Canonical Form of the Wave Equations for Massless Particles and Their Observables*

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Canonical forms of the wave equations for the neutrino and for the photon have been derived and operators for the position and other dynamical variables have been suggested.

INTRODUCTION

A SYNTHESIS of one-particle relativistic wave equations describing charged particles of finite mass and finite spin has been given by Foldy.[1] This synethesis results in a "canonical" form which is highly suitable for the physical interpretation of the particles. An obvious extension of Foldy's analysis has been carried out by Fronsdal[2] for the massless particles, the neutrino and the photon; he makes use of Wigner's[3] method for the determination of the unitary irreducible representations of the inhomogeneous Lorentz group which results in the construction of one-component representations for the above mentioned particles. Using the procedure of Foldy and Wouthuysen,[4] "position operators" are then suggested for these particles but are found unsatisfactory as they do not transform as polar vectors.

The question of position operators for these massless particles has been discussed by many authors, notable among them being Pryce,[5] Newton and Wigner,[6] and Acharya and Sudarshan.[7] The position operator suggested by Pryce for the photon is

$$\mathbf{q}=\mathbf{x}-(\mathbf{s}\times\mathbf{p})/p^2, \tag{1}$$

which enjoys all the properties desired of a position operator[8] (see the next paragraph) except the condition of commutability of the components. Newton and Wigner deduce the localized states and the position operators for particles by using the quantum theoretical description of the measuring process and the relativistic invariance of the wave function. This method is successful for all particles with finite mass and spin and for massless particles with spins 0 and $\frac{1}{2}$. But within their formalism no localized state exists for photons and consequently a position operator is not definable. Acharya and Sudarshan, accepting the statement that "the photon can not have a localized state in the sense of being an eigenfunction of the three components of the vector position operator," investigate as to what maximum configurational description one can assign in such a case. This leads to a 'front' description for the position operator for the photon which again suffers from the defect of noncommutability of the components.

The purpose of the present paper is twofold. First we want to give a canonical description for the massless particles. Then the canonical description due to its extreme simplicity will be exploited to assign operators for the dynamical variables. In this study we shall employ the method recently developed by Mathews and the present author[8] who specifically require the position operator: (i) to satisfy the usual canonical commutation relation with momentum; (ii) to be defined separately on the positive and negative energy states and consequently to give rise to the correct relativistic velocity; (iii) to transform like a polar vector under space rotations and reflections and to be invariant under time inversion; (iv) to have commuting components. In the usual representation[9] (D representation) we take the four-component neutrino equation and the modified Kemmer equation[10,11] for the photon for our present work. Of course it is to be noted that space reflections are also included so as to make the representation irreducible. It might look curious that we prefer to work with the above equations instead of the now fashionable Weyl equation for the neutrino or Good's[12] equation for the photon, the reason being that the free particles are not intrinsically parity nonconserving and in fact parity is meaningful only relative to the fields with which the particle interacts. Hence our choice of wave equation is reasonable as long as one deals with free particles.

NEUTRINO

The four-component wave equation for the neutrino is obtained from the Dirac equation just by setting the mass equal to zero. Of course this procedure to obtain the zero mass equation is not generally true if one chooses the general wave equation as

$$[\beta_\mu\partial_\mu+M]\psi=0. \tag{2}$$

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[5] M. H. L. Pryce, Proc. Roy. Soc. (London) **A195**, 62 (1948).
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[9] P. M. Mathews and A. Sankaranarayanan, Progr. Theoret. Phys. (Kyoto) **26**, 1 (1961).
[10] Harish-Chandra, Proc. Roy. Soc. (London) **A186**, 502 (1946).
[11] S. A. Bludman, Phys. Rev. **107**, 1163 (1957).
[12] R. H. Good, Jr., Phys. Rev. **105**, 1914 (1957).

In fact for the particle to have zero rest mass, M in Eq. (2) must be singular,[13] and the obvious possibility of making M zero is not correct since the resulting equation would not be irreducible. But in the Dirac case, M is a multiple of the unit matrix and hence it must be zero to be singular. Kemmer's equation is a typical case as we will see in the next section.

