state and the low-lying states. But since the most strongly interacting configurations are the ones which we have considered here, we feel that this may not make any big improvement in the results and will rather make this calculation more complicated. Probably the best way will be to use a different two-body potential. The nucleon-nucleon potential which we are using here has been fitted by the bound state properties of the very light nuclei, $2 \leqslant A \leqslant 4$. It may be that these parameters are not as good for the nuclear levels which we are considering here. A new set of parameters can be chosen by a least-square fit of the ground-state properties and low-lying states of one of the nuclei in which the closed $1p$ shell is either missing one or two nucleons or has one or two additional nucleons outside. Once these parameters are fixed they can be used for other nuclei and by comparing the results with the known experimental values we can check the accuracy of this type of calculation. Since in the Hamiltonian of this calculation we have not included a two-body spin-orbit interaction, therefore, another possibility which must be considered is that the nuclear Hamiltonian should contain an explicit two-body spin-orbit interaction.

ACKNOWLEDGMENT

One of the authors (N. U.) would like to thank Dr. C. E. Porter for a number of helpful discussions.

PHYSICAL REVIEW VOLUME 134, NUMBER 2B 27 APRIL 1964

Improved Superconductivity Approximation for the Pairing Interaction in Nuclei

YUKIHISA NOGAMI*

Division of Pure Physics, National Research Council, Ottawa, Canada and Department of Physics, Battersea College of Technology, London, England

(Received 23 December 1963)

The pairing interaction in nuclei is investigated by using a model Hamiltonian $H-\lambda_1 N-\lambda_2 N^2$, where H is the original Hamiltonian and N is the nucleon-number operator. The introduction of the term $\lambda_2 N^2$ enables one to suppress the effect of the nucleon-number fluctuation, which is inherent in the Bardeen-Cooper-Schrieffer (BCS) approximation and is a main source of its inaccuracy. A prescription is given to determine the parameters λ_1 and λ_2. The method is first illustrated in the case where all levels are degenerate, and then applied to realistic cases with nondegenerate levels. Quadrupole and other interactions are not considered. Excited states with seniority zero as well as those with nonzero seniority are discussed. The results obtained by this method are compared with those of Kisslinger and Sorenson and of Kerman, Lawson, and Macfarlane. For the ground-state energy an excellent accuracy is attained easily. It is observed that the BCS states obtained by Kisslinger and Sorenson are much better approximations to eigenstates of our model Hamiltonian rather than to those of their Hamiltonian. A new light is shed on the problem as to why the projected and renormalized BCS states are very good approximations to the true eigenstates.

1. INTRODUCTION

THE physical ideas and mathematical techniques developed in the theory of superconductivity[1] have been applied to the problem of the pairing interaction in nuclei to explain low-lying energy levels of heavy nuclei.[2] According to this theory, a system of nucleons which have pairing correlations between them can approximately be described as an assembly of free quasiparticles which are connected to the original nucleons by means of the Bogoliubov-Valatin transformation. It is characteristic of this method that the Bogoliubov-Valatin transformation is not commutable with the nucleon-number operator, and consequently the wave function which results does not correspond to a system having a definite number of nucleons. Energies and other quantities which are calculated with this wave function are then interpreted as averages of the corresponding quantities over a set of neighboring nuclei.

Once the Bogoliubov-Valatin transformation is exercised, it seems difficult, if not impossible, to remove the nucleon number fluctuation from the wave function without losing the essential merit of the theory, the energy gap. Lipkin[3] has suggested, however, that it would be possible to eliminate this effect from energy eigenvalues. He has proposed to use the model Hamiltonian

$$\mathcal{H}=H-f(N) \tag{1.1}$$

where H is the original Hamiltonian, N is the nucleon

* Present address: Department of Physics, Battersea College of Technology, London, England.

[1] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957); N. N. Bogoliubov, Zh. Eksperim. i Teor. Fiz. **34**, 58 (1958) [English transl.: Soviet Phys.—JETP **7**, 41 (1958)]; J. G. Valatin, Nuovo Cimento **7**, 843 (1958). This theory will be referred to as the BCS theory.

[2] A. Bohr, B. R. Mottelson, and D. Pines, Phys. Rev. **110**, 936 (1958); S. T. Belyaev, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **31**, No. 11 (1959); L. S. Kisslinger and R. A. Sorenson, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **32**, No. 9 (1960). KS will hereafter mean the last paper or its authors.

[3] H. J. Lipkin, Ann. Phys. (N. Y.) **9**, 272 (1960).

number operator, and $f(N)$ is a function of N which will be specified below. We consider interactions only between like nucleons but not between proton and neutron. Since any eigenstate of H is an eigenstate of N, it is also an eigenstate of $\mathcal{H}$, although the converse is not true in general. Denote by $\Phi_0(n)$ the lowest eigenstate of H with an eigenvalue $E_0(n)$,

$$H\Phi_0(n)=E_0(n)\Phi_0(n), \quad N\Phi_0(n)=n\Phi_0(n). \quad (1.2)$$

Then we have

$$\mathcal{H}\Phi_0(n)=\{E_0(n)-f(n)\}\Phi_0(n). \quad (1.3)$$

If $f(N)$ is chosen in such a way that $\Phi_0(n)$ with different n are all degenerate eigenstates of $\mathcal{H}$, in other words, if $\{E_0(n)-f(n)\}$ is independent of n, then the model ground state which will be a superposition of $\Phi_0(n)$'s belongs to the same eigenvalue of $\mathcal{H}$ as that of the true ground state which has a definite number of nucleons. As far as the energy of the ground state is concerned, this model Hamiltonian defines an eigenvalue problem which is equivalent to the original one, and the nucleon-number fluctuation will not do any harm.

The problem is then to find the function $f(N)$. If $f(N)$ is a slowly varying function of N, a power series expansion

$$f(N)=\lambda_1 N+\lambda_2 N^2+\cdots, \quad (1.4)$$

with

$$\lambda_m=\lambda_m(0), \quad \lambda_m(n)=(m!)^{-1}d^m f(n)/dn^m \quad (1.5)$$

will be useful. We suppress the argument n of $\lambda_m(n)$ when $n=0$ or the n dependence is irrelevant. Perhaps $f(N)$ will be accessible only in this form. For a system of very large number of particles, the linear approximation to $f(N)$ as in the BCS theory will be sufficient. The number of nucleons outside of the closed shells is, however, not very large and $f(N)$ may considerably deviate from its linear approximation. An extensive calculation was done by Kisslinger and Sorenson[2] (KS) using the BCS theory and many detailed properties of low-lying levels of singly-closed-shell nuclei were explained. Kerman, Lawson, and Macfarlane[4] (LKM) examined the accuracy of KS's result and found that the ground-state energies are usually not given to better than 500 keV, while excitation energies of the low-lying states are correct within 200 keV. The inaccuracy seems to be mainly due to the nucleon-number fluctuation.

