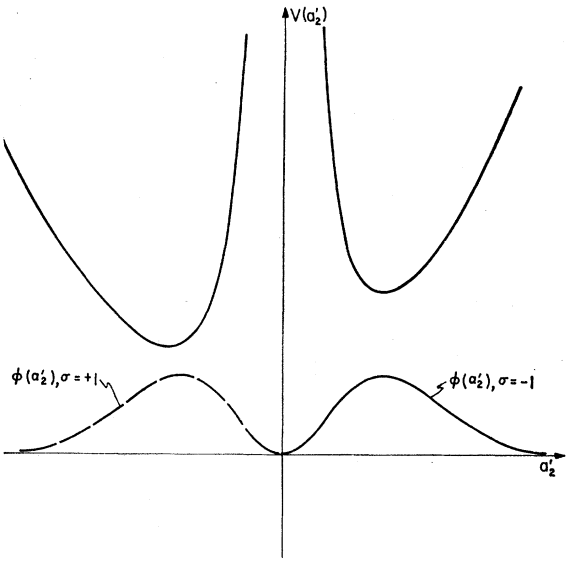

FIG. 6. The same as Fig. 4(B), redrawn in an orthogonal coordinate system. The wave function for the a_2 vibrations is indicated: for $\sigma=-1$, full line; for $\sigma=+1$, dashed line. For the mode $\eta_1-\eta_{-1}$ left and right must be interchanged. $\sigma=+1$ indicates that the wave function is nonvanishing on the side of the lower potential minimum.

which influence only the energies: One minimum is shifted up, the other one down. The change of energies is given approximately by replacing the a_2' in the linear term by the appropriate equilibrium values, Eq. (37a). Then the equations to be solved are

$$\left\{-\frac{\hbar^2}{2(2B)}\frac{\partial^2}{\partial a_2{}'^2}+\frac{K^2\hbar^2}{16Ba_2{}'^2}+C_2a_2{}'^2\pm A_{\pm 1}\right\}\varphi_{K,\pm 1,n_2}{}^I$$
$$=\tilde{\epsilon}_{K,\pm 1,n_2}\varphi_{K,\pm 1,n_2}{}^I, \tag{40}$$
$$A_{\pm 1}=\hbar\omega_1 G_2\bar{a}_{2,K,\pm 1}\approx\frac{\hbar\omega_1 G_2}{2}(3K)^{1/2}(\epsilon/E_\gamma)^{1/2}\beta_0,$$

and Eq. (35b) is not changed, of course. The solutions of both Eqs. (40) and (35b) are[1,6]

$$\varphi_{K,d,n_2}{}^I(a_2')=\frac{1}{\Gamma(l_K+\frac{3}{2})}[2\lambda^{l_K+\frac{3}{2}}\Gamma(l_K+\tfrac{3}{2}+n_2)/n_2!]^{1/2}$$
$$\times a_2{}'^{l_K+1}e^{-\frac{1}{2}\lambda a_2{}'^2}{}_1F_1(-n_2,\ l_K+\tfrac{3}{2};\ \lambda a_2{}'^2), \tag{41}$$

where

$$l_K=-\tfrac{1}{2}+\tfrac{1}{2}(1+K^2)^{1/2},\quad \text{for odd } K$$
$$=-\tfrac{1}{2}+\tfrac{1}{2}K,\quad \text{for even } K$$
$$\lambda^2=\frac{4BC_2}{\hbar^2}\quad\text{and}\quad a_2'\geq 0.$$

The function φ in (41) is normalized to unity in the range $0\leq a_2'\leq\infty$. It is consistent to have φ finite only on one side, i.e., either for a_2' positive or for a_2' negative, since the potential barrier $1/a_2{}'^2$ is impenetrable. How-

ever, the state where, e.g., $d=+1$ and φ is finite for $a_2'>0$ is degenerate with the state, where $d=-1$ and φ is finite for $a_2'<0$ (see Fig. 6). Therefore, one must take a suitable linear superposition of these two possibilities. This point will be discussed below, Sec. IIIA. We introduce a new quantum number σ to label these linear combinations. Then the energies are given by[1,6]

$$\tilde{\epsilon}_{K,\sigma,n_2}=(2n_2+l_K+\tfrac{3}{2})E_\gamma+\sigma A_\sigma, \quad n_2=0, 1, 2, \cdots. \quad (42)$$

$\sigma=0$ for even K and $\sigma=\pm 1$ for odd K. (Here σ characterizes the higher or lower minimum of Fig. 4.) For odd K the two modes $(\eta_1\pm\eta_{-1})$ are degenerate.