The four-component neutrino[14] equation is given by

$$\boldsymbol{\alpha}\cdot\mathbf{p}\psi=i(\partial\psi/\partial t). \quad (3)$$

Now there exists a Foldy-Wouthuysen type transformation[15] U given by

$$U^{\pm 1}=\exp[\pm(\pi/4)\beta(\boldsymbol{\alpha}\cdot\mathbf{p})/p] \quad (4)$$

$$=1/\sqrt{2}[1\pm\beta(\boldsymbol{\alpha}\cdot\mathbf{p})/p], \quad (5)$$

which would project the Eq. (3) to a canonical form. The transformed Hamiltonian in the new representation (c representation) is

$$U\boldsymbol{\alpha}\cdot\mathbf{p}U^{-1}=\beta p, \quad (6)$$

where $p=|\mathbf{p}|$. The canonical form of the neutrino equation is

$$\beta p\phi=i(\partial\phi/\partial t), \quad (7)$$

where $\phi=U\psi$.

Equation (7) lends itself easily for physical interpretation. Hence we shall study the assignment of operators in this representation. The operators in the usual representation would be gotten by means of an inverse transformation. A position operator[16] which satisfies all the properties required above excepting (iv) is

$$\mathbf{X}^c=\mathbf{x}+(A+B\beta)(\boldsymbol{\sigma}\times\mathbf{p}), \quad (8)$$

where A and B are functions of p. When the condition of commutability of components is imposed one gets a set of four operators for $\mathbf{X}^c$ for different values of A and B which are

$$\begin{array}{ll} A=0 & B=0 \\ A=-(1/p^2) & B=0 \\ A=-(1/2p^2) & B=-(1/2p^2) \\ A=-(1/2p^2) & B=(1/2p^2). \end{array} \quad (9)$$

[13] H. J. Bhabha, Rev. Mod. Phys. **21**, 451 (1949).

[14] It may be worth mentioning that the recent experiment of G. Danby, J-M. Gaillard, K. Goulianos, L. M. Lederman *et al.* [Phys. Rev. Letters **9**, 36 (1962)] has established that two kinds of neutrinos exist or at least they are different states of one four-component neutrino. This, in a way, justifies our working with the four-component equation.

[15] This transformation has originally been given by Mathews and the present author when they studied the observables of a Dirac particle in the extreme relativistic representation. In fact, this could easily be gotten from the Foldy-Wouthuysen transformation by making the mass equal to zero

$$\{U^{\pm 1}\}_{m\to 0}=\left\{\frac{E+m\pm\beta\boldsymbol{\alpha}\cdot\mathbf{p}}{[2E(E+m)]^{1/2}}\right\}_{m\to 0} =\frac{1}{\sqrt{2}}\left(1\pm\frac{\beta\boldsymbol{\alpha}\cdot\mathbf{p}}{p}\right).$$

[16] For detailed calculation see Ref. 8.

In the usual representation the position operator is given by

$$\mathbf{X}^D=U^{-1}\mathbf{X}^cU =\mathbf{x}+\frac{i\beta\boldsymbol{\alpha}}{2p}-\frac{i\beta(\boldsymbol{\alpha}\cdot\mathbf{p})\mathbf{p}+(\boldsymbol{\sigma}\times\mathbf{p})p}{2p^3} +(A/p)[i\beta\boldsymbol{\alpha}p^2-i\beta(\boldsymbol{\alpha}\cdot\mathbf{p})\mathbf{p}]+B(\boldsymbol{\sigma}\times\mathbf{p})\beta, \quad (10)$$

where A and B take the values given by Eq. (9).

Regarding other dynamical variables it may be observed that the momentum and the total angular momentum operators are uniquely determined in the usual representation by their relations to the infinitesimal generators of translations and rotations, respectively. In fact this infinitesimal translation operator is to be identified with the momentum operator within our description as it is the only operator which is invariant with respect to displacement and transforms like a four-vector under homogeneous Lorentz transformation. Corresponding to each of the four possibilities for $\mathbf{X}^D$ we have an orbital and a spin angular momentum operator which are defined respectively as $\mathbf{X}^D\times\mathbf{p}$ and $\mathbf{J}-\mathbf{X}^D\times\mathbf{p}$, where $\mathbf{J}$ is the total angular momentum. These angular momentum operators obey the usual commutation relations. The various operator representatives are given in Table I.