In this paper we study the pairing interaction in nuclei using a model Hamiltonian

$$\mathcal{H}=H-\lambda_1 N-\lambda_2 N^2. \quad (1.6)$$

A prescription is given to determine the coefficients λ_1 and λ_2. Strictly, for excited states different values of λ_1 and λ_2 than those for the ground state will be required, but we use for simplicity the same λ_1 and λ_2 as determined with respect to the ground state. The method is first illustrated in Sec. 2 in the case where all levels are degenerate[5] and then applied to realistic cases with nondegenerate levels in Sec. 3. Quadrupole and other interactions are not considered in this paper. Only spherical nuclei are treated but the extension to deformed nuclei will be straightforward. The result is compared with those of KS and KLM (Sec. 3B). It is observed that the effect of the nucleon-number fluctuation is satisfactorily suppressed and an excellent accuracy is attained for the ground state. In addition to the ground state and excited states with nonzero seniority, our theory can describe excited states with seniority zero concerning which the BCS approximation says nothing (Sec. 3C). The spurious states can satisfactorily be removed. Odd nuclei are discussed in Sec. 3D.

In the BCS approximation the residual pairing interaction between quasiparticles is ignored, but it is in fact not very small. Because of this residual interaction the number of quasiparticles is not conserved and the quasiparticle vacuum state, which is regarded as the ground state, is prevented from being a good eigenstate of the BCS Hamiltonian.[6] An advantage of our model Hamiltonian is that the part of the residual interaction between quasiparticles which does not conserve the number of quasiparticles is strongly suppressed compared with that in the BCS Hamiltonian. Another important feature is that our equation which determines the wave function deviates only slightly from the corresponding equation of the BCS theory. Our wave function will therefore agree closely with that of BCS. On the basis of these facts, a new light is thrown on the problem of the projected and renormalized BCS states[4,7] (Sec. 4).

2. DEGENERATE MODEL

We consider the case where n nucleons are in the configuration $(j)^n$. There is no essential difference between this and a more general case where there are several degenerate levels with different j's. The pairing Hamiltonian is

$$H=-G\sum_{m,m'>0}(-)^{j-m}(-)^{j'-m'}a_m^{\dagger}a_{-m}^{\dagger}a_{-m'}a_{m'}, \quad (2.1)$$

where G is the (positive) coupling constant and $a_m^{\dagger}(a_m)$ is the creation (annihilation) operator for a nucleon specified by the magnetic quantum number m.

The model Hamiltonian is

$$\mathcal{H}=H-\lambda_1 N-\lambda_2 N^2, \quad (2.2)$$

where $N=\sum a_m^{\dagger}a_m$ (the sum is over $m\gtrless 0$) is the nucleon number operator, and λ_1 and λ_2 are unknown as

[4] A. K. Kerman, R. D. Lawson, and M. H. Macfarlane, Phys. Rev. **124**, 162 (1961). KLM will hereafter mean this paper or its authors.

[5] A brief discussion of this case was given by Y. Nogami, Progr. Theoret. Phys. (Kyoto) **29**, 938 (1963).

[6] We mean by the BCS Hamiltonian the one of the form $H-\lambda N$.

yet. Let us rewrite $\mathcal{H}$ in terms of the quasiparticle operators which are defined by the Bogoliubov-Valatin transformation (with positive m),

$$\alpha_m = ua_m - (-)^{j-m}va_{-m}^\dagger,$$
$$\alpha_{-m} = ua_{-m} + (-)^{j-m}va_m^\dagger, \tag{2.3}$$

whre u and v are real and satisfy $u^2+v^2=1$. Then $\mathcal{H}$ takes the form

$$\mathcal{H} = \mathcal{H}_{00}+\mathcal{H}_{11}+\mathcal{H}_{20}+\mathcal{H}_{22}+\mathcal{H}_{31}+\mathcal{H}_{40}, \tag{2.4}$$

where the subscripts refer to the number of creation and annilihation operators, respectively. Explicit expressions of $\mathcal{H}_{nm}$ can be obtained as a special case of Eqs. (3.8)–(3.12). The terms $\mathcal{H}_{22}$, $\mathcal{H}_{31}$, and $\mathcal{H}_{40}$ are the residual interactions between quasiparticles. The term $\mathcal{H}_{20}$ is the so-called dangerous term, but $\mathcal{H}_{40}$ is also dangerous.

We want to determine $\lambda_1(n)$ and $\lambda_2(n)$ by a successive approximation. The ith approximation to $\lambda_1(n)$ and $\lambda_2(n)$ will be denoted by $\lambda_1^{(i)}(n)$ and $\lambda_2^{(i)}(n)$, respectively.

The first step: Putting $\lambda_2^{(1)}(n)=0$ and ignoring $\mathcal{H}_{40}$, we calculate the ground-state energy by the usual BCS method. The term $\mathcal{H}_{20}$ can be eliminated by taking $\lambda_1^{(1)}(n)=-\frac{1}{2}G(\Omega-n+n\Omega^{-1})$, where $\Omega=j+\frac{1}{2}$, and n is the nucleon number which is set equal to $\langle N\rangle_{\text{vac}}$, the expectation value of N in the ground state, i.e., the quasiparticle vacuum state. The first approximation to the ground-state energy $E_0^{(1)}(n)$ is given as $E_0^{(1)}(n)=-\frac{1}{4}Gn(2\Omega-n+n\Omega^{-1})$.

The second step: We put $\lambda_2^{(2)}(n)=\frac{1}{2}(d^2/dn^2)E_0^{(1)}(n)=\frac{1}{4}G(1-\Omega^{-1})$. The condition $\mathcal{H}_{20}=0$ then gives $\lambda_1^{(2)}(n)=-\frac{1}{2}G\{\Omega+1-\Omega^{-1}(n+1-n\Omega^{-1})\}$. Again $\mathcal{H}_{40}$ is ignored. The ground state energy is $E_0^{(2)}(n)=-\frac{1}{4}Gn\{2\Omega-n+2-\Omega^{-1}(2-n\Omega^{-1})\}$.

Repeating this procedure, we get at the ith step,

$$\lambda_1^{(i)}(n)=-\tfrac{1}{2}G\{\Omega+1-\Omega^{-i+1}(n+1-n\Omega^{-1})\},$$
$$\lambda_2^{(i)}(n)=\tfrac{1}{4}G(1-\Omega^{-i+1}), \tag{2.5}$$
$$E_0^{(i)}(n)=-\tfrac{1}{4}Gn\{2\Omega-n+2-\Omega^{-i+1}(2-n\Omega^{-1})\}. \tag{2.6}$$

In the limit $i\to\infty$, we obtain

$$\lambda_1(n)=-\tfrac{1}{2}G(\Omega+1),\quad \lambda_2(n)=\tfrac{1}{4}G, \tag{2.7}$$

which are independent of n, and

$$E_0(n)=-\tfrac{1}{4}Gn(2\Omega-n+2) \tag{2.8}$$

which is the exact ground-state energy.[8] As is seen from Eq. (2.6) the error in the energy is reduced by the factor Ω as the approximation is pushed one step further.