A. The Complete Level Scheme

We now can discuss the complete level scheme of the Hamiltonian Eq. (23). For this purpose we write down the total wave functions and energies.

$$\Psi_{K,n_0,n_2,\sigma}{}^{I,M}=\left[\frac{2I+1}{16\pi^2}\right]^{1/2}\frac{1}{\sqrt{2}}\Big\{(\mathfrak{D}_{MK}{}^I-\sigma(-)^{I+K}\mathfrak{D}_{M-K}{}^I)\frac{\eta_1-\sigma\eta_{-1}}{\sqrt{2}}\varphi_{K,1,n_2}{}^I(a_2')$$

$$+(-)^{\frac{1}{2}(K-1)}(\mathfrak{D}_{MK}{}^I+\sigma(-)^{I+K}\mathfrak{D}_{M-K}{}^I)\frac{\eta_1+\sigma\eta_{-1}}{\sqrt{2}}\varphi_{K,-1,n_2}{}^I(-a_2')\Big\}u_{K,1,n_0}{}^I(a_0'),$$

$$\sigma=\pm 1, \quad I=K, K+1, K+2, \cdots, \quad K=1, 3, 5, \cdots \text{ and } a_2'\geq 0. \quad (43)$$

$$E_{K,n_0,n_2,\sigma}{}^I=[I(I+1)-(K^2+1)]\frac{\hbar^2}{2J_0}+\hbar\omega_1-\left(\frac{\hbar\omega_1 G_0}{E_\beta}\right)^2\tfrac{3}{2}\beta_0{}^2\epsilon+(n_0+\tfrac{1}{2})E_\beta+\sigma A_\sigma+(2n_2+l_K+\tfrac{3}{2})E_\gamma, \quad \sigma=\pm 1. \quad (44)$$

$$\Psi_{K,n_0,n_2,\sigma=0}{}^{I,M}=\left[\frac{2I+1}{16\pi^2(1+\delta_{K0})}\right]^{1/2}(\mathfrak{D}_{MK}{}^I-(-1)^{I+K}\mathfrak{D}_{M-K}{}^I)\eta_0 u_{K,0,n_0}{}^I(a_0')\frac{1}{\sqrt{2}}(\varphi_{K,0,n_2}{}^I(a_2')$$

$$+(-)^{K/2}\varphi_{K,0,n_2}{}^I(-a_2')), \quad K=0, 2, 4, \cdots,$$

$$I=1, 3, 5, \cdots, \quad \text{for } K=0,$$

$$=K, K+1, K+2, \cdots, \quad \text{for } K\neq 0. \quad (45)$$

$$E_{K,n_0,\sigma=0}{}^I=(I(I+1)-K^2)\frac{\hbar^2}{2J_0}+\hbar\omega_0-\left(\frac{\hbar\omega_0 F}{E_\beta}\right)^2\tfrac{3}{2}\beta_0{}^2\epsilon+(n_0+\tfrac{1}{2})E_\beta+(2n_2+l_K+\tfrac{3}{2})E_\gamma. \quad (46)$$

We have chosen the linear combination in Eq. (43) and Eq. (45) in such a way that the following additional symmetry is fulfilled: A rotation through $\frac{1}{2}\pi$ around the z' axis (see Fig. 3) and a simultaneous replacement of a_2 by $-a_2$ [see Eq. (12)] should leave the wave function invariant.[29] Since the mentioned rotation changes

$$\mathfrak{D}_{MK}{}^I(\eta_1\pm\eta_{-1})\rightarrow e^{i(\pi/2)(K-|d|)}\mathfrak{D}_{MK}{}^I(\eta_1\mp\eta_{-1}),$$

one checks that Eq. (43) and Eq. (45) fulfill this symmetry if we choose

$$\varphi_d(-a_2')=\varphi_{-d}(a_2').$$

The energy spectrum in the giant resonance region is shown in Fig. 1. We now consider first the 1^- levels. Going up in energy the first state is the pure giant resonance n_0 along the major axis of the ellipsoid. The next levels are this same resonance with β and γ vibrations on top of it (the starting levels of the first three bands in Fig. 1). Some MeV above these levels follow the giant resonances η_1 and η_{-1} where the neutron-proton fluid oscillates along the minor axis. Two states of this type occur corresponding to the two new equilibrium shapes in Fig. 4. Above these levels come quadrupole vibrational states with one and more quadrupole quanta excited on top of the dipole quanta.