It is to be noted that all four are possible candidates to represent the position of the neutrino and there seems to be no reason to prefer one over another. The above operators regarding the position of the neutrino and their localized states have already been reported by Mathews and the author.[13] We have included those results in the discussion for the completion of our canonical synthesis.

PHOTON

For the description of the photon we shall use the Kemmer equation for a massless particle. As we observed earlier, this equation is not obtained merely by setting mass equal to zero. The Kemmer equation has been properly adapted by Harish-Chandra[10] and Bludman[11] so as to describe the photon. In the following we shall give a brief account of the derivation of the Hamiltonian form of the wave equation for the photon, its reduction to 'canonical' form and the assignment of operators for the dynamical variables.

The Kemmer equation is given by

$$[\beta_\mu\partial_\mu+M]\psi=0, \quad (11)$$

where the β_μs are matrices satisfying the well-known Duffin-Kemmer algebra. As in the Dirac case, one can not get the wave equation for zero rest mass just by making $M=0$, since the components of the wave function would then become infinite. In fact, in the Duffin-Kemmer theory the representation of the Lorentz transformations is reducible and M need not be a

TABLE I. Operator representatives for observables.[a]

	Neutrino	Photon
Position	$\mathbf{x}+\frac{i\beta\boldsymbol{\alpha}}{2p}-\frac{i\beta(\boldsymbol{\alpha}\cdot\mathbf{p})\mathbf{p}+(\boldsymbol{\sigma}\times\mathbf{p})p}{2p^3}$	$\mathbf{x}-\frac{\boldsymbol{\beta}}{p}+\frac{(\boldsymbol{\beta}\cdot\mathbf{p})\mathbf{p}}{p^3}-\frac{\mathbf{s}\times\mathbf{p}}{p^2}$
	$+\frac{A}{p}[i\beta\boldsymbol{\alpha}p^2-i\beta(\boldsymbol{\alpha}\cdot\mathbf{p})\mathbf{p}]+B\beta(\boldsymbol{\sigma}\times\mathbf{p})$	$+\left[A+B\frac{\boldsymbol{\alpha}\cdot\mathbf{p}}{p}\right]\frac{(\boldsymbol{\beta}\times\mathbf{p})\times\mathbf{p}}{p}$
Momentum	$\mathbf{p}$	$\mathbf{p}$
Velocity	$\left(\frac{\boldsymbol{\alpha}\cdot\mathbf{p}}{p}\right)\left(\frac{\mathbf{p}}{p}\right)$	$\left(\frac{\boldsymbol{\alpha}\cdot\mathbf{p}}{p}\right)\left(\frac{\mathbf{p}}{p}\right)$
Orbital angular momentum	$\mathbf{x}\times\mathbf{p}+\frac{i\beta\boldsymbol{\alpha}\times\mathbf{p}}{2p}-\frac{(\boldsymbol{\sigma}\times\mathbf{p})\times\mathbf{p}}{2p^2}$	$\mathbf{x}\times\mathbf{p}-\frac{\boldsymbol{\beta}\times\mathbf{p}}{p}-\frac{(\mathbf{s}\times\mathbf{p})\times\mathbf{p}}{p^2}$
	$+Api\beta(\boldsymbol{\alpha}\times\mathbf{p})+B\beta(\boldsymbol{\sigma}\times\mathbf{p})\times\mathbf{p}$	$-\left[A+B\frac{\boldsymbol{\alpha}\cdot\mathbf{p}}{p}\right]p(\boldsymbol{\beta}\times\mathbf{p})$
Spin angular momentum	$\frac{(\boldsymbol{\sigma}\cdot\mathbf{p})\mathbf{p}}{2p^2}-\frac{i\beta\boldsymbol{\alpha}\times\mathbf{p}}{2p}-Api\beta(\boldsymbol{\alpha}\times\mathbf{p})$	$\frac{(\mathbf{s}\cdot\mathbf{p})\mathbf{p}}{p^2}+\frac{(\boldsymbol{\beta}\times\mathbf{p})}{p}+\left[A+B\frac{\boldsymbol{\alpha}\cdot\mathbf{p}}{p}\right]p(\boldsymbol{\beta}\times\mathbf{p})$
	$-B\beta(\boldsymbol{\sigma}\times\mathbf{p})\times\mathbf{p}$	