The above calculation is simple because the wave function, or u and v, remain unchanged throughout the successive procedure. They are given correctly from the outset. One may wonder that this situation is peculiar to the degenerate model and the above procedure may not be useful in nondegenerate models. It will be shown, however, in the next section that the same situation arises in nondegenerate cases in a good approximation. With λ_1 and λ_2 given by Eq. (2.7), we have

$$\mathcal{H}_{00}=E_0(n)-\lambda_1 n-\lambda_2 n^2=0,$$
$$\mathcal{H}_{20}=\mathcal{H}_{31}=\mathcal{H}_{40}=0, \tag{2.9}$$

and

$$\mathcal{H}=\mathcal{H}_{11}+\mathcal{H}_{22}$$
$$=-G\sum_{m,m'>0}(-)^{j-m}(-)^{j'-m'}\alpha_m^\dagger\alpha_{-m}^\dagger\alpha_{-m'}\alpha_{m'}$$
$$-\lambda_1\sum_{m\gtrless 0}\alpha_m^\dagger\alpha_m-\lambda_2\Big(\sum_{m\gtrless 0}\alpha_m^\dagger\alpha_m\Big)^2. \tag{2.10}$$

Here the important point is that not only the dangerous term $\mathcal{H}_{20}$ but also all the quasiparticle nonconserving terms, $\mathcal{H}_{31}$ and $\mathcal{H}_{40}$, have disappeared, so that the number of quasiparticles becomes a good quantum number of $\mathcal{H}$. The BCS vacuum state is an exact eigenstate of $\mathcal{H}$, whereas it is only an approximate eigenstate of the BCS Hamiltonian (with $\lambda_2=0$).

It is interesting to note that the right-hand side of Eq. (2.10) is obtained from Eq. (2.2) by replacing a in it with α. This means that $\mathcal{H}$ is invariant with respect to the Bogoliubov-Valatin transformation. This can be checked as follows. The Bogoliubov-Valatin transformation can be written as[9]

$$\alpha_{\pm m}=\exp(iS)a_{\pm m}\exp(-iS), \tag{2.11}$$

where

$$S=-i\theta\sum_{m>0}(-)^{j-m}(a_m^\dagger a_{-m}^\dagger-a_{-m}a_m)$$
$$=-i\theta\sum_{m>0}(-)^{j-m}(\alpha_m^\dagger\alpha_{-m}^\dagger-\alpha_{-m}\alpha_m) \tag{2.12}$$

with $\cos\theta=u$ and $\sin\theta=v$. S satisfies

$$i\theta^{-1}[\mathcal{H},S]=-(G\Omega+2\lambda_1+4\lambda_2)\sum_{m>0}(-)^{j-m}$$
$$\times(a_m^\dagger a_{-m}^\dagger+a_{-m}a_m)+(G-4\lambda_2)$$
$$\times\sum_{m>0}(-)^{j-m}(a_m^\dagger a_{-m}^\dagger N+Na_{-m}a_m), \tag{2.13}$$

which vanishes because of Eq. (2.7). Therefore $\mathcal{H}(\alpha)=\exp(iS)\mathcal{H}(a)\exp(-iS)=\mathcal{H}(a)$. What is true for $\mathcal{H}(a)$ is also true for $\mathcal{H}(\alpha)$ with the interchange of the words nucleon and quasiparticle. The eigenvalues of $\mathcal{H}$ are degenerate with respect to the quasiparticle number as well as to the nucleon number. This simple symmetry between nucleon and quasiparticle does not hold in general cases with nondegenerate levels.

Our next task is to obtain energies of excited states.

[7] A. F. de Miranda and M. A. Preston, Nucl. Phys. **44**, 529 (1963).

[8] G. Racah, Phys. Rev. **76**, 1352 (1949); B. R. Mottleson, in *The Many Body Problem*, edited by C. DeWitt and P. Nozieres (Dunod Cie., Paris, 1959), p. 283.

[9] K. Yoshida, Phys. Rev. **111**, 1255 (1958).

Denote by $\Phi_s(n)$ the eigenstate of H with seniority $s(=2,4,\cdots,n)$, and by $\Psi_s(n)$ the model state which contains s unpaired quasiparticles. Then we have

$$\mathcal{H}\Phi_s(n)=\{E_s(n)-\lambda_1 n-\lambda_2 n^2\}\Phi_s(n), \quad (2.14)$$

$$\mathcal{H}\Psi_s(n)=(-\lambda_1 s-\lambda_2 s^2)\Psi_s(n). \quad (2.15)$$

The eigenvalue in Eq. (2.15) does not depend on the number of paired quasiparticles in the state $\Psi_s(n)$. Since $\Phi_s(n)$ and $\Psi_s(n)$ are supposed to be degenerate eigenstates of $\mathcal{H}$, we get

$$E_s(n)=\lambda_1(n-s)+\lambda_2(n^2-s^2)$$
$$=-\tfrac{1}{4}G(n-s)(2\Omega-n-s+2). \quad (2.16)$$

This gives the complete spectrum of the degenerate model.[8]

3. NONDEGENERATE MODEL

A. Basic Equations

We consider the Hamiltonian

$$H=\sum_j \sum_{m\gtrless 0} E_j a_{jm}{}^\dagger a_{jm}-G\sum_{j,j'}\sum_{m,m'>0}(-)^{j-m}(-)^{j'-m'}$$
$$\times a_{jm}{}^\dagger a_{j-m}{}^\dagger a_{j'-m'}a_{j'm'} \quad (3.1)$$

where E_j are the single j-shell particle levels, G is the coupling constant, and $a_{jm}{}^\dagger(a_{jm})$ is the creation (annihilation) operator for a nucleon specified by j and m. Our model Hamiltonian is given by Eq. (2.2) where $N=\sum_j\sum_{m\gtrless 0}a_{jm}{}^\dagger a_{jm}$. Operators of quasiparticles are introduced by means of the Bogoliubov-Valatin transformation (with positive m)

$$\alpha_{jm}=u_j a_{jm}-(-)^{j-m}v_j a_{j-m}{}^\dagger,$$
$$\alpha_{j-m}=u_j a_{j-m}+(-)^{j-m}v_j a_{jm}{}^\dagger, \quad (3.2)$$

where u_j and v_j are real and satisfy $u_j{}^2+v_j{}^2=1$.

Introduce the notations

$$\Delta=G\sum_j \Omega_j u_j v_j, \quad (3.3)$$

$$\epsilon_j=E_j+(4\lambda_2-G)v_j{}^2, \quad (3.4)$$

$$\lambda=\lambda_1+2\lambda_2(n+1), \quad (3.5)$$

$$\mathfrak{N}_j=\sum_{m\gtrless 0}\alpha_{jm}{}^\dagger\alpha_{jm}, \quad \mathfrak{A}_j=\Omega_j{}^{-1/2}\sum_{m>0}(-)^{j-m}\alpha_{j-m}\alpha_{jm} \quad (3.6)$$

where $\Omega_j=j+\frac{1}{2}$, and n is the nucleon number which is set equal to the expectation value of N in the quasivacuum state.

$$n=\langle N\rangle_{\rm vac}=2\sum_j \Omega_j v_j{}^2. \quad (3.7)$$

In terms of the quasiparticle operators, $\mathcal{H}$ can be written in the form of Eq. (2.4). The first term is a c number,

$$\mathcal{H}_{00}=2\sum_j \Omega_j v_j{}^2 E_j-G^{-1}\Delta^2-G\sum_j \Omega_j v_j{}^4$$
$$-\lambda_1 n-\lambda_2 n^2-4\lambda_2\sum_j \Omega_j(u_j v_j)^2. \quad (3.8)$$