The transition energies are found by subtracting from the energies (44) and (46) the ground-state energy which can be obtained by replacing in (46) $\hbar\omega_0$ by zero and inserting $I=K=n_0=n_2=0$; $l_K=-\frac{1}{2}$.

In this discussion we have so far neglected the rotation-vibration interaction and the Coriolis term. The first one is very small and it is to be expected to change things by negligible amounts (except for very high spins). The second one is of the order of the rotational energy. At high spins the Coriolis force is therefore expected to mix several bands $(\eta_0;\eta_1\pm\eta_{-1})$. However, even for high spins the effect of this mixing can cause an energy shift of the bands of only a few hundred keV.

IV. THE DIPOLE OPERATOR AND THE ABSORPTION CROSS SECTION

A. The Dipole Operator

Our program is now to calculate the dipole operator in terms of the creation and annihilation operators $\beta_v{}^+$, β_v of dipole quanta and to establish its additional dependence on the shape of the intrinsic nucleus. We

need for this calculation, as we will see, the potential energy of the dipole oscillations. Since we assume harmonic dipole oscillations it must be of the form

$$V_\mu(\xi_\mu)=\tfrac{1}{2}C_\mu|\xi_\mu|^2, \tag{47}$$

where ξ_μ characterizes the amplitude of dipole vibration connected with $\beta_\mu^+|0\rangle$. The proton distribution is given by

$$(\rho_p)_\mu=\rho_p(0)(1+\xi_\mu F_\mu j_1(k_\mu r)Y_{1\mu}), \tag{48}$$

where F_μ is a normalization factor, $\rho_p(0)$ is the unperturbed proton distribution. We calculate the energy constant C_μ in the hydrodynamical model

$$\tfrac{1}{2}C_\mu|\xi_\mu|^2=\kappa\int_v\frac{|\rho_0-2\rho_p|^2}{\rho_0}dV, \tag{49}$$

where κ is the symmetry energy parameter in the Bethe-Weizsäcker semiempirical binding energy formula and where ρ_0 is the unperturbed matter distribution and the integration goes over the volume of the intrinsic nucleus. We obtain with Eq. (48) expanding in the small parameters a_μ, Eq. (4):

$$\begin{aligned}\tfrac{1}{2}C_\mu|\xi_\mu|^2&=\text{const}+\kappa\frac{4\rho_p^2(0)}{\rho_0}|\xi_\mu|^2F_\mu^2\int_v j_1^2(k_\mu r)|Y_{1\mu}|^2r^2drd\Omega\\&=\text{const}+\frac{4\kappa\rho_p^2(0)}{\rho_0}\frac{F_\mu^2|\xi_\mu|^2}{2k_\mu^3}\Big[G(k_\mu R_0)\\&\quad+\frac{\partial G(\rho)}{\partial\rho}\Big|_{a_{v=0}}(k_\mu R_0)a_0\tau_\mu\Big],\end{aligned} \tag{50}$$

where

$$G(\rho)=\rho^3[j_1^2(\rho)-j_0(\rho)j_2(\rho)]$$

and

$$\tau_\mu=(5/4\pi)^{1/2}(1020|10)(1\mu20|1\mu).$$

So we get

$$C_\mu=\frac{4\kappa\rho_p^2(0)}{\rho_0}\frac{F_\mu^2}{k_\mu^2}[G(k_\mu R_0)+R_\mu(k_\mu R_0)a_0]\equiv Q_\mu F_\mu^2, \tag{51}$$

where

$$R_\mu=\frac{\partial G(\rho)}{\partial\rho}\Big|_{a_{v=0}}(k_\mu R_0)\tau_\mu.$$