[a] The values of A and B are given by Eqs. (9) and (28) for the neutrino and photon.

multiple of unity as in the Dirac case. Consequently, to describe massless particles M in Eq. (11) must be a nonvanishing singular matrix. Then, accordingly, we can define the following relations:

$$(1-M)\beta_\mu=\beta_\mu M \quad\text{and}\quad (1-M)M=0. \tag{12}$$

Now multiplying by $(1-M)$ on the left of Eq. (11) and using the relations (12), we get

$$\beta_\mu\partial_\mu(M\psi)=0. \tag{13}$$

The corresponding Klein-Gordan equation for the massless particle is

$$\square(M\psi)=0, \tag{14}$$

guaranteeing a plane-wave solution for Eq. (11). Multiplying (11) by $\partial_\rho\beta_\rho\beta_\nu$ on the left and using the properties of β matrices, we get

$$\partial_\nu(M\psi)=\partial_\rho\beta_\rho\beta_\nu(M\psi). \tag{15}$$

Now choosing $\nu=4$, Eq. (15) reduces to

$$\partial_4(1-\beta_4{}^2)(M\psi)-\boldsymbol{\partial}\cdot\boldsymbol{\beta}\beta_4(M\psi)=0. \tag{16}$$

Again a premultiplication of Eq. (11) by $M\beta_4$ and the application of Eq. (12) give

$$\partial_4\beta_4{}^2(M\psi)+\boldsymbol{\partial}\cdot\beta_4\boldsymbol{\beta}(M\psi)=0. \tag{17}$$

On adding (16) and (17) one gets

$$i(\partial/\partial t)(M\psi)=\boldsymbol{\alpha}\cdot\mathbf{p}(M\psi), \tag{18}$$

where

$$\boldsymbol{\alpha}=i(\beta_4\boldsymbol{\beta}-\boldsymbol{\beta}\beta_4). \tag{19}$$

Equation (18) gives the Hamiltonian form of the photon equation. The initial condition for the above equation is

$$[\boldsymbol{\beta}\cdot\boldsymbol{\partial}\beta_4{}^2+(1-\beta_4{}^2)M]\psi=0. \tag{20}$$

Now making $M\psi=\phi$, Eq. (18) can be written as

$$i(\partial\phi/\partial t)=\boldsymbol{\alpha}\cdot\mathbf{p}\phi. \tag{21}$$

This may be compared with the neutrino equation and the similarity is indeed gratifying. One can choose $\phi=M\psi=(\mathbf{E},\mathbf{H},0,0)$ and then $\nabla\cdot\mathbf{E}=0$. This can be easily achieved by making the choice $M=\beta_5{}^2$ where $\beta_5=(1/4!)\epsilon_{\mu\nu\lambda\rho}\beta_\mu\beta_\nu\beta_\lambda\beta_\rho$. This choice of M satisfies the relation (12).

Now we can define a Foldy-Wouthuysen type unitary transformation V which transforms (21) to a "canonical" form and consequently $\boldsymbol{\alpha}\cdot\mathbf{p}$ will assume a diagonal form $\beta_4 p$ and the Hermitian conjugation relations will be conserved. The transformation matrix which does this job is chosen to be

$$V^{\pm1}=\exp[\pm i(\pi/2)(\boldsymbol{\beta}\cdot\mathbf{p})/p] \tag{22}$$

$$=1\pm i(\boldsymbol{\beta}\cdot\mathbf{p})/p-(\boldsymbol{\beta}\cdot\mathbf{p})^2/p^2, \tag{23}$$