Here the last term is equal to $-\lambda_2(\langle N^2\rangle_{\rm vac}-\langle N\rangle_{\rm vac}{}^2)$, which is the correction for energy due to the nucleon-number fluctuation. The other terms in $\mathcal{H}$ are

$$\mathcal{H}_{11}=\sum_j\{(u_j{}^2-v_j{}^2)(\epsilon_j-\lambda)+\lambda_2+2\Delta u_j v_j\}\mathfrak{N}_j, \quad (3.9)$$

$$\mathcal{H}_{20}=\sum_j \Omega_j{}^{1/2}\{2u_j v_j(\epsilon_j-\lambda)-\Delta(u_j{}^2-v_j{}^2)\}$$
$$\times(\mathfrak{A}_j{}^\dagger+\mathfrak{A}_j), \quad (3.10)$$

$$\mathcal{H}_{22}=-\sum_{j,j'}\{Gu_j v_j u_{j'}v_{j'}+\lambda_2(u_j{}^2-v_j{}^2)(u_{j'}{}^2-v_{j'}{}^2)\}$$
$$\times N(\mathfrak{N}_j,\mathfrak{N}_{j'})-\sum_{j,j'}(\Omega_j\Omega_{j'})^{1/2}\{G(u_j{}^2u_{j'}{}^2+v_j{}^2v_{j'}{}^2)$$
$$+8\lambda_2 u_j v_j u_{j'}v_{j'}\}\mathfrak{A}_j{}^\dagger\mathfrak{A}_{j'}, \quad (3.11)$$

$$\mathcal{H}_{31}=\sum_{j,j'}\{Gu_j v_j(u_{j'}{}^2-v_{j'}{}^2)-4\lambda_2(u_j{}^2-v_j{}^2)u_{j'}v_{j'}\}$$
$$\times(\mathfrak{A}_{j'}{}^\dagger\mathfrak{N}_j+\mathfrak{N}_j\mathfrak{A}_{j'}), \quad (3.12)$$

$$\mathcal{H}_{40}=\sum_{j,j'}(\Omega_j\Omega_{j'})^{1/2}\{\tfrac{1}{2}G(u_j{}^2v_{j'}{}^2+v_j{}^2u_{j'}{}^2)-4\lambda_2 u_j v_j u_{j'}v_{j'}\}$$
$$\times(\mathfrak{A}_j{}^\dagger\mathfrak{A}_{j'}{}^\dagger+\mathfrak{A}_j\mathfrak{A}_{j'}). \quad (3.13)$$

In Eq. (3.11), $N(\ )$ stands for a normal product. The number of quasiparticles is conserved by $\mathcal{H}_{11}$ and $\mathcal{H}_{22}$, but not by $\mathcal{H}_{20}$, $\mathcal{H}_{31}$ and $\mathcal{H}_{40}$.

The coefficients u_j and v_j are chosen in such a way that the dangerous term $\mathcal{H}_{20}$ vanishes. We assume that the ground state is given by the quasiparticle vacuum state $\Psi_{\rm vac}$ and its energy by $\mathcal{H}_{00}+\lambda_1 n+\lambda_2 n^2$. The condition $\mathcal{H}_{20}=0$ is equivalent to minimizing $\mathcal{H}_{00}$. This procedure goes in parallel with the usual BCS theory.[10] The result is the equation

$$2(\epsilon_j-\lambda)u_j v_j-\Delta(u_j{}^2-v_j{}^2)=0 \quad (3.14)$$

which is equivalent to the gap equation

$$\tfrac{1}{2}G\textstyle\sum_j \Omega_j\{(\epsilon_j-\lambda)^2+\Delta^2\}^{-1/2}=1 \quad (3.15)$$

combined with Eq. (3.3). The u_j and v_j are given as

$$u_j{}^2=\tfrac{1}{2}[1+(\epsilon_j-\lambda)\{(\epsilon_j-\lambda)^2+\Delta^2\}^{-1/2}],$$
$$v_j{}^2=\tfrac{1}{2}[1-(\epsilon_j-\lambda)\{(\epsilon_j-\lambda)^2+\Delta^2\}^{-1/2}]. \quad (3.16)$$

These equations have exactly the same form as the corresponding ones in the usual BCS theory. The only difference is the appearance of the term $4\lambda_2 v_j{}^2$ in ϵ_j (3.4). The role of the chemical potential is played by λ (3.5) which is determined so that the condition (3.7)

[10] There is a subtle difference. In the BCS theory one minimizes $\mathcal{H}_{00}$ taking its variation with respect to v_j. *Later* one chooses λ so that Eq. (3.7) is satisfied. These two things cannot be done simultaneously because the Lagrangian multiplier λ is supposed to be independent of the variational parameter v_j, whereas λ depends on v_j through Eq. (3.7). In our approximation, however, these two procedures can be done simultaneously because λ_1 and λ_2 can be regarded as constants. Bayman's method [Nucl. Phys. **15**, 33 (1960)] can be reformulated to give our procedure.

is satisfied. Using Eq. (3.14), $\mathcal{H}_{11}$ can be rewritten as

$$\mathcal{H}_{11}=\sum_j[\{(\epsilon_j-\lambda)^2+\Delta^2\}^{1/2}+\lambda_2]\mathfrak{N}_j. \quad (3.17)$$

If the function $f(N)$ were known here which would completely eliminate the effect of the nucleon-number fluctuation, it would be unnecessary to invoke the condition (3.7). The values of $\lambda_1(n)$, $\lambda_2(n)$, $\cdots$ should then be independent of n. We saw an example in the preceeding section. There, exact $\lambda_1(n)$ and $\lambda_2(n)$ given by Eq. (2.7) are independent of n. The average number of nucleons in $\Psi_{\rm vac}$ is irrelevant, although the condition (3.7) has been used in the successive steps which has led to the exact answer. Eq. (2.5) shows that the dependence of $\lambda_1(n)$ and $\lambda_2(n)$ on n decreases as the approximation is improved.

KS have shown that their λ, which corresponds to our λ (3.5), increases almost linearly with increasing n. As we shall see in Sec. 3.B our $\lambda_2(n)$ is not sensitive to the change of n, implying the approximate linear dependence of λ on n. A merit of the second quantization formalism is that the same Hamiltonian can describe systems of arbitrary number of particles. The parameters in the Hamiltonian are usually supposed to be independent of the number of particles in the system. In this sense the BCS theory which involves the number-dependent parameter λ is not quite satisfactory.

In solving their gap equation KS put $\epsilon_j=E_j$, ignoring the self-energy correction $-Gv_j^2$. According to KLM, however, although this self-energy correction has an appreciable effect on λ and Δ, its influence on energies and wave functions is negligible. This is an important character of the gap equation. Our ϵ_j has another additional term $4\lambda_2{}^2v_j^2$. As will be seen in Sec. 3.B, λ_2 seems to be larger but only a little larger than $\frac{1}{4}G$, thus $(4\lambda_2-G)v_j^2$ is smaller than Gv_j^2. Presumably it will be permissible to put $\epsilon_j=E_j$ as it is so in KS's case.[10a]

Once we put $\epsilon_j=E_j$, then λ_1 and λ_2 appear in the combination λ (3.5) in the gap equation and consequently our u_j, v_j and λ agree with those of KS. This will furnish us with an enormous simplicity in our successive approximation.