If one takes $a_0\approx\frac{1}{3}$ and $k_\mu R_0\approx 2.08$ one estimates that

$$[R(k_\mu R_0)a_0/G(k_\mu R_0)]\sim\tfrac{1}{5}\cdots\tfrac{1}{6}.$$

The dipole operator in the intrinsic system is defined as

$$\hat{D}_\mu=\int_v\rho_p r Y_{1\mu}d\tau. \tag{52}$$

With Eq. (48) we get after some calculations similar to those above

$$\hat{D}_\mu=\rho_p(0)F_{-\mu}S_{-\mu}\xi_{-\mu}=\rho_p(0)\left[\frac{\hbar\omega_{-\mu}}{2Q_{-\mu}}\right]^{1/2}S_{-\mu}(\beta_{-\mu}^+-\beta_{-\mu}), \tag{53}$$

where we have used Eq. (51) and introduced annihilation and creation operators of the dipole quanta by means of $\xi_\mu=(\hbar\omega_\mu/2C_\mu)^{1/2}(\beta_\mu^+-\beta_\mu)$, and where $\omega_\mu(\mu=0,\pm1)$ is given by Eq. (18). We have introduced further

$$\begin{aligned}S_\mu&=\frac{1}{k_\mu^4}[\bar{G}(k_\mu R_0)+\bar{R}(k_\mu R_0)a_0],\\\bar{G}(\rho)&=-3(\rho\cos\rho-\sin\rho)-\rho^2\sin\rho;\\\bar{R}&=\frac{\partial\bar{G}(\rho)}{\partial\rho}\Big|_{a_{v=0}}k_\mu R_0\tau_\mu,\end{aligned} \tag{54}$$

where τ_μ is given by Eq. (50).

Equation (54) is again an expansion of the integral, Eq. (52), in terms of the surface parameters. It is easy to check that the second term of Eq. (54) is of the order $\frac{1}{5}$ smaller than the first one. The normalization factor F_μ has dropped out in Eq. (53). This form of the intrinsic dipole operator is still not suitable for calculations since it contains the quadrupole vibrational coordinates a_v in a complicated form. Inserting the ω_μ from Eq. (17) and Q_μ from Eq. (51) we obtain finally after some expansions

$$\hat{D}_\mu=\left[\frac{\hbar\omega_{-\mu}(0)}{2Q_{-\mu}(0)}\right]^{1/2}S_{-\mu}(0)(\beta_{-\mu}^+-\beta_{-\mu})[1+f_{-\mu}a_0'], \tag{55}$$

where

$$\begin{aligned}f_\mu&=\frac{1}{2}\Big[G_0+\frac{2\bar{R}(k_\mu R_0)}{\bar{G}(k_\mu R_0)-\bar{R}(k_\mu R_0)\beta_0}\\&\quad-\frac{R(k_\mu R_0)}{G(k_\mu R_0)+R(k_\mu R_0)\beta_0}\Big],\\Q_\mu(0)&=\frac{4\kappa\rho_p^2(0)}{\rho_0}R_0^3\frac{[G(k_\mu R_0)+R(k_\mu R_0)\beta_0]}{(k_\mu R_0)^3},\\S_\mu(0)&=\rho_p(0)R_0^3\cdot R_0\frac{[\bar{G}(k_\mu R_0)+\bar{R}(k_\mu R_0)\beta_0]}{(k_\mu R_0)^4},\end{aligned} \tag{56}$$

and $\hbar\omega_\mu(0)$ is given by Eq. (18). Terms linear in a_2' do not contribute to transitions from the ground state.

The expression Eq. (55) shows explicitly how the dipole operator depends in lowest approximation on the quadrupole coordinates a_v'. Since the f_v are pure coefficients of the order 1, one can already see that the contribution of the latter terms to the dipole cross section are of the order $\frac{1}{10}$ compared with that of the "pure" giant dipole operator. Equation (55) is the dipole operator in intrinsic coordinates. In the laboratory system it is obtained by rotating it with the $\mathfrak{D}_{\mu\nu}^1$ matrices.