and the canonical form of the equation (c representation) is

$$i(\partial\phi'/\partial t)=\beta_4 p\phi', \tag{24}$$

where

$$\beta_4 p = V\boldsymbol{\alpha}\cdot\mathbf{p}V^{-1} \tag{25}$$

and β_4 is represented by a diagonal matrix

$$\beta_4 = \begin{bmatrix} 1 & & & & \cdots & & & & 0 \\ & 1 & & & & & & & \\ & & 1 & & & & & & \\ \cdot & & & -1 & & & & & \cdot \\ \cdot & & & & -1 & & & & \cdot \\ \cdot & & & & & -1 & & & \cdot \\ & & & & & & 0 & & \\ & & & & & & & 0 & \\ & & & & & & & & 0 \\ 0 & & & & \cdots & & & & 0 \end{bmatrix}. \tag{26}$$

In the c representation, again as in the case of the neutrino, one can write a general operator for position. Such an operator takes the same form as Eq. (8) and is given by

$$\mathbf{X}^c = \mathbf{x} + (A + B\beta_4)\mathbf{s}\times\mathbf{p}, \tag{27}$$

where A and B are functions of p and of degree -2 to have the correct dimension of position. Again $\mathbf{X}^c$ satisfies all the requirements (i) to (iii). The condition of commutability of components restricts the freedom of A and B drastically, but by no means determines a unique solution. In fact, exactly as in the case of the neutrino, A and B take four different sets of values satisfying the commutability condition. A straightforward evaluation of the commutator gives the following results:

$$\begin{aligned} A &= 0 & B &= 0 \\ A &= (-2/p^2) & B &= 0 \\ A &= -(1/p^2) & B &= 1/p^2 \\ A &= -(1/p^2) & B &= -(1/p^2). \end{aligned} \tag{28}$$

In the usual representation the operators are given by

$$\mathbf{X}^D = V^{-1}\mathbf{X}^c V = V^{-1}[\mathbf{x} + (A + B\beta_4)\mathbf{s}\times\mathbf{p}]V$$

$$= \mathbf{x} - \frac{\boldsymbol{\beta}}{p} + \frac{(\boldsymbol{\beta}\cdot\mathbf{p})\mathbf{p}}{p^2} - \frac{\mathbf{s}\times\mathbf{p}}{p^2} + \left[A + B\frac{\boldsymbol{\alpha}\cdot\mathbf{p}}{p}\right]\frac{(\boldsymbol{\beta}\times\mathbf{p})\times\mathbf{p}}{p}. \tag{29}$$

Thus, the four different position operators are given by substituting the various values of A and B of Eq. (28) in Eq. (29). As we observed in the neutrino case, there appears to be no reason to prefer one position operator over the others as long as one is dealing with free particles. Thus, the multiplicity of the possible position operators seems to be a common feature for all the existing massless particles. Observations made regarding the angular momentum and other operators in the case of the neutrino also follow here and the results are given in Table I. It is interesting to note that the position operators for $A = B = 0$ coincide with the position operators suggested by Mathews and the author[9,17] in the extreme-relativistic representation of the Dirac and Kemmer equations. This strengthens the view that on physical grounds particles with high energy behave like massless particles. Probably an interaction of the particle with external fields might have a bearing on a particular choice of the four position operators.

CONCLUSION

In the foregoing, it is established that Foldy's synthesis of the wave equations can easily be extended to massless particles and the possible position operators have also been obtained. Of the four possibilities for the position, the simplest one, that is just **x** in the c representation, corresponds to the "mean-position operator" of Foldy and Wouthuysen. Of course it is to be observed that none of the four operators is covariant, and it does not seem necessary to insist that observables which have no direct relationship to the generators of the Lorentz group should have a covariant character.[18] In fact, as Wigner[19] remarks, "it would be erroneous to infer that the relativistic transformation properties of a system will define all its physical properties." In particular, the configuration of the particle which does not affect the transformation properties remains physically significant.

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[17] P. M. Mathews and A. Sankaranarayanan, Progr. Theoret. Phys. (Kyoto) **27**, 1063 (1962).

[18] The author is grateful to Dr. P. M. Mathews for making this observation.

[19] E. P. Wigner, Nuovo Cimento **3**, 517 (1956).