B. Successive Approximation

Before describing the successive procedure, it will be appropriate to discuss a special choice of $\lambda_2(n)$. If $4\lambda_2=G$, the $(j\rightarrow j)$ elements of $\mathcal{H}_{31}$ and $\mathcal{H}_{40}$ vanish. This is already a considerable improvement over the BCS approximation. Moreover, $\epsilon_j=E_j$ holds exactly. What KS solved is in fact the gap equation of this case.

In the nondegenerate model, it is obviously impossible to completely eliminate $\mathcal{H}_{31}+\mathcal{H}_{40}$ with any value of $\lambda_2(n)$. A $(j\rightarrow j'\neq j)$ element of $\mathcal{H}_{31}$ or $\mathcal{H}_{40}$ may vanish for some $\lambda_2(n)>\frac{1}{4}G$, but then their $(j\rightarrow j)$ elements will survive. For some $\lambda_2(n)>\frac{1}{4}G$, however, the effect of $\mathcal{H}_{31}+\mathcal{H}_{40}$ may be strongly suppressed in the average as compared with that in the BCS approximation which assumes $\lambda_2(n)=0$.

Now we want to determine $\lambda_1(n)$ and $\lambda_2(n)$ by a successive procedure as was illustrated in Sec. 2. Instead of starting with $\lambda_2{}^{(1)}(n)=0$, we may assume first that

$$\lambda_2{}^{(1)}(n)=\tfrac{1}{4}G, \quad (3.18)$$

and calculate $\mathcal{H}_{00}+\lambda_1{}^{(1)}n+\lambda_2{}^{(1)}n^2$, the ground-state energy, which we denote by $E_0{}^{(1)}(n)$. From Eq. (3.8) we see that

$$E_0{}^{(1)}(n)=E_0{}^{\rm KS}(n)-G\sum_j\Omega_j(u_jv_j)^2$$
$$=2\sum_j\Omega_jv_j{}^2E_j-G^{-1}\Delta^2-\tfrac{1}{2}Gn. \quad (3.19)$$

Here $E_0{}^{\rm KS}$ stands for the ground-state energy obtained by KS using the BCS approximation. Now one can easily show that

$$dE_0{}^{(1)}(n)/dn=\lambda-\tfrac{1}{2}G. \quad (3.20)$$

Combining this with Eq. (3.5) one gets

$$dE_0{}^{(1)}(0)/dn=\lambda_1{}^{(1)},\ \tfrac{1}{2}d^2E_0{}^{(1)}(0)/dn^2=\lambda_2{}^{(1)}=\tfrac{1}{4}G. \quad (3.21)$$

With the exception of the degenerate model, λ_m does not vanish for $m>2$ in general, and hence $\lambda_2(n)$ deviates from $\frac{1}{4}G$ for $n>0$. It is, therefore, a better approximation to expand $f(n)$ around the nucleon number of the nucleus under consideration than to expand it around $n=0$. The λ_2 in the formulas (3.8$\sim$13) and (3.17) should then be understood as $\lambda_2(n)$.

Next step is to assume

$$\lambda_2{}^{(2)}(n)=\tfrac{1}{2}(d^2/dn^2)E_0{}^{(1)}(n)\approx\{E_0{}^{(1)}(n+2)$$
$$+E_0{}^{(1)}(n-2)-2E_0{}^{(1)}(n)\}/8. \quad (3.22)$$

Or we may use the more convenient formula

$$\lambda_2{}^{(2)}(n)=\tfrac{1}{2}d\lambda/dn$$
$$=\tfrac{1}{2}(\sum_j\Omega_jE_j{}^{-3})\{\Delta^2(\sum_j\Omega_jE_j{}^{-3})^2$$
$$+(\sum_j\Omega_j(E_j-\lambda)E_j{}^{-3})^2\}^{-1}, \quad (3.23)$$

which can be derived by using Eqs. (3.7) and (3.15). An advantage of this formula over Eq. (3.22) is that it is free from the error which may be caused by the error in the neighboring nuclei.[11] We used Eq. (3.22) because

[10a] *Note added in proof.* This has been confirmed by solving the gap equation without ignoring the self-energy correction. This will be reported in Nuclear Physics by Y. Nogami and I. J. Zucker.

[11] KS gave only two or three figures to λ and Δ, which may introduce an error of the order of 20 keV in the energy. Because the error in $E_0{}^{(1)}(n)$ may be amplified through Eq. (3.22), KS's values of λ and Δ are not sufficiently accurate for the present purpose. We did not attempt to refine KS's result but adopted slightly different values of λ, namely $\lambda=0.60$ and 1.08 for $n=6$ and 8, respectively, instead of $\lambda=0.59$ and 1.09. The value of Δ is insensitive to these changes. If we took $\lambda=0.59$ for $n=6$, then $\langle N\rangle_{\rm vac}=5.98<6$, while for all other n, $\langle N\rangle_{\rm vac}$ deviates the other way. This fluctuation in the error causes an appreciable error in $\lambda_2{}^{(2)}$ through Eq. (3.22). For $n=8$, $\lambda=1.09$ gave $\langle N\rangle_{\rm vac}=8.05$ which is less accurate than the one adopted here. In determining λ_2 by Eq. (3.22), we put $\langle N\rangle_{\rm vac}=n$. To obtain $E_{\rm KLM}$[12] we used the original values of λ and Δ of KS.

TABLE I. The ground-state energy (MeV) of the even-even nickel isotopes Ni^{58}-Ni^{66}. E_{KLM} is the exact energy.[a] E_{KS} is KS's result.[b] $E_0^{(1)}$ and $E_0^{(2)}$ are of our first and second approximations, respectively. The coupling constant G was taken to be 0.331 MeV and the positions of the single-particle levels were chosen as $E(p_{3/2})=0$, $E(f_{5/2})=0.78$ MeV, $E(p_{1/2})=1.56$ MeV, and $E(g_{9/2})=4.52$ MeV. The fourth figure of the energy has little significance.

A	n	E_{KS}	$E_0^{(1)}$	$E_0^{(2)}$	E_{KLM}
58	2	−1.127	−1.382	−1.458	−1.494
60	4	−1.510	−1.904	−2.037	−2.111
62	6	−1.087	−1.540	−1.750	−1.751
64	8	0.223	−0.207	−0.440	−0.508
66	10	2.476	2.165		1.698

[a] See Ref. 12.
[b] See Ref. 11.

we did not notice Eq. (3.23) before the calculation. Since both formulas give almost the same values for $\lambda_2^{(2)}$, we did not repeat the calculation using Eq. (3.23).