$$D_\mu=\textstyle\sum_\nu\mathfrak{D}_{\mu\nu}^1(\theta)\hat{D}_\nu. \tag{57}$$

B. The γ-Absorption Cross Section

The giant resonance is usually studied by γ-absorption experiments: A bremsstrahlung beam falls on the target and the photon absorption is measured. We treat the problem by assuming that at $t=0$ an electromagnetic field of frequency ω is switched on, and we determine the behavior of the system for times long compared with the lifetime of the excited levels. Now let Ψ_i be the ground state and Ψ_f the excited state. The ground state is stable; i.e., it does not decay. The excited state, however, decays partly into the continuum and mostly to other configurations which are not of the giant resonance type. Let us describe the total system by Eq. (22)

$$H=H_0+H_1+H_{\rm rad}, \tag{58}$$

where H_0 is the rot-vib-dip Hamiltonian which we have discussed so far. H_1 shall describe the decay of the dipole levels, i.e., there will exist nonvanishing off-diagonal matrix elements of H_1 which connect the dipole levels with other excited levels.[30] $H_{\rm rad}$ is the interaction with the electromagnetic field.

$$H_{\rm rad}=e(4\pi/3)^{1/2}\mathbf{E}\cdot\mathbf{D}, \tag{59}$$

where $\mathbf{E}$ is the electric field, $\mathbf{D}$ is the dipole operator, Eq. (57), and e the electron charge, and

$$\mathbf{E}=\boldsymbol{\varepsilon}E_0\sin\omega t. \tag{60}$$

If a_i, a_f are the amplitudes of the states Ψ_i, Ψ_f and if Ψ_ν is a complete orthonormal set and eigenfunctions of H_0, then we get in the usual manner

$$\dot{a}_i=\frac{(H_{\rm rad})_{if}}{2i\hbar}a_f e^{i(\omega_{if}-\omega)t}, \tag{61}$$

$$a_f=\frac{(H_{\rm rad})_{fi}}{2i\hbar}a_i e^{i(\omega_{fi}-\omega)t}+\sum_{\nu\neq i}\frac{(H_1)_{f\nu}}{it}a_\nu e^{i\omega_{f\nu}t}. \tag{62}$$

In the first equation we have neglected the transitions from the ground state to other excited states, because they are "off resonance." By definition, the Hamiltonian H_1 does not contain matrix elements to the ground state. Therefore, the sum in the second equation is restricted to $\nu\neq 1$. This sum describes, so to say, the thermalization of the energy contained in the excited state Ψ_f. Following Wigner and Weisskopf,[21,31] we take this energy dissipation into account in a certain averaged way by the ansatz

$$\dot{a}_f=\frac{(H_{\rm rad})_{fi}}{2i\hbar}a_i e^{i(\omega_{fi}-\omega)t}-\frac{\Gamma}{2\hbar}a_f, \tag{63}$$

i.e., we assume that the different phases in the sum of Eq. (62) average out.[30] We have a system of coupled equations where a_i, a_f have to fulfill the initial conditions

$$\begin{aligned} a_i(t=0)&=1,\\ a_f(t=0)&=0. \end{aligned} \tag{64}$$

We can solve Eq. (63) by inserting for a_i its initial value, Eq. (62) and we get

$$a_f(t)=\frac{(H_{\rm rad})_{fi}}{2i\hbar[i(\omega_{fi}-\omega)-\Gamma/2\hbar]}e^{i(\omega_{fi}-\omega)t}-e^{-(\Gamma/2)t}. \tag{65}$$

The probability for observing the excited state Ψ_f after a long time is given by

$$|a_f(\infty)|^2=\frac{|(H_{\rm rad})_{fi}|^2}{4\hbar^2}\frac{1}{(\omega_{fi}-\omega)^2-\Gamma^2/4\hbar^2}. \tag{66}$$

The γ-absorption cross section $\sigma(\hbar\omega)$ is defined as the energy absorbed during the lifetime $\hbar/\Gamma$ of the excited state divided by the incoming energy flux per unit area and second.

$$\sigma(E)=\frac{|a_f(\infty)|^2E_{fi}}{(E_0^2c/8\pi)(\hbar/\Gamma)}=\frac{2\pi}{\hbar}\frac{1}{E_0^2c}|(H_{\rm rad})_{fi}|^2\times\frac{\Gamma}{(E_{fi}-E)^2+\Gamma^2/4}. \tag{67}$$

This formula exhibits the resonance behavior of the cross section as a Breit-Wigner-type formula. It is derived, as we see, by a straightforward procedure. It is interesting to discuss on this basis the autocorrelation effects of our system.[19] By this we mean the following: Since the rotational times for low spins are of the order $\hbar/100$ keV and the decay time of the giant resonance is of the order $\hbar/\Gamma\approx\hbar/2$ MeV (which is our measuring time) we expect that the system "sees" the intrinsic nucleus. Indeed, we see from Eq. (67) that since $\Gamma\approx 2$ MeV the resonance energy E_{fi} can be shifted by amounts $\Delta E\ll\Gamma$, i.e., we can neglect the rotational energies. If, however, the rotational energies become of the order of Γ or even larger, then we can detect these shifts according to Eq. (67). In this latter case we do not see the "intrinsic" nucleus but the nucleus of the laboratory system.