We neglect the self-energy correction for E_j (3.4) and put $\epsilon_j=E_j$, then as was noted before the gap equation and consequently u_j and v_j of the second step remain the same as those of the first step. This is an essential simplifying factor of our successive procedure. We ought to solve the gap equation once and only once. To proceed further is almost trivial. The ith approximation to the ground-state energy takes the form

$$E_0^{(i)}(n)=E_0^{KS}(n)-\lambda_2^{(i)}\sum_j \Omega_j(u_jv_j)^2. \quad (3.24)$$

This method has been applied to the even-even nickel isotopes, Ni^{58}-Ni^{66}, which have been treated approximately by KS and exactly by KLM. The same coupling constant G and single-particle energies E_j as those of KS and KLM have been used, namely $G=0.331$ MeV, $E(p_{3/2})=0$, $E(f_{5/2})=0.78$ MeV, $E(p_{1/2})=1.56$ MeV, and $E(g_{9/2})=4.52$ MeV. In Table I the ground-state energies of KS and of our first and second approximations are listed and compared with KLM's exact answer E_{KLM}.[12] It is seen that an excellent accuracy is attained at the second step. There is an irregularity in the error, namely $E_0^{(2)}$ is very close to E_{KLM} at $n=6$ whereas it is less so for $n\neq 6$. We believe this comes from the insufficient accuracy of the λ and Δ used. To proceed further requires little effort, but it will not make much sense unless a more accurate solution of the gap equation is available. The value of $\lambda_2^{(2)}(n)$ and other relevant quantities are listed in Table II.[11]

TABLE II. Quantities relevant to the calculation. The unit for λ, Δ and $\lambda_2^{(2)}$ is MeV. λ and Δ were taken from KS but λ for $n=6$ and 8 were slightly modified.[a] $(\Delta n)^2=\langle N^2\rangle_{vac}-\langle N\rangle_{vac}^2$. The coupling constant G and the single-particle energies are the same as those in Table I.

n	λ	Δ	$\langle N\rangle_{vac}$	$\frac{1}{4}(\Delta n)^2$	$4\lambda_2^{(2)}$
2	−0.31	0.80	2.01	0.77	0.430
4	0.14	1.04	4.02	1.19	0.443
6	0.60	1.15	6.02	1.37	0.484
8	1.08	1.14	8.01	1.30	0.520
10	1.64	0.99	10.03	0.94	

[a] See Ref. 11.

[12] KLM listed the difference between their exact energy and KS's approximate one, but not the exact energy itself. Hence, we first obtained KS's energy using λ and Δ given by KS and then added the difference to it to get KLM's exact energy, which is listed in Table I. The fourth figures will be of little significance.

It has been assumed throughout that the ground state is given by the quasiparticle vacuum state Ψ_{vac}, which is not an exact eigenstate of $\mathcal{H}$ because $\mathcal{H}_{40}\neq 0$. However, the fact that $E_0^{(2)}$ is very close to the exact energy E_{KLM} implies that the term $\mathcal{H}_{40}$ has been strongly suppressed so that Ψ_{vac} is a good approximation to the ground state of $\mathcal{H}$. As its form suggests, $\mathcal{H}_{31}$ will also be well suppressed at the same time. It should be noted that this Ψ_{vac}, which is the same as KS's is a rather poor approximation to the ground state of the BCS or KS Hamiltonian which does not contain the term $\lambda_2 N^2$.

C. Excited States

Having determined λ and $\lambda_2(n)$ by the preceeding procedure, we may ignore $\mathcal{H}_{31}$ and $\mathcal{H}_{40}$. The Hamiltonian to be considered now is

$$\mathcal{H}'=\mathcal{H}_{11}+\mathcal{H}_{22} \quad (3.25)$$

which conserves the number of quasiparticles. The vacuum state Ψ_{vac} is the exact ground state of $\mathcal{H}'$, and excited states are constructed by operating $\alpha^\dagger$'s on Ψ_{vac}.

Let us introduce an operator

$$\mathcal{A}^\dagger(j_1j_2JM)=\sum_{m_1,m_2}(j_1j_2m_1m_2|JM)\alpha_{j_1m_1}^\dagger\alpha_{j_2m_2}^\dagger \quad (3.26)$$

of which a special case is $\mathcal{A}^\dagger(jj00)=\mathcal{A}_j^\dagger$. An excited state of seniority 2 is given by

$$\Psi_2(j_1j_2JM)=\mathcal{A}^\dagger(j_1j_2JM)\Psi_{vac} \quad (3.27)$$

where $J\neq 0$ and the suffix 2 refers to the seniority. $\Psi_2(j_1j_2JM)$ with $J\neq 0$ is an eigenstate of $\mathcal{H}'$ because $\mathcal{A}_j\Psi_2(j_1j_2JM)=0$ for $J\neq 0$. Denote by $\Delta E_2(j_1j_2)$ the excitation energy of $\Psi_2(j_1j_2JM)$ with $J\neq 0$ which are degenerate with respect to J and M. Then we get

$$\Delta E_2(jj)=2\{(\epsilon_j-\lambda)^2+\Delta^2\}^{1/2}+2(4\lambda_2-G)(u_jv_j)^2, \quad (3.28)$$

and

$$\Delta E_2(j_1j_2)=\{(\epsilon_{j_1}-\lambda)^2+\Delta^2\}^{1/2}+\{(\epsilon_{j_2}-\lambda)^2+\Delta^2\}^{1/2}+2\{2\lambda_2(u_{j_1}^2v_{j_1}^2+u_{j_2}^2v_{j_2}^2)-Gu_{j_1}v_{j_1}u_{j_2}v_{j_2}\}, \quad (3.29)$$

for $j_1\neq j_2$.

Excitation energies of low-lying excited states of Ni^{58}-Ni^{64} have been estimated by Eqs. (3.28) and (3.29). The $\lambda_2^{(2)}(n)$ listed in Table II has been used for λ_2. The result is shown in Fig. 1 together with those of KS and KLM. Our excitation energies are nearly the same as KS's. This means that the residual pairing

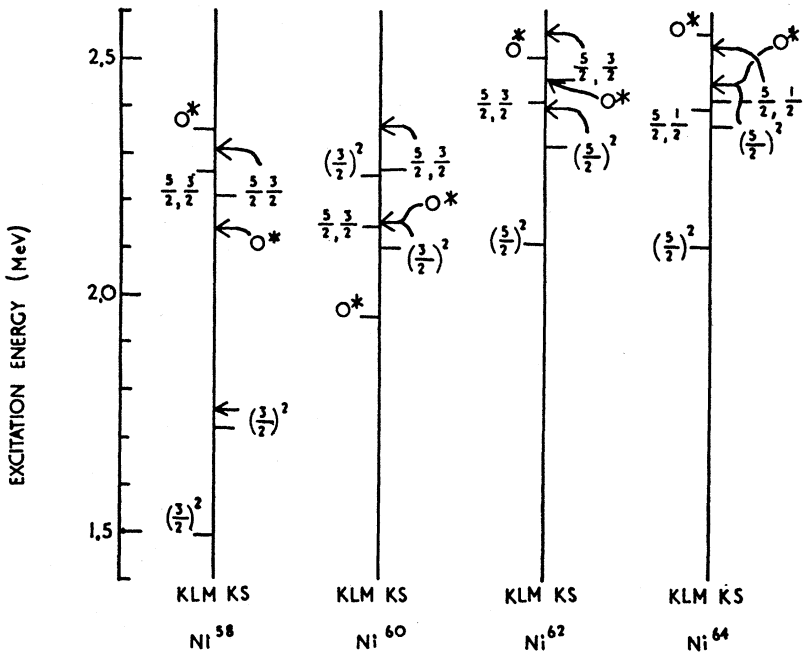

FIG. 1. Low-lying excited states in the even-even nickel isotopes. The notation $(\frac{5}{2},\frac{3}{2})$, etc., denotes the single-particle levels occupied by the unpaired particles, while 0* labels the first seniority=zero state. The entries above KLM are exact positions obtained by Kerman *et al.*,[4] while those above KS were given by Kisslinger and Sorenson.[2] Our result is shown by arrows. The same coupling constant and single-particle energies as in obtaining Table I and $\lambda_2^{(2)}$ given in Table II have been used.

interaction between quasiparticles which is neglected in the BCS theory is in fact not important for the excited states with nonzero seniority. It seems that the error is mainly due to the nucleon number fluctuation.

The excited state with seniority zero is constructed by adding paired quasiparticles to the vacuum. This is a sort of collective excitation caused by the last term in $\mathfrak{H}_{22}$ (3.11). In the degenerate model discussed in Sec. 2 eigenvalues of $\mathfrak{H}$ are independent of the number of paired quasiparticles. Here it is not true as it is, but is related to the appearance of spurious states.

Let us look for an operator Q which gives an excited state with seniority zero by

$$\Psi_0'=Q\Psi_0 \tag{3.30}$$

where Ψ_0 is the model ground state which is the lowest eigenstate of $\mathfrak{H}$. The Schrödinger equation is

$$[\mathfrak{H},Q]\Psi_0=\Delta E_0 Q\Psi_0, \tag{3.31}$$

where ΔE_0 is the excitation energy. Now it is important to note that there are a set of spurious states which are degenerate with Ψ_0, i.e., which satisfy Eq. (3.31) with $\Delta E_0=0$. As was pointed out by Anderson and by Baranger[13] this is due to the fact that the nucleon number operator

$$N=\sum_j\{2\Omega_j v_j^2+(u_j^2-v_j^2)\mathfrak{N}_j +2\Omega_j^{1/2}u_jv_j(\mathfrak{A}_j^\dagger+\mathfrak{A}_j)\} \tag{3.32}$$

is commutable with $\mathfrak{H}$, and hence the states $N^m\Psi_0$ $(m=0,1,2,\cdots)$ are degenerate eigenstates of $\mathfrak{H}$. These spurious states $(m\neq 0)$ appear because of the uncertainty of the nucleon number and may be discarded.

Our approximation is to ignore $\mathfrak{H}_{31}$ and $\mathfrak{H}_{40}$ and to replace Ψ_0 by $\Psi_{\rm vac}$, and to look for an excited state

$$\Psi_0'=Q\Psi_{\rm vac} \quad \text{with} \quad Q=\sum_j c_j\mathfrak{A}_j^\dagger, \tag{3.33}$$

which satisfies

$$[\mathfrak{H}',Q]\Psi_{\rm vac}=\Delta E_0 Q\Psi_{\rm vac}. \tag{3.34}$$

It is interesting to see that, if we replace the coefficient of $\mathfrak{A}_j^\dagger\mathfrak{A}_{j'}$ in H_{22} (3.11)

$$G(u_j^2u_{j'}^2+v_j^2v_{j'}^2)+8\lambda_2u_jv_ju_{j'}v_{j'} \tag{3.35}$$

by G, Eq. (3.34) can be reduced to the simple equation

$$G\sum_j\Omega_j\{\Delta E_2(jj)-G\Omega_j-\Delta E_0\}^{-1}=1 \tag{3.36}$$

which may be solved graphically. This is not very misleading because the value of Eq. (3.35) is in fact fairly close to G. However, we solve Eq. (3.34) accurately. This can be reduced to a standard problem for a computer, namely to solving the secular equation

$$|h_{jj'}-\Delta E_0\delta_{jj'}|=0, \tag{3.37}$$

where

$$h_{jj}=\langle\mathfrak{A}_j\mathfrak{H}'\mathfrak{A}_j^\dagger\rangle_{\rm vac}=\Delta E_2(jj)-\Omega_j \times\{G+2(4\lambda_2-G)(u_jv_j)^2\}, \tag{3.38}$$

and for $j\neq j'$

$$h_{jj'}=\langle\mathfrak{A}_j\mathfrak{H}'\mathfrak{A}_{j'}^\dagger\rangle_{\rm vac}=-(\Omega_j\Omega_{j'})^{1/2}\{G(u_j^2u_{j'}^2+v_j^2v_{j'}^2) +8\lambda_2u_jv_ju_{j'}v_{j'}\}. \tag{3.39}$$

[13] P. W. Anderson, Phys. Rev. **112**, 1900 (1958); M. Baranger, *ibid.* **120**, 957 (1960). See also J. Högaasen-Feldman, Nucl. Phys. **28**, 258 (1961).

TABLE III. The excitation energies (MeV) of states with seniority zero in the even-even nickel isotopes Ni^{58}–Ni^{64}. These are the roots of Eq. (3.37). The lowest state with ΔE_{sp} is spurious.

n	ΔE_{sp}	$\Delta E'$	$\Delta E''$	$\Delta E'''$
2	0.041	2.14	3.88	8.50
4	0.063	2.16	3.38	7.67
6	0.113	2.46	2.87	6.77
8	0.154	2.45	2.94	5.77

Eq. (3.34) combined with Eq. (3.33) has a spurious solution. Since Eq. (3.34) is an approximate equation, the excitation energy of the spurious state does not vanish exactly, but its deviation from zero may give a measure of the accuracy of our approximation.

Four solutions of Eq. (3.37), of which the lowest one is spurious, are listed in Table III. The excitation energies are denoted by $\Delta E_{\rm sp}$, $\Delta E_0'$, $\Delta E_0''$ and $\Delta E_0'''$. The spurious excitation energy $\Delta E_{\rm sp}$ is satisfactorily small. The excitation energies are compared with KLM's in Fig. 1. The agreement is rather good. The normalized coefficients c_j of Eq. (3.33) are listed in Table IV. The normalized coefficients of the state

$$\sum_j \Omega_j^{1/2} u_j v_j \mathfrak{A}_j^\dagger \Psi_{\rm vac} \tag{3.40}$$

which will be closer to an exact spurious state $\sum_j \Omega_j^{1/2} \times u_j v_j \mathfrak{A}_j^\dagger \Psi_0$ than our spurious solution is, are also shown in Table IV.

Strictly, the smallness of the $\Delta E_{\rm sp}$ obtained shows that the commutator $[N, \mathcal{H}_{31}+\mathcal{H}_{40}]$ has been suppressed, but it does not necessarily imply that $\mathcal{H}_{31}+\mathcal{H}_{40}$ has been suppressed. However, the above comparison (Fig. 1) with KLM's exact answer will justify the neglect of $\mathcal{H}_{31}$ and $\mathcal{H}_{40}$.