C. Estimates of the Absorption Cross Section

The strength of the absorption cross section is given by the square of the dipole matrix elements. We do not

[30] We assume in the sense of the random-phase approximation that the interaction of the giant resonance levels due to H_1 averages out to zero,

$$\Sigma_\nu\frac{\langle\Psi_\alpha,H_1\Psi_\nu\rangle\langle\Psi_\nu,H_1\Psi_\beta\rangle}{E_\alpha-E_\nu}\approx 0.$$

[31] A similar procedure was used by S. Flügge for the derivation of the Breit-Wigner-formula [S. Flügge, Z. Naturforsch. **1**, 121 (1946); **3a**, 97 (1948)].

perform detailed calculations and comparison with experiments here. Such calculations are to be published elsewhere. However, we would like to give an estimate of the relative strength of the most important levels; i.e., essentially the excitation of the two giant modes along the minor axis and the excitation of the first a_2' vibrations. These cross sections are different only because the overlap integrals of the a_2' vibrational wave functions differ. The vibrational function is given by Eq. (41).

The overlap integral leading from the ground state to the excited states is

$$\frac{1}{2}\int_{-\infty}^{0}\varphi_{000}{}^{I}(-a_2')\varphi_{1,-1,n_2}{}^{I}(-a_2')da_2'+\frac{1}{2}\int_{0}^{\infty}\varphi_{000}{}^{I}(a_2')\varphi_{1,1,n_2}{}^{I}(a_2')da_2'$$

$$=\frac{1}{\Gamma(l_1+\frac{3}{2})}\left[\frac{2\lambda^{l_1+\frac{5}{2}}\Gamma(l_1+\frac{5}{2}+n_2)}{n_2!}\right]^{1/2}\int_{0}^{\infty}a_2'^{l_1+\frac{3}{2}}e^{-\lambda a_2'^2}{}_1F_1(-n_2,\,l_1-\tfrac{3}{2};\,\lambda a_2'^2){}_1F_1(0,1;\,\lambda a_2'^2)da_2'. \quad (68)$$

Here exists the general formula[32] (in which p is the arbitrary),

$$\int_0^\infty e^{-\lambda a_2'^2}a_2'^{l_{K_1}+l_{K_2}+2+p}{}_1F_1(-n_1;\,l_{K_1}+\tfrac{3}{2};\,\lambda a_2'){}^2{}_1F_1(-n_2;\,l_{K_2}+\tfrac{3}{2};\,\lambda a_2'){}^2da_2'$$

$$=\tfrac{1}{2}\lambda^{-\frac{1}{2}(l_{K_1}+l_{K_2}+p+3)}\Gamma[\tfrac{1}{2}(l_{K_1}+l_{K_2}+p+3)]\Gamma(l_{K_2}+\tfrac{3}{2})\Gamma[n_2+\tfrac{1}{2}(l_{K_2}-l_{K_1}-p)]\{\Gamma(n_2+l_{K_2}+\tfrac{3}{2})$$

$$\times\Gamma[\tfrac{1}{2}(l_{K_2}-l_{K_1}-p)]\}^{-1}{}_3F_2(-n_1,\,\tfrac{1}{2}(l_{K_1}+l_{K_2}+p+3),\,\tfrac{1}{2}(l_{K_1}-l_{K_2}+p+2);\,l_{K_1}+\tfrac{3}{2},$$

$$-n_2+\tfrac{1}{2}(l_{K_1}-l_{K_2}+p+2);\,1)$$

$$=\tfrac{1}{2}\lambda^{-\frac{1}{2}(l_{K_1}+l_{K_2}+p+3)}\Gamma[\tfrac{1}{2}(l_{K_1}+l_{K_2}+p+3)]\Gamma(l_{K_1}+\tfrac{3}{2})\Gamma(l_{K_2}+\tfrac{3}{2})\Gamma[n_1+\tfrac{1}{2}(l_{K_1}-l_{K_2}-p)]\Gamma[n_2+\tfrac{1}{2}(l_{K_2}-l_{K_1}-p)]$$

$$\times\{\Gamma(n_1+l_{K_1}+\tfrac{3}{2})\Gamma(n_2+l_{K_2}+\tfrac{3}{2})\Gamma[\tfrac{1}{2}(l_{K_1}-l_{K_2}-p)]\Gamma[\tfrac{1}{2}(l_{K_2}-l_{K_1}-p)]\}^{-1}{}_3F_2(-n_1,\,-n_2,$$

$$\times\tfrac{1}{2}(l_{K_1}+l_{K_2}+p+3);\,1-n_1+\tfrac{1}{2}(l_{K_2}-l_{K_1}+p),\,1-n_2+\tfrac{1}{2}(l_{K_1}-l_{K_2}+p);\,1). \quad (69)$$

In deriving the second, more symmetrical expression above, use is made of the fact that n_1 is an integer.[33]

Note that four of the above factorials are, with the abbreviation $(a)_n=\Gamma(a+n)/\Gamma(a)$, simply

$$\frac{(\frac{1}{2}(l_{K_1}-l_{K_2}-p))_{n_1}(\frac{1}{2}(l_{K_2}-l_{K_1}-p))_{n_2}}{(l_{K_1}+\frac{3}{2})_{n_1}(l_{K_2}+\frac{3}{2})_{n_2}}.$$

In (69) we have $(\frac{1}{2}(l_{K_2}-l_{K_1}-p))_{n_2}/(l_{K_2}+\frac{3}{2})_{n_2}$ as factor.

Thus, we can write

$$\int_0^\infty a_2'^p\varphi_{K_1,d,n_1}{}^{I}(a_2')\varphi_{K_2,d',n_2}{}^{I}(a_2')da_2'=[n_1!n_2!\Gamma(n_1+l_{K_1}+\tfrac{3}{2})\Gamma(n_2+l_{K_2}+\tfrac{3}{2})]^{-1/2}$$

$$\times(\tfrac{1}{2}(l_{K_1}-l_{K_2}-p))_{n_1}(\tfrac{1}{2}(l_{K_2}-l_{K_1}-p))_{n_2}\lambda^{-\frac{1}{2}p}\Gamma[\tfrac{1}{2}(l_{K_1}+l_{K_2}+p+3)]$$

$$\times{}_3F_2(-n_1,\,-n_2,\,\tfrac{1}{2}(l_{K_1}+l_{K_2}+p+3);\,1-n_1+\tfrac{1}{2}(l_{K_2}-l_{K_1}+p),\,1-n_2(l_{K_1}-l_{K_2}+p);\,1), \quad (70)$$

and (68) becomes

$$\int_0^\infty\varphi_{000}{}^{I}(a_2')\varphi_{1,d,n_2}{}^{I}(a_2')da_2'$$

$$=\tfrac{1}{2}(l_1+\tfrac{1}{2})\Gamma[n_2+\tfrac{1}{2}(l_1+\tfrac{1}{2})]$$

$$\times[n_2!\Gamma(n_2+l_1+\tfrac{3}{2})]^{-1/2}. \quad (68')$$

For the ratio of the absorption cross section leading from the ground state to the lowest state of one of the bands (Fig. 6), on the one hand, and to the first a_2' vibrational satellite on the other hand we find

$$\frac{|\langle\varphi_{0,0,0^0}|\,\varphi_{1,\pm1,0^1}\rangle|^2}{|\langle\varphi_{0,0,0^0}|\,\varphi_{1,\pm1,1^1}\rangle|^2}\approx\frac{10}{1}, \quad (71)$$

i.e., the absorption cross section to the pure "transverse" dipole state, i.e., that associated with an excitation along the short axis, is about 10 times stronger than that to the vibrational satellite of this state. The transitions to both "transverse" dipole states are of

[32] L. J. Slater, *Confluent Hypergeometric Functions* (Cambridge University Press, Cambridge, 1960), p. 54, Eq. (3.7.4).