TABLE IV. The normalized amplitudes of the various components in the spurious state and in the excited states with seniority zero in Ni^{58}–Ni^{66}. The column "Eq. (3.40)" shows the normalized amplitudes of the state (3.40). Other four are for four solutions of Eq. (3.34) with the excitation energies ΔE_{sp}, $\Delta E'$, $\Delta E''$, and $\Delta E'''$, respectively.

j	Eq. (3.40)	Sp. sol.	1st	2nd	3rd
$n=2$					
$p_{3/2}$	0.751	0.751	−0.650	−0.087	−0.082
$f_{5/2}$	0.584	0.595	0.743	−0.263	−0.158
$p_{1/2}$	0.225	0.218	0.139	0.958	−0.123
$g_{9/2}$	0.212	0.187	0.083	0.071	0.976
$n=4$					
$p_{3/2}$	0.643	0.622	−0.777	−0.076	−0.066
$f_{5/2}$	0.677	0.709	0.612	−0.313	−0.160
$p_{1/2}$	0.271	0.271	0.135	0.944	−0.130
$g_{9/2}$	0.236	0.194	0.066	0.069	0.976
$n=6$					
$p_{3/2}$	0.535	0.508	−0.852	−0.108	−0.057
$f_{5/2}$	0.731	0.776	0.515	−0.333	−0.148
$p_{1/2}$	0.328	0.320	0.081	0.933	−0.141
$g_{9/2}$	0.269	0.193	0.040	0.078	0.977
$n=8$					
$p_{3/2}$	0.450	0.422	−0.110	−0.898	−0.052
$f_{5/2}$	0.734	0.782	−0.432	0.428	−0.134
$p_{1/2}$	0.404	0.416	0.892	0.094	−0.151
$g_{9/2}$	0.309	0.194	0.073	0.025	0.978

D. Odd Nuclei

The odd nucleus or the state of seniority one may be given by

$$\Psi_1(j)=\alpha_j^\dagger \Psi_{\rm vac}. \tag{3.41}$$

The interactions $\mathcal{H}_{31}$ and $\mathcal{H}_{40}$ being ignored, $\Psi_1(j)$ is an eigenstate of $\mathcal{H}$ and the additional quasiparticle moves independently in the orbit (jm). The difference between the energy of the seniority-one state and that of the seniority-zero state is given by

$$\Delta E_1(j)=\{(\epsilon_j-\lambda)^2+\Delta^2\}^{1/2}+\lambda_2. \tag{3.42}$$

In our approximation the ground-state energies of the even and odd nuclei are given as

$$\begin{aligned} E_0(n)&=\mathcal{H}_{00}+\lambda_1 n+\lambda_2 n^2, && \text{(even } n\text{)} \\ E_1(n)&=\mathcal{H}_{00}+\lambda_1 n+\lambda_2 n^2+\Delta E_1. && \text{(odd } n\text{)} \end{aligned} \tag{3.43}$$

Here $\mathcal{H}_{00}$ is very small and in particular its dependence on n is negligible. The even-odd mass difference is then given as

$$\begin{aligned} P_n&=M(n)-\tfrac{1}{2}\{M(n+1)+M(n-1)\} \\ &=E_1(n)-\tfrac{1}{2}\{E_0(n+1)+E_0(n-1)\} \\ &=\{(\epsilon_j-\lambda)^2+\Delta^2\}^{1/2}, \end{aligned} \tag{3.44}$$

which agrees with KS's result. For excited states of odd nuclei, our result does not show any remarkable deviation from KS. Recently Fano *et al.*[14] have attempted to account for the observed pairing energies or the empirical even-odd mass differences in several light spherical nuclei and deformed nuclei by using realistic interactions combined with the BCS plus Hartree-Fock model, but their resulting theoretical pairing energies are in general much smaller than the observed values. They then suggested that the most important reason for this discrepancy will be given by the effect of the residual interaction between the quasiparticles.

In our treatment the major part of the residual pairing interaction between quasiparticles has been taken into account, nevertheless we have got the same even-odd mass difference as KS. The discrepancy of Fano *et al.* may imply that the effect of the residual interaction between quasiparticles on the excitation energies is important if realistic interaction are taken instead of our simple pairing interaction.

4. PROJECTED BCS STATES

It has been shown by KLM that projecting out and normalizing that part of the BCS state that corresponds

[14] G. Fano, J. Sawicki, and A. Tomasini, Nuovo Cimento **29**, 309 (1963).

to the correct number of nucleons yields a remarkably close approximation to the true eigenfunction.[4,7] This may sound surprising in view of the fact that the BCS state does not give a very good approximation to the energy. Now we may understand this as follows.

Suppose $\Phi_0(n)$ defined by Eq. (1.2) are all degenerate ground eigenstates of $\mathcal{H}$, and that the BCS vacuum state Ψ_{vac} agrees with Ψ_0 which is the lowest eigenstate of $\mathcal{H}$. Then Ψ_{vac} is a superposition of $\Phi_0(n)$'s,

$$\Psi_{\text{vac}}=\sum_{n'} c_{n'}\Phi_0(n'). \tag{4.1}$$

The point is that no excited state enter here, and after projecting out obviously only $\Phi_0(n)$ remains, which is the true ground state having desired nucleon number n. The preceeding analyses show that this is realized in a good approximation. Decisive factors in our argument are that (i) the gap equation and consequently the wave function are little influenced by the introduction of the term $\lambda_2 N^2$, and that (ii) the other dangerous term $\mathcal{H}_{40}$ is strongly suppressed by suitably choosing λ_2. Because of (i), the BCS vacuum state is very close to our vacuum state. Actually in the course of the successive approximation to get λ_2, we have assumed that they are the same. Because of (ii) the vacuum state becomes a very good approximation to Ψ_0.

For the excited states with nonzero seniority our treatment does not give better results than KS, but it may be possible to readjust λ_1 and λ_2 so that accurate energies of excited states are obtained but at the sacrifice of the accuracy for the ground state. Then the above argument may be applied to the excited states. At the present, however, it still sounds intriguing that the projected and renormalized BCS excited states are very close to the true excited states.

5. CONCLUSION

We have investigated the pairing interaction in nuclei using the model Hamiltonian $\mathcal{H}=H-\lambda_1 N-\lambda_2 N^2$, where the parameters λ_1 and λ_2 are chosen so that the effect of the nucleon number fluctuation on the ground-state energy is eliminated.

An excellent accuracy has been attained for the ground-state energy, and at the same time the quasiparticle nonconserving interaction has been strongly suppressed. On the basis of this result an explanation has been given to the problem of the projected BCS states. For the excitation energies of the excited states with nonzero seniority our approximation has given more or less the same result as that of KS. The excited states with seniority zero have been obtained easily with fairly good accuracy. It is concluded that the introduction of the term $\lambda_2 N^2$ makes the theory more powerful and transparent without causing mathematical complication.

ACKNOWLEDGMENTS

The author is grateful to Dr. I. J. Zucker for his kind help in the computations, and to Dr. F. J. Bloore for helpful discussions in the early stages of this work. This work was begun at the National Research Council and finished at Battersea College of Technology. The author would like to express his sincere thanks to Dr. Ta-You Wu and to Dr. L. R. B. Elton for their hospitality at the National Research Council and Battersea College of Technology, respectively.