[33] W. N. Bailey, *Generalized Hypergeometric Series* (Cambridge University Press, Cambridge, 1935). The relevant equation in the notation used by Bailey (see p. 22) is $\Gamma(\alpha_{123})\Gamma(\alpha_{124})Fp(0;4,5)=(-1)^m\Gamma(\alpha_{025})\Gamma(\alpha_{015})Fn(5;0,2)$.

the same strength, as to be expected from sum-rule arguments. This can easily be seen from the integrals Eq. (68) which now go over the negative region of a_2'. Up to a phase factor the integrals are the same.

The question arises about the corrections to these results arising from the a_v'-dependent terms in the dipole operator Eq. (55). It is easy to see that these corrections are of the order $(\epsilon/E_\gamma)\beta_0 \approx \frac{1}{10}$ to $(1/20)$ and they are thus small.

Finally, we estimate the ratio of the absorption cross sections of the β_0^+ giant resonance and its satellite, i.e., the β_0^+ giant resonance with an a_0' vibration on top of it. The wave functions for these states are given by Eq. (34) and the changes in the intrinsic deformation β_{0v} are very small.

$$\beta_{0,1}=\frac{\hbar\omega_1 G_0}{C_0}=\frac{\hbar\omega_1 G_0}{E_\beta}\frac{\epsilon}{E_\beta} \quad 3\beta_0^2\approx\frac{1}{10} \text{ to } \frac{1}{100}$$

in most cases. Therefore, we can neglect the β_{0v} and put them equal to zero. Then the only possibility for exciting the a_0' vibration comes from the a_0'-dependent term in the dipole operator, which is of the order

$$\left[\frac{\hbar^2}{2BE_\beta}\right]^{1/2}=\left[\tfrac{3}{2}\beta_0^2\frac{\epsilon}{E_\beta}\right]^{1/2}\sim\frac{1}{10} \text{ to } \frac{1}{20}.$$

Therefore, the ratio of the excitation matrix elements is of this order and the cross sections differ by the factor 1/100.

V. DISCUSSIONS AND RESULTS

We have developed a dynamical collective theory of the giant resonances in heavy deformed nuclei. The extension from the earlier work[2] consists in the inclusion of rotations and quadrupole vibrations of the intrinsic nucleus. We have constructed the complete Hamiltonian on the basis of the adiabatic assumption, i.e., the rotations and vibrations are taken to be slow compared to the giant resonance oscillations. The additional degree of freedom of the intrinsic nucleus, the dipole oscillations, are handled similarly to the coupling of an odd particle to the core in the Nilsson model.[24,25] The complete Hamiltonian contains coupling terms between the rotation, the quadrupole vibrations and the giant dipole oscillations. We calculated the solutions of the Hamiltonian neglecting the Coriolis term and the rotation-vibration interaction.

A series of rotational and vibrational bands on the top of the various modes of the giant resonance is predicted. An interesting result is that the intrinsic nucleus, which is axially symmetric in its ground state, gets deformed and stabilizes in two new nonaxially symmetric equilibrium shapes when a giant resonance mode perpendicular to the major axis is excited. The two new nonaxial equilibrium shapes are separated energetically by $\sim$1.0–2.5 MeV. The nucleus does not change its equilibrium shape if a giant resonance mode along the major axis is excited (at least for the lowest rotational giant band).

We have constructed the dipole operator in terms of the collective coordinates. This dipole operator contains terms which can lead from the ground state to the 1^- states of the different giant resonance bands. The formula for the differential absorption cross section is derived under the assumption that the part of the Hamiltonian which is responsible for the damping of the giant resonance can be treated as a perturbation. However, we did not perform detailed calculations concerning the absorption cross section. A paper concerning such calculations and an analysis of some experimental data is in preparation.[34]

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[34] *Note added in proof.* See M. Danos and W. Greiner, Phys. Letters 8, 113 (1964). In the meantime a more accurate solution of the a_2 vibrations, Eq. (35a) has been obtained by L. C. Maximon (private communication). He finds that the splitting of the upper giant resonance mode is about 1.5 times larger than that predicted by (46) and (48) and reported in the above Phys. Letters paper. Similarly, the dipole strengths are changed somewhat. We thank L. C. Maximon for permission to quote his results prior to publication.